EKT&EC — Problem List 2: to be solved 11.03

There are more problems than we will manage to solve next time. Some of them will be postponed for the next time.

Problem 1 Suppose X is a G space, $H < G$, then

$$G \overset{H}{\times} X \simeq G/H \times X$$

as G spaces. On LHS G-acts on the first factor, on RHS the action is diagonal.

(As a corollary: On $G \overset{H}{\times} G/H \simeq G/H \times G/H$.)

Problem 2 Consider a stronger definition of G-CW-complex: the cells are of the form

$$G/H \times \overset{\circ}{D}(V)\,, \qquad \text{i.e. the } H\text{-representations } V \text{ are trivial.}$$

Using [Bredon, *Compact transformation groups*, Theorem 3.3, p. 182] show that every manifold with smooth G-action has such decomposition.

Show that if X has cells of the form $G \overset{H_i}{\times} \overset{\circ}{D}(V_i)$, then it admits a finer decomposition in with the trivial actions of H_i on V_i.

I suggest we neglect the characteristic (gluing) maps and only consider decomposition into open cells.

Problem 3 Let $K = \mathbb{C}[t, t^{-1}]$, $\mathcal{O}_+ = \mathbb{C}[t]$, $\mathcal{O}_- = \mathbb{C}[t^{-1}]$. Show that any matrix $A \in \mathrm{GL}_n(K)$ can be written as

$$A = B_+ D B_-\,,$$

where $B_\pm \in GL_n(\mathcal{O}_\pm)$ and D is a diagonal matrix $D = diag(t^{k_1}, t^{k_2}, \ldots, t^{k_n})$. Deduce that every vector bundle defined by polynomial transition functions is isomorphic to a sum of line bundles.

Hint: first solve the problem for $\mathbb{C}[[t]][t^{-1}]$. *Start with* $n = 2$.

Problem 4 Describe vector bundles over the sphere S^2 up to topological equivalence.

Problem 5 Suppose $G = S^1$ acts on S^2 by rotation. Describe S^1-equivariant bundles on S^2.

Problem 6 Find a holomorphic vector bundle over $\mathbb{P}^1 \times \mathbb{C}$, such that the restrictions to $\mathbb{P}^1 \times \{0\}$ and to $\mathbb{P}^1 \times \{1\}$ are not isomorphic as holomorphic vector bundles.

Problem 7 Let V be a representation of G containing each representation as a direct summand. Show that for a compact G-CW-complex $[X, \mathrm{Gr}_n(V)]_G = Vect^n_G(X)$. Here $[X, Y]_G$ stands for G-homotopy classes.