Algebry i grupy Liego

29.2.2024

http://www.mimuw.edu.pl/%7Eaweber/

1 Examples first

1.1 Topological groups [Bredon: Introduction to compact transformation groups, chapter 0]

1.2 Let G be a topolpgical T_1 space with continuous map

$$m : G \times G \to G\,, \qquad \nu : G \to G$$

satisfying the axiom of a group
– the multiplication μ,
– taking the inverse ν.

• In other words G is a ,,group object" in the category of topological spaces.

1.3 Examples
- discrete groups
- $\mathbb{K}_+$, $\mathbb{K}^*$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (quaternions)
- compact torus $(S^1)^r$
- complex torus $(\mathbb{C}^*)^r$
- S^3 as a subgroup of $\mathbb{H}^*$
- $U(n)$, $SU(n)$ subgroups of $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$
- $O(n)$, $SO(n)$ subgroups of $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$
- $Sp(n)$ the subgroup of $GL_n(\mathbb{H})$ preserving the norm $|v|^2 = \sum_{i=1}^n |v_i|^2$
- matrix groups preserving a given quadratic form (or other structure, e.g. the octonionic multiplication)
- $O(m,n)$, the subgroup of $GL_{m+n}(\mathbb{R})$ preserving a nondegenerate symmetric form of the type (m,n).
- groups of isometries of a compact Riemannian manifold (can be realized as a matrix group)
- Heisenberg group N/Z where

$$N = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

cannot be realized as a matrix group

1.4 Exercise: $U(n)$, $SU(n)$, $SO(n)$, $Sp(n)$ are connected, $O(n)$ has two components

1.5 Exercise: $\pi_1(U(n)) = \mathbb{Z}$, $\pi_1(SU(n)) = 1$, $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$ (long exact sequence of homotopy groups needed)

1.6 Exercise: Elements of $Sp(n)$ preserve the form $\mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ given by $(v,w) = \sum_{i=1}^n v_i \overline{w}_i$.

1.7 Two approaches to Lie groups
- study of compact Lie groups
- study of complex algebraic reductive groups (definition later)

1.8 Noncompact or nonreductive groups are more difficult; theory of nilpotent or solvable groups is a separate subject.

1.9 But any connected Lie group G contains a maximal compact subgroup K (which is unique up to a conjugation) and as a topological space $G \simeq K \times \mathbb{R}^n$. (Cartan-Iwasawa-Malcev Theorem)

1.10 For every connected complex linear semisimple (to be defined later) group we have a decomposition (as a topological space)

$$G = K \times A \times N$$

where K is maximal compact, $A \simeq \mathbb{R}^k$, N is a nilpotent group, $\simeq \mathbb{R}^\ell$ as a topological space. This is Iwasawa decomposition. The special case is the Gram-Schmidt orthogonalization process

$$GL_n(\mathbb{R}) = O(n) \times (\mathbb{R}_{>0})^n \times N$$

where N consist of the upper-triangular matrices with 1's at the diagonal.

1.11 Every compact Lie group can be embedded into $U(n)$ as a closed subgroup.

1.12 Classification of compact connected groups [Cartan]: every such G is of the form $\tilde{G}/A$, where A is a finite abelian group and $\tilde{G} = \prod_{i=1}^k H_i$ and H_i is a torus $(S^1)^r$ or a simple[1] simply-connected compact group, which is of the form

- $SU(n)$ (Type A_{n-1})

  — $X \in M_{n\times n}(\mathbb{C})$, $\det(X) = 1$, $\overline{X}^T X = I$

- $\widetilde{SO(n)} = Spin(n)$ (Type B_m for $n = 2m+1$ or Type D_n for $n = 2m$). Here $\widetilde{SO(n)}$ means the two-fold cover.

  — $X \in M_{n\times n}(\mathbb{R})$, $\det(X) = 1$, $X^T X = I$

- $Sp(n)$ (Type C_n)

  — $X \in M_{n\times n}(\mathbb{H})$, $\overline{X}^T X = I$

- Exceptional group of the type E_6, E_7, E_8, G_2 or F_4

1.13 Definitions of the compact simple Lie groups have common pattern, while the field varies

- $\mathbb{C}$ – Type A_n (preserving hermitian product)
- $\mathbb{R}$ – Type B_n and D_n (preserving scalar product)
- $\mathbb{H}$ – Type C_n (preserving scalar product in the quaternionic space)
- octonions $\mathbb{O}$ are related to exceptional groups, e.g. $G_2 = Aut(\mathbb{O})$ preserving scalar product

1.14 For each compact Lie group G there exists a complex Lie group $G_\mathbb{C}$, the complexification of G, in which G is the maximal compact subgroup. The group $G_\mathbb{C}$ is defined by a polynomial formula in $GL_N(\mathbb{C})$ for some N

[1]Simple Lie group means that the every proper normal subgroups is finite.

- $SL(n,\mathbb{C})$ (Type A_{n-1})

- $\widetilde{SO_n(\mathbb{C})} = Spin_n(\mathbb{C})$ (Type B_m for $n = 2m+1$ or Type D_n for $n - 2m$), where $SO_n(\mathbb{C})$ is a subgroup of $SL_n(\mathbb{C})$ preserving a fixed nondegenerate symmetric form.

- $Sp_n(\mathbb{C})$ (Type C_n), where $Sp_n(\mathbb{C})$ is a subgroup of $GL_{2n}(\mathbb{C})$ preserving a fixed nondegenerate antisymmetric form.

- Complex exceptional group of the type E_6, E_7, E_8, G_2 or F_4, eg. $(G_2)_{\mathbb{C}} \subset GL_7(\mathbb{C})$ is the group preserving certain exterior 3-form.

1.15 Exercise: The real symplectic group $Sp_n(\mathbb{R}) \subset GL_{2n}(\mathbb{R})$ (appears in real symplectic geometry or in classical mechanics) is noncompact and its maximal compact subgroup is equal to $U(n)$.

Topological properties

1.16 Exercise: show that the continuity of multiplication and the inverse is equivalent to:

- equivalently $\phi : G^2 \to G^2$, $\phi(g,h) = (g, gh)$ is a homeomorphism

1.17 Exercise: Suppose $H \leq G$ is a subgroup. Then the action $G \times G/G \to G/H$ is continuous.

1.18 Theorem. Suppose $H \leq G$ is a subgroup. Then

• the quotient $\pi : G \to G/H$ map is open:

∘ To show that $\pi(U)$ is open in G/H one has to check, that $\pi^{-1}\pi(U)$ is open in G. That is so since $\pi^{-1}\pi(U) = \bigcup_{h\in H} Uh$.

1.19 If H is closed, then the space G/H is regular [every closed subset F of G/H and a point p not contained in gH admit non-overlapping open neighborhoods; hence G/H is Hausdorff].

∘ We can assume $g = 1$. The coset $1H$ is closed in G/H. To find desired nieghbourhoods it is enough to find open sets $U, V \subset G$, $1 \in U \cap V$, such that

$$(*) \qquad V \cap \Big(\bigcup_{k\in\pi^{-1}(F)} Ug \Big) = \emptyset\,.$$

(since by 1.18 the image $\pi(V)$ is open). Consider the map $\alpha : G^2 \to G$, $\alpha(g,h) = g^{-1}h$. The set $\alpha^{-1}(G \setminus \pi^{-1}(F))$ is open in G^2 and contains $(1,1)$. Thus (by the definition of the product topology) there exist open sets U, V such that $(1,1) \in U \times V$ and $U \times V \subset \alpha^{-1}(G \setminus \pi^{-1}(F))$. This means: for $g \in U$, $h \in V$: $g^{-1}h \notin \pi^{-1}(F)$. Hence (*) holds as desired.

1.20 Corollary: Any topological group is a regular topological space.

1.21 Theorem: Let G_0 be the connected component of $1 \in G$. (It is the biggest set, which is connected and contains 1.) Then

• G_0 is a closed normal subgroup.

• If G_0 is open, then G/G_0 is a discrete group.

∘ Proof that G_0 is closed. Suppose $H = \overline{G_0}$, $g \in H \setminus G_0$. Then $G_0 \cup \{g\}$ is not connected (since it is bigger than G_0), thus there exist $U \ni g$, $U \cap G_0 = \emptyset$. This contradicts that g lies in the closure.

1.22 If G is a connected topological group, then any open neighbourhood of the identity generates the group.

$\circ$ Let U be a neighbourhood of 1. We can assume U is connected. The group $H = \langle U \rangle$ generated by U is connected and open. Suppose $g \notin H$. Then $gU \cap H = \emptyset$. (Otherwise $k \in gU \cap K$, so $g = kh^{-1}$, $h \in U$.) It follows, that the set $V = \bigcup_{g \notin K} gU$ is a disjoint from K. Thus $G = K \sqcup V$ which contradicts connectedness.

1.23 Exercise: A normal discrete subrgroup of a connected group lies in its center.

Lie groups

From now on we assume that G is a C^∞-manifold, μ, ν are smooth. Homomorphism of Lie groups are assumed to be smooth maps. (It is enough to assume that μ is C^1, smoothness follows. Smoothness of ν follows from continuity.)

1.24 Any subgroup of a Lie group, which is a submanifold, is closed. The converse is also true (closed $\Rightarrow$ submanifold), but it is harder. By a closed Lie subrgroup we understand a subgroup, which is a submanifold.

1.25 Note that there are subsets, which are subgroups, and admit a structure of a Lie group. E.g.

$$\phi : \mathbb{R} \to S^1 \times S^1 \,,$$

$$\phi(t) = (e^{2\pi t}, e^{2\pi i \lambda t})$$

for $\lambda \notin \mathbb{Q}$. Here the map ϕ is not a homeomorphism on the image. The group $\phi(\mathbb{R})$ is not a Lie subgroup according to our convention.