

Equivariant K-theory and Elliptic cohomology

MIMUW, Spring semester 2026
Andrzej Weber

I will assume that the participants of the course are well familiar with the classical cohomology theory.

Introduction

Summary of Cohomology Theories on the category of Compact Spaces

0.1 Let \mathbf{Top}_\star denote the category of compact topological spaces with a distinguished point and continuous maps. We assume that the spaces are homeomorphic to polyhedra (or CW-complexes). By \star we denote the one point space or the distinguished point in the topological space X .

0.2 Depending on the circumstances we assume that the spaces are “decent”. We will discuss technical issues later. It is most convenient to assume that the spaces we consider are CW-complexes.

0.3 Let \mathbf{hTop}_\star be the homotopy category.

A short formulation of cohomology axioms

0.4 A cohomology theory consists of:

- A sequence of contravariant functors

$$\tilde{H}^n : \mathbf{hTop}_\star \rightarrow \mathbf{Ab}, \quad n \in \mathbb{Z},$$

- Natural transformations (connecting homomorphisms)

$$\delta : \tilde{H}^n(A) \rightarrow \tilde{H}^{n+1}(X/A)$$

for each pair (X, A) with $A \subseteq X$ closed.

0.5 In **standard cohomology theory** the following axioms are satisfied

- (i). Exactness Axiom

$$\cdots \rightarrow \tilde{H}^n(X/A) \rightarrow \tilde{H}^n(X) \rightarrow \tilde{H}^n(A) \xrightarrow{\delta} \tilde{H}^{n+1}(X/A) \rightarrow \cdots$$

- (ii). Dimension Axiom

$$\tilde{H}^n(S^0) = \begin{cases} R & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \quad \text{a fixed abelian group/ring (the coefficients)}$$

Here S^0 denotes the 0-dimensional sphere, i.e. two point space, one of the points is distinguished.

0.6 This data determines the theory.

0.7 We write:

$$H^n(X) = \tilde{H}^n(X^+), \text{ where } X^+ = X \sqcup \star,$$

$$H^n(X, A) = \tilde{H}^n(X/A) \text{ for } A \subset X \text{ closed (cofibration),}$$

$$H^*(X) = \bigoplus_{n \in \mathbb{Z}} H^n(X),$$

If we want to stress the coefficients (e.g. in \mathbb{Z}) we write $H^*(X; \mathbb{Z})$.

0.8 We deduce the classical set of Eilenberg-Steenrod axioms

- exactness axiom
- excision axiom
- dimension axiom
- for noncompact spaces one has to assume an additional axiom

$$H^*\left(\bigsqcup_{\alpha} X_{\alpha}\right) = \prod_{\alpha} H^*(X_{\alpha}).$$

0.9 Let

$$A \vee B = A \times \star \cup \star \times B$$

and

$$A \wedge B = A \times B / A \vee B.$$

- Show that $S^1 \wedge X$ is homotopy equivalent to $\Sigma X = [0, 1] \times X / \sim$

$$(0, x) \sim (0, y), \quad (1, x) \sim (1, y)$$

0.10 Generalized cohomology theory: There is only one axiom (i). We reject the dimension axiom.

0.11 Exactness Axiom:

$$\cdots \rightarrow \tilde{h}^n(X/A) \rightarrow \tilde{h}^n(X) \rightarrow \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \rightarrow \cdots$$

- No dimension axiom !

0.12 The knowledge of $\tilde{h}^*(S^0)$ does not determine the theory: If $\tilde{h}^*(S^0)$ is torsion free then

$$\tilde{k}^*(X) := \tilde{H}^*(X; \mathbb{Z}) \otimes \tilde{h}^*(S^0)$$

is another cohomology theory. The theories h and k have the same value for $X = S^0$, but in general they are not isomorphic.

$$\text{K-theory:} \quad \tilde{K}^*(-) \not\simeq H^*(-) \otimes \tilde{K}^*(S^0) \simeq \bigoplus_{k \in \mathbb{Z}} H^{*+2k}(-),$$

$$\text{Unitary cobordism:} \quad \tilde{U}^*(-) \not\simeq \tilde{H}^*(-) \otimes \tilde{U}^*(S^0).$$

0.13 Exercise: show that

- $\tilde{h}^*(\star) = 0$
- $\tilde{h}^*(A \wedge B) \simeq \tilde{h}^*(A) \oplus \tilde{h}^*(B)$
- $\tilde{h}^k(X) \simeq \tilde{h}^{k+1}(S^1 \wedge X)$.

0.14 Cohomology theories are represented by spectra: (see Brown's representability theorem, [R. Switzer, *Algebraic topology - homotopy and homology*, Classics in Mathematics, Berlin, New York: Springer-Verlag, pp. 152–157]).

0.15 G -spaces

Let G be a compact group, $G - \mathbf{hTop}_{\star}$ the category of compact G -spaces with a distinguished point, which is preserved by G . Additionally one has to exclude topological pathologies. The notion of G -CW-complexes is most convenient. For spaces like manifolds, algebraic varieties with smooth/algebraic group actions the topological complications do not appear.

0.16 Basic equivariant construction:

- For an orthogonal representation $H \rightarrow \text{Aut}(V)$ let $D(V)$ denote the unit ball in V and $S(V) = \partial D(V)$ the unit sphere.
- For $H \subset G$ and a H -space Y let

$$G \overset{H}{\times} Y = G \times Y / \sim, \quad (gh, y) \sim (g, hy) \text{ for } h \in H$$

We have for a G -space Z

$$\text{Map}_H(X, Z) = \text{Map}_G(G \overset{H}{\times} X, Z).$$

- Applying $\overset{H}{\times}$ we construct a building block, an „equivariant cell” – a tubular thickening $G \overset{H}{\times} D(V)$ of the orbit G/H , whose boundary $G \overset{H}{\times} S(V)$ is a sphere bundle over G/H .

0.17 A G -space X admits a G -CW decomposition if there is a filtration

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that for each k there are given a subgroup H_k , its orthogonal representation $H_k \rightarrow GL(V_k)$ and a G -homeomorphism

$$X_k \simeq X_{k-1} \cup_{f_k} (G \overset{H_k}{\times} D(V_k))$$

where $f_k : G \overset{H_k}{\times} S(V_k) \rightarrow X_{k-1}$ is the gluing data, „the characteristic map”.

- The assumption that there is a distinguished fixed point is made for convenience. If the action of G on X has no fixed point, we can artificially add a point considering $X^+ = X \sqcup \star$.

0.18 To dive deeper into the structure of G -spaces, see *slice theorem* [Bre72, §II.5].

- Example: equivariant tubular neighbourhood.
- Peculiarities of algebraic tori acting on algebraic varieties, [Car02, §4.1].

Generalized equivariant cohomology theory

0.19 Exactness Axiom

$$\cdots \rightarrow \tilde{h}^n(X/A) \rightarrow \tilde{h}^n(X) \rightarrow \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \rightarrow \cdots$$

0.20 Suspension is more involving, and it is included into axioms:

- Let $V \simeq \mathbb{R}^d$ be a linear representation of G , and $S_V = V \sqcup \{\infty\}$, i.e. one point compactification of V , which is a sphere with G action. It has at least two points: 0 and ∞ .
- Let $|V|$ denote the real dimension of V .
- There is given a family of **compatible** isomorphisms, see [tD71]

$$\tilde{h}^n(X) \simeq \tilde{h}^{n+|V|}(S_V \wedge X).$$

(Easier to say $h^n(X) \simeq h^{n+|V|}(S_V \times X, \{\infty\} \times X)$.)

0.21 Compatibilities of the equivariant suspensions:

- Behavior with respect to taking the direct sum of representations. There is given a family of isomorphisms

$$\tilde{h}^*(S_V \wedge S_W \wedge X) \simeq \tilde{h}^*(S_W \wedge S_V \wedge X) \simeq \tilde{h}^*(S_{V \oplus W} \wedge X),$$

which satisfy a (easy to guess) coherences involving signs.

- Commutation of δ with $S_V \wedge$:

We leave details for later.

0.22 Transformation of theories. Although coefficients of the theory do not determine it, but in the presence of transformation of theories one can conclude the isomorphism:

- If $k^* \rightarrow h^*$ is a transformation of generalized equivariant cohomology theories and suppose that for every orbit G/H the induced map

$$(\spadesuit) \quad k^*(G/H) \xrightarrow{\simeq} h^*(G/H)$$

is an isomorphism. Then for any G -space X

$$k^*(X) \xrightarrow{\simeq} h^*(X).$$

- Enough to assume (\spadesuit) for orbits appearing in X
- That is so because here the building blocks is not the point \star , but the orbits G/H .

Instead of developing an abstract theory let us study in detail K -theory, [Seg68].

1 Vector bundles

See [HJJS08].

1.1 (Non-equivariant) vector bundles and constructions on bundles – recollection.

We fix $\mathbb{K} = \mathbb{R}, \mathbb{C}$ (possibly \mathbb{H}), but preferably \mathbb{C} .

Definition: a vector bundle over \mathbb{K} is a map $p : E \rightarrow X$ together with continuous operations

$$+ : E \times E \rightarrow E, \quad \cdot : \mathbb{K} \times E \rightarrow E, \quad 0 : X \rightarrow E,$$

such that:

- $p : E \rightarrow X$ is a local trivial fibration,
- the maps $+$ and \cdot preserve the fibers of p , the map 0 is a section of p ,
- each fiber $E_x = p^{-1}(x)$ is a vector space w/r to $+$ and \cdot .

1.2 Examples

- Trivial bundle $X \times \mathbb{K}^n$. If $n = 1$ such bundle is often denoted by $\mathbb{1}_X$ or θ_X .
- Bundles known from differential geometry:
 - If X is a manifold over \mathbb{R} or \mathbb{C} , then we have the tangent vector bundle TX , cotangent vector bundle T^*X , also denoted by Ω_X^1 , but this can be confused with the sheaf of sections.
 - Exterior powers of the cotangent bundle — do not confuse with the shaves of forms. (But in practice often denoted by the same symbol Ω_X^p .)
 - If $Y \subset X$ is a submanifold: normal bundle $\nu_{Y/X} = (TX|_Y)/TY$, conormal bundle $\nu_{Y/X}^* = \ker(T^*X|_Y \rightarrow T^*Y)$,
 - If $f : X \rightarrow Y$ is a submersion, then we have the relative tangent bundle, $Tf \subset TX$, consisting of vectors tangent to the fibers.
- Tautological bundles: Let V be a vector space, $\dim V = n < \infty$ and let $\text{Gr}_k(V)$ be the Grassmann manifold parametrizing k -dimensional subspaces. We have
 - $S = \{(W, v) \in \text{Gr}_k(V) \times V : v \in W\}$ the tautological bundle, it is a subbundle of the trivial bundle $\underline{V} = \text{Gr}_k(V) \times V$.
 - $Q = \underline{V}/S$ - the quotient bundle.
 - As topological bundles $\underline{V} \simeq S \oplus Q$, (an isomorphism determined e.g. by a choice of a scalar/hermitian product in V). But as holomorphic bundles we have only an exact sequence.
 - In particular for $k = 1$, $\text{Gr}_1(V) = \mathbb{P}(V)$ the tautological line bundle is denoted $\mathcal{O}_{\mathbb{P}(V)}(-1)$, and the dual bundle by $\mathcal{O}_{\mathbb{P}(V)}(1)$.

1.3 For the fixed base X the class of isomorphism classes of vector bundles of given dimension n is a set, moreover (see 2.8)

$$\text{Vect}^n : \mathbf{Top}^{op} \rightarrow \mathbf{Set}$$

is a functor, which factors through \mathbf{hTop}^{op} (see 2.9).

- We often skip \mathbb{K} in the notation, meaning $\mathbb{K} = \mathbb{C}$.

1.4 Richer categories: smooth, holomorphic, algebraic bundles.

- Associated sheaves of sections.

1.5 Vector bundles can be identified with projective modules over $C(X; \mathbb{K})$.

1.6 Operation on bundles

- Direct sum $E \oplus F = E \times_X F$
- The set $\bigcup_{n \geq 0} \text{Vect}^n(X)$ is a semigroup.
- Tensor product — this demands use of defining cocycles
- On $\bigcup_{n \geq 0} \text{Vect}^n(X)$ we have a structure of a semiring
- Hom-bundle, in particular the dual bundle
- Any functor e.g. symmetric and exterior powers,

1.7 For a subbundle $F \subset E$ we have the quotient E/F

1.8 In general a morphism of bundles $f : E \rightarrow F$ might have nonconstant rank, so kernel and cokernel is not defined unless we pass to sheaves (assuming some richer structure, e.g. holomorphic).

1.9 Equivariant vector bundles

- We assume that G acts on E and X and the map p is G equivariant. Moreover, if $g(x) = y$ then the map $g|_{E_x} : E_x \rightarrow E_y$ is linear.

- In particular, if x is a fixed point, then E_x is a linear representation of G .

1.10 What are the bundles over G/H ? Given a representation $H \rightarrow \text{GL}(V)$ define a bundle over G/H

$$G \overset{H}{\times} V = G \times V / \sim, \quad (gh, v) \sim (g, hv).$$

1.11 Theorem: Let $W \rightarrow G/H$ be a vector bundle, $V = W_{eH}$ the fiber over eH , this is an H -representation. Then the natural map

$$G \overset{H}{\times} V \rightarrow W, \quad [g, v] \mapsto gv$$

is an isomorphism of G -bundles.

1.12 Corollary:

$$\text{Vect}_G^n(G/H) \simeq \text{Vect}_H^n(pt).$$

1.13 Generalization for H -space

$$\text{Vect}_G^n(G \overset{H}{\times} X) \simeq \text{Vect}_H^n(X).$$

2 Vector bundles and K-theory

2.1 Definition via cocycle

- Given a covering $\{U_i\}_{i \in I}$ and gluing functions $g_{ij} : U_i \cap U_j \rightarrow \text{GL}_n(\mathbb{C})$ such that

$$g_{ij}g_{jk} = g_{ik} \quad \text{on } U_i \cap U_j \cap U_k$$

(cocycle condition). We define

$$E = \left(\bigcup_{i \in I} \{i\} \times U_i \times \mathbb{C}^n \right) / \sim$$

$$(i, x, v) \sim (j, x, g_{ij}(x)v) \quad \text{for } x \in U_{ij}.$$

- Passing to a subcovering allows to compare cocycles defined for different coverings, like in the Čech cohomology.

- Two cocycles $\{g_{ij}\}$ and $\{g'_{ij}\}$ define isomorphic bundles if (on a finer covering) there exist functions $h_i : U_i \rightarrow \text{GL}_n(\mathbb{C})$ such that

$$g'_{ij} = h_i g_{ij} h_j^{-1} \quad \text{on } U_i \cap U_j.$$

- In other words

$$\text{Vect}^n(X) := H^1(X; \mathcal{C}(X, \text{GL}_n(\mathbb{C}))).$$

This is the „nonabelian” cohomology with coefficients in the sheaf of groups $\mathcal{C}(X, \text{GL}_n(\mathbb{C}))$ of continuous functions with values in $\text{GL}_n(\mathbb{C})$.

2.2 The same mechanism applies to equivariant bundle. But as a basic ”trivial bundles” are the bundles of the form

$$G \overset{H}{\times} (S \times V) \rightarrow G \overset{H}{\times} S = U,$$

where S is a H -space and V is a representation of H .

- We have to assume that the sets of the covering are as above, hence they are preserved by G and the cocycle is G -equivariant.

- In particular, suppose that the action of G on X is trivial, and the bundle itself is trivial. But it does not mean that we can take as a trivializing covering consisting of X .

- Example: Suppose $\mathbb{1}^n = E_1 \oplus E_2$ with the action of $G = S^1$ by two different characters on the summands.

2.3 Theorem: *Every bundle is a quotient of a trivial bundle (under standing assumption that X is compact).*

Proof (for the nonequivariant case): Let $n = \text{rk}(E)$. Suppose the bundle E has trivialization on a finite covering $\{U_i\}_{i \in I}$:

$$\varphi_i : U_i \times \mathbb{C}^n \rightarrow E|_{U_i}.$$

Assume that ψ_i is a partition of unity for the covering $\{U_i\}$. The map

$$\begin{aligned} X \times \bigoplus_{i \in I} \mathbb{C}^n &\rightarrow E \\ (x, \{v_i\}_{i \in I}) &\mapsto \sum_{i \in I} \psi_i(x) \varphi_i(x, v_i) \end{aligned}$$

is well defined and surjective. □

2.4 Better point of view: We have a natural map

$$\begin{aligned} C(X; E) \times X &\rightarrow E \\ (s, x) &\mapsto s(x). \end{aligned}$$

We pick a finite dimensional vector space $V \subset C(X; E)$, which subjects to every E_x (in the proof above the space V is spanned by ψ_i times a basis section of $E|_{U_i}$).

2.5 Proof in the equivariant case: one has two construct a finite dimensional subspace

$$V \subset W := C(X; E)$$

of the space of the global sections, such that

- at every point $x \in X$ the restriction $V \rightarrow E_x$ is a surjection,
- V is G -invariant.
- This is possible due to a version of Peter-Weyl theorem: \spadesuit *Suppose W is a topological vector space which is Hausdorff, complete and locally convex. Suppose G (a compact group) acts continuously. Then*

$$W_a = \{v \in W : \dim(\text{span } Gv) < \infty\}$$

is dense in W . [Mos61, 2.16]

- (The subscript a in the notation evokes to the Peter-Weyl classical statement, that polynomial functions are dense in all continuous functions $C(G) = C(G; \mathbb{C})$.)

- By the theorem above, at each point $x \in X$ we can choose sections spanning the fiber E_x and belonging to W_a . By compactness of X we can choose finitely many sections spanning fibers at each point. □

2.6 Proof of (\spadesuit): if $X = G$ and $E = X \times \mathbb{C}$, then this is the Peter-Weyl theorem:

$$C(G)_a \text{ is dense in } C(G)$$

with the norm sup. Suppose $w \in W$, $\phi \in C(G)$. Let

$$w_\phi = \int_G \phi(g)g(w) dg \quad (\text{Haar measure}).$$

Let $U \subset W$ be a convex neighbourhood of 0 in V . We will construct an element $w_\psi \in W_a$, such that $w - w_\psi \in 2U$. Let $V \subset G$ be a neighbourhood of e , such that $g(w) - w \in U$ for $g \in V$. Let $\psi : G \rightarrow \mathbb{R}_+$, $\text{supp}(\psi) \subset V$, $\int_G \psi(g)dg = 1$. Then

$$w_\psi - w = \int_G (\psi(g)g(w) - w)dg = \int_G \psi(g)(g(w) - w)dg \in U$$

by convexity. The map

$$\begin{aligned} C(G) &\rightarrow W, \\ \phi &\mapsto w_\phi \end{aligned}$$

is continuous. Chose $\phi \in C(G)_a$, such that

$$w_\psi - w_\phi = \int_G (\psi(g) - \phi(g))g(w) \in U$$

by density of $C(G)_a \subset C(G)$. Hence

$$w - w_\phi = w - w_\psi + w_\psi - w_\phi \in 2U.$$

The vector w_ϕ is contained in a finite dimensional representation, since for $h \in G$

$$h(w_\phi) = h \int_G \phi(g)g(w)dg = \int_G \phi(g)hg(w)dg = \int_G \phi(gh^{-1})g(w)dg = w_{h\phi}.$$

But $h\phi \in \text{span}\{g\phi : g \in G\} = \text{span}\{\phi_1, \phi_2, \dots, \phi_m\}$ which is a finite dimensional space by assumption on ϕ . By the definition of w_ϕ (linearity)

$$w_{h\phi} \in \text{span}\{w_{\phi_1}, w_{\phi_2}, \dots, w_{\phi_m}\}.$$

□

2.7 Corollary: Every G -equivariant vector bundle is a subbundle of a trivial bundle $V_X = X \times V$, where V is a representation of G and the action of G on $X \times V$ is diagonal.

- For every G bundle E there exist a G -bundle F , such that $E \oplus F \simeq V_X$.
- Every G -bundle of rank n is isomorphic to the pull-back (TBA) from the tautological bundle over $\text{Gr}_n(V)$ for an appropriate G -representation V .
- Allowing V to be of infinite dimension it is possible to fix an universal space V . It is necessary to assume that any representation of G is a direct summand of V infinitely many times.

Pull-back

2.8 For $f : X \rightarrow Y$ and a vector bundle $p : E \rightarrow Y$ let f^*E (also denoted by $f^!E$) is defined by

$$f^*E = E \times_Y X = \{(v, x) \in E \times X : p(v) = f(x)\}.$$

- We have $(f^*E)_x \simeq E_{f(x)}$.

2.9 Theorem: If $f, g : X \rightarrow Y$ are homotopic, then $f^*E \simeq g^*E$.

(I copy the proof from [AB64]. Assumption: X is a compact Hausdorff space.)

2.10 Lemma: Suppose E, F are vector bundles over X and let $Y \subset X$ be a closed subspace, such that $E|_Y \simeq F|_Y$. Then there exists an open set $U \supset Y$, such that $E|_U \simeq F|_U$.

• Consider the bundle $\text{Hom}(E, F)$. The isomorphism over Y defines a section of $\text{Hom}(E, F)|_Y$. This section can be extended on each open set of the trivializing cover, and patched together using partition of unity. We obtain a global section $\phi : X \rightarrow \text{Hom}(E, F)$. The condition „ $\phi_x : E_x \rightarrow F_x$ is an isomorphism” is an open condition. □

- In the equivariant case the same proof works, , we can extend nonequivariantly and after that average over the group G - which is compact by assumption.

2.11 Lemma: *Let $E \rightarrow X \times [0, 1]$ be a vector bundle, $\iota_t : X \rightarrow X \times [0, 1]$ the inclusion $x \mapsto (x, t)$, Then $\iota_t^* E \simeq \iota_s^* E$ for all $s, t \in [0, 1]$.*

• Let $F = p^* \iota_t^* E$, where $p : X \times [0, 1] \rightarrow X$ is the projection. Over $X_t := X \times \{t\}$ we have an isomorphism $E|_{X_t} \simeq F|_{X_t}$. Thus $E|_{X_s} \simeq F|_{X_s}$ for s sufficiently close to t . This means that $E|_{X_s} \simeq E|_{X_t}$. Using compactness of $[0, 1]$ and transitivity of the relation \simeq we obtain the claim of the lemma. \square

- Theorem 2.9 follows from Lemma in the standard way.
- For the equivariant case we have to assume that the homotopy is G -invariant.

References

- [AB64] Michael Atiyah and Raoul Bott. On the periodicity theorem for complex vector bundles. *Acta Math.*, 112:229–247, 1964.
- [Bre72] Glen E. Bredon. *Introduction to compact transformation groups*, volume Vol. 46 of *Pure and Applied Mathematics*. Academic Press, New York-London, 1972.
- [Car02] James B. Carrell. Torus actions and cohomology. In *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, volume 131 of *Encyclopaedia Math. Sci.*, pages 83–158. Springer, Berlin, 2002.
- [HJJS08] D. Husemöller, M. Joachim, B. Jurčo, and M. Schottenloher. *Basic bundle theory and K-cohomology invariants*, volume 726 of *Lecture Notes in Physics*. Springer, Berlin, 2008. With contributions by Siegfried Echterhoff, Stefan Fredenhagen and Bernhard Krötz.
- [Mos61] G. D. Mostow. Cohomology of topological groups and solvmanifolds. *Ann. of Math. (2)*, 73:20–48, 1961.
- [Seg68] Graeme Segal. Equivariant K -theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.
- [tD71] Tammo tom Dieck. Lokalisierung äquivarianter Kohomologie-Theorien. *Math. Z.*, 121:253–262, 1971.
- [Wei13] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K -theory.