Lie Groups and Algebras 2024

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6 Lecture 04.04. Examples, complexification

6.1 (Summary) Every compact Lie group G contains a maximal torus $T \simeq (S^1)^r$.

- All maxima tori are conjugate
- $\mathfrak{t} = \mathfrak{g}^T$
- If G is connected, $g \in G$, then there exist $h \in G$, such that $g \in hTh^{-1}$. In other words

$$\bigcup_{h \in G} hTh^{-1} = G\,.$$

- The Weyl group $N(T)/T$ is finite and acts on T.
- If $g, h \in T$ are conjugate in G, then they lie in the same W-orbit.

6.2 Example: $U(n) \subset \mathrm{GL}_n(\mathbb{C})$. The maximal torus consists of diagonal matrices.

- Lie algebra: since $U(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \;:\; \bar{A}^T A = I\}$,

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}_n(\mathbb{C}) \;:\; \bar{A}^T + A = 0\}$$

The maximal torus is of rank r, a distinguished example of a maximal torus of consists of diagonal matrices. The action of the matrix $t = diag(t_1, t_2, \dots, t_n) \in T$ on $\mathfrak{gl}_n(\mathbb{C})$:

$$t\{a_{ij}\}t^{-1} = \{t_i t_j^{-1} a_{ij}\}\,.$$

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{i \neq j} V_{ij}\,.$$

Here $\mathfrak{t}_{\mathbb{C}}$ consists of diagonal matrices with entries $a_{ii} \in \mathbb{C}$, while the Lie algebra of T consists of the matrices with purely imaginary entries. The space V_{ij} consists of matrices having everywhere 0 except the a_{ij}. This is a decomposition of complex T-representations. The associated weights re equal to $t_i - t_j$.

- The Lie algebra $\mathfrak{u}(n)$ decomposes as

$$\mathfrak{t} = \mathfrak{u}(n)^T \oplus \bigoplus_{i<j} V'_{ij}$$

where $V'_{ij} = (V_{ij} \oplus V_{ji}) \cap \mathfrak{u}(n)$

6.3 Examples: $SO(n) \subset \mathrm{GL}_n(\mathbb{R})$. Let $r = \lfloor \frac{n}{2} \rfloor$. An example of a maximal torus consists of 2×2-block diagonal matrices with rotations in each block. There are r blocks, and if $2r < n$, then the S-E corner entry is equal to 1. For example for $n = 3$

$$\begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- Exercise: use elementary linear algebra to show that it indeed is a maximal torus.

•

$$\mathfrak{so}(n) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \ : \ A^T + A = 0\}$$

To see decomposition of $\mathfrak{so}(n)$ as T representation it is convenient to pass to the complexofication. Let $SO_n(\mathbb{C})$ be the group preserving the complex 2-linear form defined by the matrix I. Let us focus on the case $n = 2r$. It is convenient to change coordinates, so that the 2-linear form is given by the matrix $Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. The associated quadratic form is equal to

$$\sum_{i=1}^{r} x_i x_{i+r}$$

In new coordinates the complexification of the torus consists of the diagonal matrices

$$diag(t_1, t_2, \ldots, t_r, t_1^{-1}, t_2^{-1}, \ldots, t_r^{-1})$$

Thus

$$\mathfrak{so}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(\mathbb{C}) \ : \ A^T Q + QA = 0\}$$

If $A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, then

$$Z^T + Z = 0\,, \quad Y^T + Y = 0\,, \quad W = X^T\,.$$

• The decomposition of the $\mathfrak{so}(n) \otimes \mathbb{C} = \mathfrak{so}_n(\mathbb{C})$ has the form

$$\mathfrak{t}_{\mathbb{C}} \oplus \bigoplus \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the one dimensional representation of the torus with $\alpha = t_i - t_j$ (with $i \neq j$) or $\pm(t_i + t_j)$.

• Note that $\mathfrak{so}_n(\mathbb{C}) \simeq \wedge^2 \mathbb{C}^n$ as T representations:

$$\wedge^2(\mathbb{C}^{2n}) = \wedge^2(\mathbb{C}^n_+ \oplus \mathbb{C}^n_-) = \underbrace{\wedge^2 \mathbb{C}^n_+}_{weights\ t_i + t_j} \oplus \underbrace{(\mathbb{C}^n_+ \otimes \mathbb{C}^n_-)}_{weights\ t_i - t_j} \oplus \underbrace{\wedge^2 \mathbb{C}^n_-}_{weights\ -(t_i + t_j)} .$$

• Corollary: $\mathfrak{so}_n(\mathbb{C}) \simeq \wedge^2 \mathbb{C}^n$ as $SO(n)$ representations.

6.4 Exercise: Analyse the Lie algebra $\mathfrak{sp}(n)$ and show that $\mathfrak{sp}(\mathfrak{n}) \otimes \mathbb{C} \simeq Sym^2 \mathbb{C}^n$ as T-representation.

6.5 General strategy in Lie theory:
1) complexify the Lie algebra and (if possible) find the corresponding complex Lie group.
2) study representations of the complexified Lie algebra
3) derive conclusions of the group itself.

• Complexification of Lie groups: For a given real Lie group there always exist a complex Lie group with the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. The point is to realize this group as a matrix group defined by polynomial identities.

6.6 Definition: Complex linear group is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ defined by polynomial identities.

6.7 Definition: A complex linear group is said to be reductive if the category of its representations is semisimple. This means, that every representation admits a decomposition into a direct sum of simple representations.

6.8 The groups $\mathrm{GL}_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_n(\mathbb{C})$ are reductive.

6.9 Fact: any reductive group has an embedding into $\mathrm{GL}_n(\mathbb{C})$, such that the image is invariant with respect to the Cartan involution: $\Theta : A \mapsto (\overline{A}^T)^{-1}$.

6.10 Another characterization: *the largest connected solvable normal subgroup (the radical) is an algebraic torus* $\simeq (\mathbb{C}^*)^r$.

6.11 Remark: there are no compact connected linear groups of positive dimension (any algebraic subset of $\mathbb{C}^{n^2}$ is finite or noncompact).

6.12 There are equivalences of categories

$$\begin{gathered}\{\text{Complex representations of (real) comapact connected, simplyconnected group } G\}\\ \updownarrow \\ \{\text{Complex representations of } \mathfrak{g}\}\\ \updownarrow \\ \{\text{Complex representations of } \mathfrak{g}_{\mathbb{C}}\}\\ \updownarrow \\ \{\text{Complex representations of the reductive group } G_{\mathbb{C}}\}\end{gathered}$$

The last equivalence requires explanation: it is not clear that $G_{\mathbb{C}}$ is an algebraic group. It will follow from classification.

6.13 For any compact group $G \subset U(n)$ it is clear how to define $G_{\mathbb{C}}$. Namely since

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus i\,\mathfrak{u}(n) = \mathfrak{u}(n)_{\mathbb{C}}$$

$$A = \tfrac{A-\overline{A}^T}{2} + \tfrac{A+\overline{A}^T}{2}\,,$$

hence $\mathfrak{g}_{\mathbb{C}}$ is naturally isomorphic to $\mathfrak{g}+i\,\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ and $G_{\mathbb{C}}$ is the corresponding subgroup. It is missing to show that $G_{\mathbb{C}}$ is closed and algebraic.

∘ Example of $SO_n(\mathbb{C})$: suppose $A^TA = I$. Let $B = \Theta(A) = (\overline{A}^T)^{-1}$. The equation $B^TB = I$ reads as

$$((\overline{A}^T)^{-1})^T \cdot (\overline{A}^T)^{-1} = I$$

i.e.

$$\overline{A}^{-1} \cdot (\overline{A}^T)^{-1} = I\,.$$

Hence

$$A^{-1} \cdot (A^T)^{-1} = I\,.$$

and $A^TA = I$.

6.14 The opposite direction of reasoning: Having a reductive group $G_{\mathbb{C}}$, together with embedding into $\mathrm{GL}_n(\mathbb{C})$, invariant with respect to the Cartan involution, construct the compact group $K := G_{\mathbb{C}} \cap U(n)$.

6.15 Properties of the Cartan involution $\Theta : G_{\mathbb{C}} \to G_{\mathbb{C}}$ and $\theta : \mathfrak{g} \to \mathfrak{g}$, $\theta(A) = -\overline{A}^T$, see [Knapp §1]

- the fixed points is a compact subgroup $K := G_{\mathbb{C}}^{\Theta} = G_{\mathbb{C}} \cap U(n)$
- θ is a homomorphism of Lie algebras

• the Lie algebra $\mathfrak{g}$ decomposes into eigenspaces of θ: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$

∘ $\mathfrak{k} := \mathfrak{g}_1$ is the Lie algebra of K (here $\mathfrak{k}$ is the $\mathfrak{gothic}$ k).

∘ $\mathfrak{p} := \mathfrak{g}_{-1}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$,

∘ $\mathfrak{p} = i\mathfrak{k}$ hence $\mathfrak{g} \simeq \mathfrak{k} \otimes \mathbb{C}$ as complex Lie algebras.

6.16 For $G_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$ the space $\mathfrak{p}$ consists of the hermitian (or self-adjoint) matrices matrices $A = \overline{A}^T$.

6.17 Corollary: let $\phi, \psi : G \to H$ homomorphism of complex Lie groups, G reductive, connected. If $\phi_{|K} = \psi_{|K}$ then $\phi = \psi$.

6.18 The map $K \times \mathfrak{p} \to G_{\mathbb{C}}$ given by $(g, X) \mapsto g \cdot exp(X)$ is a diffeomorphism.

6.19 Proof of 6.18 for $G_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$: by polar decomposition every invertible matrix A can be written uniquely as $A = Q\,P$, where $Q \in U(n)$ and $P = \theta(P)$ is positive definite. Any positive definite matrix P has logarithm.

• Reminder from linear algebra course: for $P = (A^*A)^{\frac{1}{2}}$, $Q = AP^{-1}$ we check $QQ^* = (AP^{-1})(P^{-1}A^*) = A(A^*A)^{-1}A^* = I$).

7 Lecture 11.04. $SL_2(\mathbb{C})$

7.1 Lie group G comes with the adjoint representation: the action by conjugation of G on G fixes e, hence we get $Ad : G \to Aut(\mathfrak{g})$.

∘ If G is connected, then $ker(Ad) = Z(G)$.

7.2 The differential of Ad, i.e. $ad : \mathfrak{g} \to End(\mathfrak{g})$ is given by the commutator $ad_X(Y) = [X, Y]$.

• We assume that $G \subset \mathrm{GL}_n(\mathbb{C})$ and check the equality for matrices.

∘ First note that $Ad_A(Y) = AYA^{-1}$. Then set $A = e^{tX}$ and differentiate:

$$\tfrac{d}{dt}(e^{tX}Ye^{-tX})_{|t=0} = (Xe^{tX}Ye^{-tX} + e^{tX}Y(-X)e^{-tX})_{|t=0} = XY - YX\,.$$

7.3 A **representations of the Lie algebra** $\mathfrak{g}$ is a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a vector space, or equivalently for any $X, Y \in \mathfrak{g}$ and any $v \in V$

$$\rho(X)\rho(Y)v - \rho(Y)\rho(X)v = \rho([X, Y])v$$

7.4 The kernel of the adjoint representation

$$ker(ad) = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g}\ [X, Y] = 0\}\,.$$

This is called the center of the Lie algebra and denoted by $Z(\mathfrak{g})$. If $Z(\mathfrak{g}) = 0$, then Ado theorem about embedding of $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ is for free; $\mathfrak{g}$ embeds in $End(\mathfrak{g})$.

Representations of $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{su}(2)$

7.5 Groups $SU(2)$, $SL_2(\mathbb{C})$, $SL_2(\mathbb{R})$ and relations between their representations.

$$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{su}(2) \oplus i\mathfrak{su}(2) = \mathfrak{sl}_2(\mathbb{R})_{\mathbb{C}}$$

7.6 Action of T allows to decompos $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ into weight spaces

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathfrak{t}^* \setminus 0} \mathfrak{g}_\alpha .$$

(α's are called roots.)

∘ In the case of $SL_2(\mathbb{C})$

$$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{t} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} \,.$$

7.7 $\mathfrak{sl}_2(\mathbb{C})$ is spanned by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

∘ $[X, Y] = H$.

∘ $[H, X] = 2X$, i.e. $X \in \mathfrak{g}_2$

∘ $[H, Y] = -2Y$, i.e. $Y \in \mathfrak{g}_{-2}$

7.8 Maximal torus $\mathbb{C}^* \hookrightarrow SL_2(\mathbb{C})$

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

decomposes any representation V into weight spaces $V = \bigoplus_{k \in \mathbb{Z}} V_k$. For $v \in V_k$:

∘ $Hv = kv$

∘ $Xv \in V_{k+2}$

∘ $Yv \in V_{k-2}$

• In general:

∘ if $\alpha, \beta \in \mathfrak{t}^*$ and

∘ $X \in \mathfrak{g}_\alpha$ (i.e. $\forall H \in \mathfrak{t}$ $[H, X] = \alpha(H)X$) ,

∘ and $v \in V_\beta$ (i.e. $\forall H \in \mathfrak{t}$ $Hv = \beta(H)v$)

then $Xv \in V_{\alpha+\beta}$.

7.9 Examples of representations of $\mathfrak{sl}_2(\mathbb{C})$:

∘ Natural representation ("defining representation") $V \simeq \mathbb{C}^2$

∘ symmetric powers of the natural representations $Sym^k(V)$

7.10 General construction of the symmetric power:

$$T^k(V) = V \otimes V \otimes \cdots \otimes V \qquad k \text{ times}$$

is a representation of the permutation group Σ_k. Let

$$sym_k = \tfrac{1}{k!} \sum_{\sigma \in \Sigma_k} \sigma \in \mathbb{C}[\Sigma_k] \,.$$

be the symmetrizing operator: $sym_k \circ sym\pi_k = sym_k$. It acts on $T^k(V)$.

$$Sym^k(V) = T^k(V)^{\Sigma_k} = im(sym_k) = coker(sym_k) \,.$$

Hence we have two descriptions of $Sym^k(V)$

∘ as Σ_k-invariant tensors

∘ as $T^k(V)$ modulo the relation $v_1 \otimes v_2 \otimes \cdots \otimes v_k \sim v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)}$

○

$$Sym^k(V^*) = \text{Polynomial functions on } V \text{ of degree } k$$

• The above construction is natural, hence for any G and a representation V of G we have well defined representation $Sym^k(V)$.

7.11 The algebra $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the subalgebra of differential operators in 2 variables generated by $X = x\frac{\partial}{\partial y}$ and $Y = y\frac{\partial}{\partial x}$, $H = [X, Y] = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. The natural representation: linear forms, $Sym^k(\mathbb{C}^2) \simeq \{k - \text{polynomial forms}\}$.
○ $x\frac{\partial}{\partial y}(x^k y^\ell) = \ell x^{k+1} y^{\ell-1}$
○ $y\frac{\partial}{\partial x}(x^k y^\ell) = k x^{k-1} y^{\ell+1}$

7.12 Highest weight vectors in the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ is a vector$v \in V$ such that $Xv = 0$.

7.13 [Fulton-Harris, §11] Theorem: irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ (or $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{sl}_2(\mathbb{Z})$) are isomorphic to $Sym^k(V)$. They are characterized by the weight of the vector $v \in ker(X)$ (highest weight vector), which is a natural number.

7.14 Key Lemma: Suppose $v \in V_m$ is a highest weight vector (i.e$Xv = 0$) then $XY^{n+1}v = (n + 1)(m - n)Y^n v$.
○ $(n = 0)$ then $XYv = [X, Y]v + YXv = Hv = mv$
○ $(n = 1)$ then $XY^2v = [X, Y]Yv + YXYv = HYv + Y(mv) = (m - 2 + m)Yv$
○ $(n = 2)$ then $XY^3v = [X, Y]Y^2v + YXY^2v = HY^2v + Y((m - 2 + m)v) = (m - 4 + m - 2 + m)Yv$
○ ...

7.15 Corollary: if $\dim V < \infty$ then $m \in \mathbb{N}$ and $Y^{m+1}v = 0$.

7.16 The representations $Sym^k V$, $k \in \mathbb{N}$ are irreducible, it has the highest vector of the weight k. This is the full list of irreducible representations of $SL_2(\mathbb{C})$ (and $SU(2)$ as well).

7.17 Every complex representation of $\mathfrak{s}l_2(\mathbb{C})$ extends to a representation of $SL_2(\mathbb{C})$:
○ By polar decomposition $SL_2(\mathbb{C}) = SU(2) \times \mathbb{R}^3 = S^3 \times \mathbb{R}^3$ as topological spaces, hence $\pi_1(SL_2(\mathbb{C})) = \pi_1(S^3) = 1$. So every representation of $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ lifts to $SL_2(\mathbb{C}) \to \mathrm{GL}(V)$.

7.18 Corollary: Every complex/real representation of $\mathfrak{s}l_2(\mathbb{R})$ extends to a representation of $SL_2(\mathbb{R})$.
○ Proof: complexify.

7.19 If $W = \bigoplus_{n\in\mathbb{N}} Sym^n(V)^{\oplus a_n}$ as a $\mathfrak{s}l_2(\mathbb{C})$-representation, then $a_n = \dim W_n - \dim W_{n+2}$.

7.20 Examples of computations $Sym^k(V) \otimes V = Sym^{k+1}(V) \oplus Sym^{k-1}(V)$.

7.21 The character of the representation $Sym^k(V)$ restricted to the maximal torus is equal to

$$t^{-k} + t^{-k+2} + \cdots + t^k = \sum_{i+j=k} (t^{-1})^i t^j = \frac{t^{k+1} - t^{-k-1}}{t - t^{-1}}$$

Z ćwiczeń:

7.22 Trace form defined for $\mathfrak{gl}_n(\mathbb{C})$:

$$B_0(X,Y) = Tr(XY).$$

7.23 Suppose $\mathfrak{g} \subset \mathfrak{u}(n)$. Then B_0 is nondegenerate on $\mathfrak{g}_{\mathbb{C}}$ since $B_0(X, \theta(X)) = B_0(X, \theta(X))$ is real and < 0 for $X \neq 0$

7.24 Killing form: $B(X,Y) = Tr(ad_X \circ ad_Y)$. This form is symmetric and G-invariant.

7.25 Killing form is nondegenerate on $\mathfrak{g}/Z(\mathfrak{g})$.
∘ Beacause this is the form from (7.22) for $G := Ad(G)$.

7.26 If G is compact, then B is nonpositive definite:
∘ Because one can choose a G-invariant metric in $\mathfrak{g}$, such that $Ad(G) \subset O(\mathfrak{g})$.

8 Lecture 18.04 – Systems of roots

8.1 Rank of the Lie group $r(G) := \dim(T)$, where T is a maximal torus.

8.2 For compact groups:
∘ $\mathfrak{t}$ is the Lie algebra of the compact maximal torus
∘ $\mathfrak{t}_{\mathbb{C}}$ the complexification
∘ $\mathfrak{t}_{\mathbb{Z}} = ker(exp : \mathfrak{t} \to T)$
∘ $\mathfrak{t}^*_{\mathbb{Z}} = \mathrm{Hom}(\mathfrak{t}_Z, \mathbb{Z}) \subset \mathfrak{t}^*$, here belong the roots of the Lie algebra

8.3 Having chosen $T \subset G$ we decompose

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathfrak{t}^*_{\mathbb{Z}} \setminus \{0\}} \mathfrak{g}_\alpha \,.$$

into eigenspaces
∘ $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$
∘

$$\mathcal{R} = \{\alpha \in \mathfrak{t}^* \setminus \{0\} \ : \ \mathfrak{g}_\alpha \neq 0\}$$

is called the root system. We will show:

8.4 More general: for any representation of G

$$V = \bigoplus_{\alpha \in \mathfrak{t}^*} V_\alpha \,.$$

∘ $\mathfrak{g}_\alpha V_\beta \subset V_{\alpha+\beta}$

8.5 For an invariant 2-linear form on $\mathfrak{g}$: if $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha \perp_\phi \mathfrak{g}_\beta$.
Proof: for $H \in \mathfrak{t}$

$$0 = \phi(Hv_\alpha, v_\beta) + \phi(v_\alpha, Hv_\beta) = \alpha(H)\phi(v_\alpha, v_\beta) + \beta(H)\phi(v_\alpha, v_\beta) = (\alpha - \beta)(H)\phi(v_\alpha, v_\beta)$$

8.6 Suppose $\mathfrak{g}$ is a Lie algebra of a compactgroup. If $\mathfrak{g}_\alpha \neq 0$ then $\mathfrak{g}_{-\alpha} \neq 0$. The invariant scalar product identifies $\mathfrak{t} \simeq \mathfrak{t}^*$. Define $H_\alpha \in \mathfrak{t}$ such that $(H_\alpha, v) = \alpha(v)$ for $v \in \mathfrak{t}$. Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ then

$$[x, y] = (x, y)H_\alpha$$

◦ Proof:

$$([x, y], h) + (y, [x, h]) = 0$$

$$([x, y], h) = (y, [h, x]) = \alpha(h)(x, y) = (H_\alpha, h)(x, y)$$

8.7 Suppose $(x, y) = \frac{2}{(\alpha,\alpha)}$, then $x, y, h_\alpha = 2H_\alpha/(\alpha, \alpha)$ is a basis of $lin(x, y, H_\alpha)$ satisfying the standard relations of $\mathfrak{sl}_2$.

$$[h_\alpha, x] = \alpha(h_\alpha)x = (H_\alpha, h_\alpha)x = \frac{2}{(\alpha, \alpha)}(H_\alpha, H_\alpha)$$

$$[x, y] = (x, y)H_\alpha = h_\alpha$$

◦ Corollary: for every root α we have constructed a copy of $\mathfrak{sl}_2 \in \mathfrak{g}_\mathbb{C}$, hance also a copy of $SL_2(\mathbb{C})$ in G (or $SU(2)$ in the compact group.) We denote such a coppy by $\mathfrak{sl}_2(\mathbb{C})_\alpha$

8.8 Let E be a real vector space with a scalar product. An abstract system of roots is a finite set $\mathcal{R} \subset E$ such that

1. The roots $\mathcal{R}$ span $\mathfrak{t}^*_\mathbb{Z} \setminus \{0\}$
2. If $\alpha \in \mathcal{R}$ then $-\alpha \in \mathcal{R}$
3. If $\beta \in lin\{\alpha\}$, then $\beta = \pm\alpha$
4. The reflections in $\alpha \in \mathcal{R}$ preserve $\mathcal{R}$
5. for $\alpha, \beta \in \mathcal{R}$ the quotient $n_{\alpha,\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

(see Kirillov Def 7.1)

8.9 We fix a G-invariant scalar product in $\mathfrak{g}$, hence we have a W-invariant scalar product in $\mathfrak{t}$ and $\mathfrak{t}^*$. If $Z(G)$ is finite, then the preferred choice is the (minus) Killing form.

8.10 Theorem. Let $\mathcal{R} \subset \mathfrak{t}^*$ be the set of roots of a compact Lie group with $Z(G)$ finite. Then it satisfies the axioms of an abstract system of roots. Moreover
◦ $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \mathcal{R}$
◦ For $\alpha, \beta \in \mathcal{R}$ the $\mathfrak{sl}_2(\mathbb{C})$ representation $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ is irreducible
◦ For $\alpha, \beta \in \mathcal{R}$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ (not only „$\subset$”)
[see Kirillov Theorem 6.44]

8.11 Examples of root systems of rank 2 (from ćwiczenia) $SO(4)$, $Sp(2)$, $SO(5)$, see [FuHa §21]

8.12 Lie group $SU(3)$ (and its complexification $SL_3(\mathbb{C})$)

- ∘ $\mathfrak{t}_{\mathbb{C}}$ =diagonal matrices with trace =0,
- ∘ the Lie algebra of the compact torus: $diag(t_1, t_2, t_3)$ s.t. $Re(t_i) = 0$, $t_1 + t_2 + t_3 = 0$
- ∘ weights $L_i : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$, i.e. $L_i \in \mathfrak{t}^*_{\mathbb{C}}$, $i = 1, 2, 3$

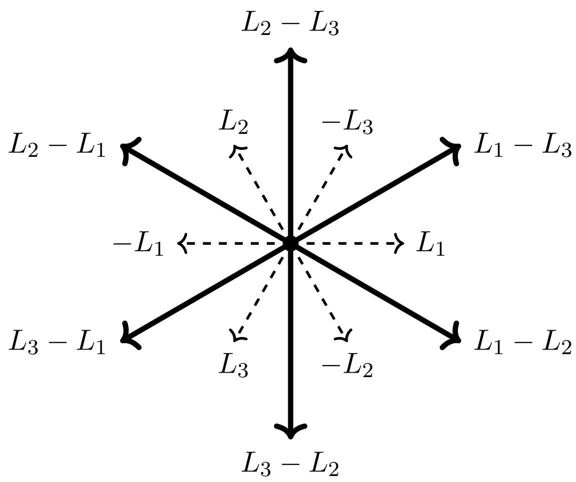
$$diag(t_1, t_2, t_3) \mapsto t_i \,.$$

There is a relation $L_1 + L_2 + L_3$.

- ∘ Roots: $\alpha_{i,j} = L_i - L_j$, $i \neq j$
- ∘ Cartan numbers

$$n_{\alpha_{1,2},\alpha_{2,3}} = 2\frac{((1,-1,0),(0,1,-1))}{((1,-1,0),(1,-1,0))} = -1$$

$\angle(\alpha_{1,2}, \alpha_{2,3}) = 2\pi/3$



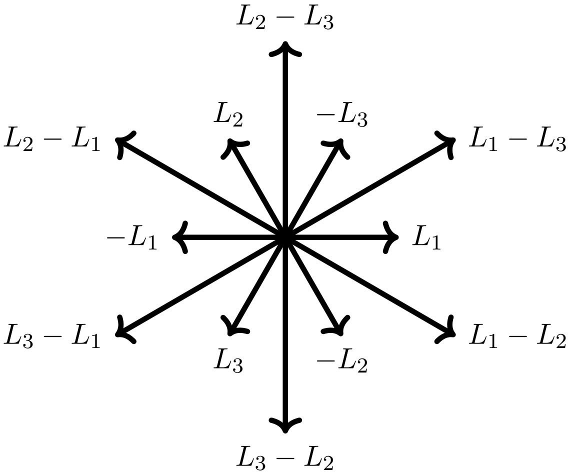
8.13 Examples of root systems of rank 2 (from ćwiczenia) $SO(4)$, $Sp(2)$, $SO(5)$, see see [FuHa §21]

- • $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
- • $\mathfrak{sp}(2)$: $\mathfrak{t}_{\mathbb{C}} \ni diag(t_1, t_2, -t_1, -t_2) \overset{L_i}{\mapsto} t_i$
- ∘ Roots $\pm 2L_i$ or $L_i - L_j$
- ∘ For $\alpha \neq \pm\beta$

$$n_{\alpha,\beta} \in \left\{ 0\ ,\ \pm 2\frac{((1,-1),(2,0))}{((1,-1),(1,-1))} = 2\ ,\ \pm 2\frac{((2,0),(1,-1))}{((2,0),(2,0))} = 1 \right\}$$

- • $\mathfrak{so}(5)$: $\mathfrak{t}_{\mathbb{C}} \ni diag(t_1, t_2, -t_1, -t_2, 0) \overset{L_i}{\mapsto} t_i$
- ∘ Roots $\pm L_i$ or $L_i - L_j$
- ∘ $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$

8.14 Exceptional G_2 (it will be later)



8.15 In general there are strong restrictions for $n_{\alpha,\beta}$. The Cartan numbers $n_{\alpha\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ satisfy

- $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2(\angle(\alpha,\beta))$,
- $n_{\alpha\beta}n_{\beta\alpha} \in \mathbb{Z}$, therefore $n_{\alpha\beta}n_{\beta\alpha} \in \{0,1,2,3\}$ for $\alpha \neq \pm\beta$.

◦ $\angle(\alpha,\beta) \in \{30^o, 45^o, 90^o, 120^o, 135^o, 150^o\}$

8.16 The Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is spanned by $\mathfrak{t}$ (the diagonal matrices of trace 0) and $E_{i,j} \in \mathfrak{g}_{L_i - L_j}$

$$[H, E_{i,j}] = (L_i(H) - L_j(H))E_{i,j}$$

i.e. if $H = diag(t_1, t_2, t_3)$, then

$$[H, E_{i,j}] = (t_i - t_j)E_{i,j}$$

8.17 With this notation we list the basic representations of $SL(3)$

- The defining representation $V = \mathbb{C}^3$

The weights L_1, L_2, L_3. The corresponding eigenvectors e_1, e_2, e_3

$$E_{1,2}e_1 = 0,\ E_{1,3}e_1 = 0,\ E_{2,3}e_1 = 0$$

The *highest weight vector* $e_1 \in \bigcap_{i>j} ker(E_{i,j})$ generates whole representation.

- The second exterior power $\wedge^2 V \simeq V^*$.

The weights: $L_1 + L_2 = -L_3$, $L_1 + L_3 = -L_2$, $L_2 + L_3 = -L_1$. The corresponding eigenvectors $e_1 \wedge e_2, e_2 \wedge e_3, e_2 \wedge e_3$. The action of $E_{i,j}$ for $i < j$

$$E_{12} : e_1 \wedge e_2 \mapsto 0 \wedge e_2 + e_1 \wedge e_1 = 0$$

$$E_{13} : e_1 \wedge e_2 \mapsto 0 \wedge e_2 + e_1 \wedge 0 = 0$$

$$E_{23} : e_1 \wedge e_2 \mapsto 0 \wedge e_2 + e_1 \wedge 0 = 0$$

The *highest weight vector* $e_1 \wedge e_2 \in \bigcap_{i>j} ker(E_{i,j})$ generates whole representation.

- $Sym^2(V) = lin\{e_1^2, e_2^2, e_3^2, e_1e_2, e_1e_3, e_2e_3\}$. The *highest weight vector* $e_1^2 \in \bigcap_{i>j} ker(E_{i,j})$ generates whole representation. Equivalently the *highest weight vector* $e_3^* \in \bigcap_{i>j} ker(E_{i,j})$ generates V^*.
- Some examples of representations of $\mathfrak{sl}_3(\mathbb{C})$. In paricular $Sym^2(V) \otimes (V)^*$, see Fulton-Harris §12-13.

Claim: every irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ is isomorphic to a subrepresentation of $Sym^\bullet(V) \otimes Sym^\bullet(\wedge^2 V)$. For a representation with the highest weight vector of the weight $(a+b)L_1 + bL_2$ take the representation generated by $v = (e_1)^a \otimes (e_1 \wedge e_2)^b$. The remaining vectors are obtained by application of the operators E_{21}, E_{31}, E_{32} given by the action of elementary matrices.

Proofs of properties of the root system 8.8

8.18 Theorem: Any compact connected Lie group of rank 1 is isomorphic to $SU(2)$ or $SO(3)$ or S^1. Proof: Let $n = \dim(G)$. G acts on $S^{n-1} \subset \mathfrak{g}$ via Ad. The action Ad fixes $= \mathfrak{t}$. Therefore $G/T \to S^{n-1}$ is a covering, so it has to be a homeomorphism. We get a fibration $S^1 = T \to G \to G/T = S^{n-1}$. If $n > 3$ the $\pi_1(T) \to \pi_1(G)$ is a monomorphism. The group G contains a subgroup H isomorphic to $SU(2)$ or $SO(3)$ with the Lie algebra $\mathfrak{t} \oplus \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{-\alpha_0}$, where α_0 the longest root. There is an element $g \in N(T) \subset H$ such that $Ad(g)_{|\mathfrak{t}} = -Id : \mathfrak{t} \to \mathfrak{t}$, so $gtg^{-1} = t^{-1}$. But in G the conjugation by g is homotopic to Id. Contradiction. Hence $n \leq 3$. $\square$

8.19 Proof of 8.8.1. More general we have $\bigcap_{\alpha \in \mathcal{R}} ker(\alpha) = T(Z(G))$.

8.20 Proof of 8.8.2. $\mathfrak{g}$ is a real representation of T. The summands of the decomposition of $\mathfrak{g}_{\mathbb{C}}$ come in pairs.

8.21 Proof of 8.8.3. For any root α let

$$\mathfrak{k}_\alpha = \mathfrak{t} \oplus \bigoplus_{\beta \text{ proportional to } \alpha} \mathfrak{g}_\beta$$

Let K generated by $exp(\mathfrak{k}_\alpha)$; the roots of the closure have the same kernel as α, hence K closed. By 8.18 $\dim K = 3$.

8.22 Proof of 8.8.4. The action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)_\alpha$ is the reflection in $ker(h_\alpha)$ (denoted by s_α). It preserves the root system.

8.23 Proof of 8.8.5. The vector space $W = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$ is a representation of $\mathfrak{sl}_2(\mathbb{C})_\alpha$. The number $n_{\alpha,\beta}$ is the weight of $v \in \mathfrak{g}_\beta$. So it is an integer

$$[h_\alpha, v] = \frac{2}{(\alpha,\alpha)}[H_\alpha, v] = \frac{2}{(\alpha,\alpha)}\beta(H_\alpha)v = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}v$$

9 Lecture 25.04. Positive and simple roots, Weyl group

9.1 We have defined the Weyl group as NT/T. It acts on $\mathfrak{t}^*$ preserving roots. For a root α we have a copy of $SU(2)$ or $SU(2)/\langle \pm I \rangle$ in G, dnoted by K_α.

∘ $N(T_{K_\alpha}) \subset N(T)$ and the nontrivial element of W_{K_α} acts as a reflection in α.

∘ We will identify the Weyl group $N(T)/T$ as the group of isometries of $\mathfrak{t}^*$ generated by s_α. For the moment we consider abstract root systems and abstract Weyl group generated by the reflections. If we deal with a Lie algebra of a compact group, then the elements s_α are realized as the effect of the action of elements from $N(T)$.

9.2 For the root systems of rank 2 (ie. $\dim E = 2$) we have

∘ $A_1 \cup A_1$ realized as the root system of $SU(2) \times SU(2)$

- ∘ A_2 realized by $SU(3)$
- ∘ B_2, also called C_2 realized by $SO(5)$ or $Sp(2)$
- ∘ G_2 given by 8.14

Positive and simple roots [Kirillov 7.4]

9.3 Dividing E into two half–spaces we decompose $\mathcal{R} = \mathcal{R}_+ \sqcup \mathcal{R}_-$.
- ∘ The division is given by the sign (α, ρ), where ρ is a generic vector of $\mathbf{t}^*$.

9.4 A positive root is simple if cannot be written as a sum of two positive roots. Every positive roots can be written as a sum of simple roots.

9.5 For two simple roots $(\alpha, \beta) \le 0$. [Kirillov, Lem. 7.11 and 7.14]
- ∘ The proof follows from the analysis of root systems of rank 2.

9.6 The set of all simple roots form a basis.
- • Obviously it spans
- • If $v = \sum a_i\alpha_i = \sum b_j\beta_j$ with $a_i, b_j \ge 0$, then $||v|| = 0$
- ∘

$$||v||^2 = \sum_{i,j} a_i b_j (\alpha_i, \beta_j) \le 0$$

- ∘ On the other hand $(v, \rho) > 0$. Contradiction.

9.7 Dynkin diagram:
- ∘ vertices = simple roots denoted in Kirillov by Π
- ∘ edges:

no edge if $n_{\alpha,\beta} = 0$

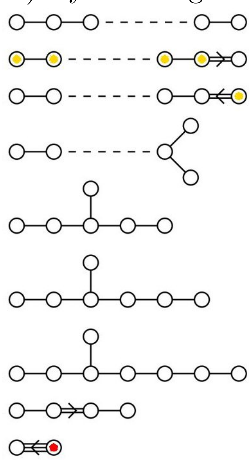
$\alpha \;\text{——}\; \beta$ if $n_{\alpha\beta}n_{\beta\alpha} = -1$

$\alpha \Leftarrow \beta$ if $n_{\alpha\beta}n_{\beta\alpha} = -2$, $|\alpha| < |\beta|$

$\alpha \Lleftarrow \beta$ if $n_{\alpha\beta}n_{\beta\alpha} = -3$, $|\alpha| < |\beta|$

9.8 All possible irreducible (i.e. connected) Dynkin diagrams. The longer roots are in colour:

$SU(n+1)$	$SL_n(\mathbb{C})$	A_n
$SO(2n+1)$	$SO_{2n+1}(\mathbb{C})$	B_n
$Sp(n)$	$Sp_n(\mathbb{C})$	C_n
$SO(2n)$	$SO_{2n}(\mathbb{C})$	D_n
		E_6
		E_7
		E_8
		F_4
$Aut(\mathbb{O})$		G_2



9.9 Having chosen division into positive and negative roots one redefine the functional defining the split:

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha \,.$$

∘ Theorem: $\rho \in P$.

9.10 Weyl Chambers = connected components of $E \setminus \bigcup_{\alpha \in \mathcal{R}} \mathcal{H}_\alpha$, where

$$\mathcal{H}_\alpha = \{\lambda \in E \ : \ (\alpha, \lambda) = 0\}$$

∘ The positive chamber:

$$C_+ = \{\lambda \in E \ : \ \forall_{\alpha \in \mathcal{R}_+} \quad (\alpha, \lambda) > 0\}$$

∘ The chamber C_+ has exactly $n = \dim E$ walls corresponding to simple roots.

∘ Aplying reflections in walls one can transform C_+ to any other chamber.

9.11 Weyl group = the group generated by the reflections s_α

$$W = \langle s_\alpha \mid \alpha \in \mathcal{R} \rangle \,.$$

9.12 Theorem:

∘ 1) W acts transitively on the set of chambers

∘ 2) W is generated by the reflections in simple roots

∘ 3) W acts freely on the set of chambers

• 1) and 2) is easy by a geometric argument

9.13 Suppose $C = w(C_+)$ let. Define the length $\ell(w)$

$$\ell(w) = |\{\alpha \in \mathcal{R}_+ \mid w(\alpha) \in \mathcal{R}_-\}| \,.$$

(Number of walls separating C from C_+.)

9.14 Theorem: If $w = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_k}$ is a shortest presentation of $w \in W$, then $k = \ell(w)$.

∘ From above follows 9.12.3. That is the stabilizer of C_+ consists of elements of w of length 0, i.e. it consists only of the identity. (Exercise 7.3 in Kirillov.)

9.15 Topological proof of 9.12.3 with the root system of a compact Lie group: if $g \in NT$ preserves the chamber C_+, one may assume $g(X) = X$ for some $X \in C_+$. The group topologically generated by $\exp(tX)$ and g is abelian, $\simeq torus \times \mathbb{Z}_n$ can be topologically generated by one element, so it is contained in a maximal torus. This torus has to be T (*). Hence $[g] = 1 \in N(T)/T$.

∘ (*) The centralizer of the torus $\exp(tX)$ has the Lie algebra equal to

$$\mathfrak{t} \oplus \bigoplus_{\alpha:\ \alpha(X)=0} \mathfrak{g}_\alpha \,.$$

9.16 Corollary: Since $W_{top} := N(T)/T$ acts freely and $W_{alg} := \langle s_\alpha : \alpha \in \mathcal{R} \rangle$ acts transitively, thus $W_{top} = W_{alg}$, i.e. two notions of the Weyl group coincide. The Weyl group acts freely and transitively on the set of Weyl chambers.

9.17 The vertices Π of the Dynkin diagram may be treated as generators of W, the number of the edges between α and β, i.e. $n_{\alpha\beta}n_{\beta\alpha}$ encodes the angle $\angle(\alpha,\beta)$. Hence the order the corresponding rotation $s_\alpha s_\beta$.

- First of all $s_\alpha^2 = 1$
- no edge: s_α, s_β commute $\iff (s_\alpha s_\beta)^2 = 1$
- one edge $(s_\alpha s_\beta)^3 = 1$ (equivalently the braid relation $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$)
- double edge $(s_\alpha s_\beta)^4 = 1$
- triple edge $(s_\alpha s_\beta)^6 = 1$

9.18 Theorem [not so obvious]: These are the relations defining W.

∘ This is an example of a Coxeter group.

10 09.05 Relations defining Lie algebra, G_2

10.1 [Kirillov §7.5] For an abstract root system we define two lattices

∘ Q – the root lattice spanned by roots α

∘ P – the weight lattice

$$P = \left\{\lambda \in E \ : \ \forall \alpha \in \mathcal{R} \quad 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z}\right\}$$

∘ the functional $\alpha^\vee = 2\frac{(\alpha,-)}{(\alpha,\alpha)} \in E^*$ does not depend on the scalar product. It is called a coroot.

∘ By the last axioms of $\mathcal{R}$ there is an inclusion $Q \subset P$.

10.2 In the situation when the root system comes from a Lie group, we have

$$Q \subset \mathfrak{t}^*_{\mathbb{Z}} \subset P$$

∘ Reminder: $\mathfrak{t}^*_{\mathbb{Z}} = \mathrm{Hom}(ker(\exp : \mathfrak{t} \to T), \mathbb{Z}) = \mathrm{Hom}_{alg}(T_{\mathbb{C}}, \mathbb{C}^*) = \mathrm{Hom}(T, S^1)$.

∘ The second inclusion holds because for $\lambda \in \mathfrak{t}^*_{\mathbb{Z}} = \mathrm{Hom}_{alg}(T_{\mathbb{C}}, \mathbb{C}^*)$

$$2\frac{(\alpha,\lambda)}{(\alpha,\alpha)} = \lambda(h_\alpha)$$

describes the multiplicity of λ restricted to the torus $T_\alpha = \exp(lin\, h_\alpha) \subset K_\alpha$

$$\mathbb{C}^* \simeq T_\alpha \subset T \xrightarrow{\lambda} \mathbb{C}^*\,.$$

Hence $\lambda(h_\alpha) \in \mathbb{Z}$).

10.3 The dual latices:

$$P^\vee \subset \mathfrak{t}_{\mathbb{Z}} \subset Q^\vee \subset \mathfrak{t}_{\mathbb{R}}$$

$$Q^\vee = \{v \in \mathfrak{t} \ : \ \forall \alpha \in \mathcal{R}\ \alpha(v) \in \mathbb{Z}\}$$

∘ If $v \in \alpha^{-1}(\mathbb{Z})$ the element $\exp(v) \in T$ acts trivially on $\mathfrak{g}_\alpha$.

∘ For $v \in Q^\vee$ the element $\exp(v)$ acts trivially on $\mathfrak{g}$, hence (since we consider connected groups)

$$\exp(v) \in Z(G)\,.$$

∘ Theorem: For semisimple Lie group: $Z(G) \simeq Q^\vee/\mathfrak{t}_{\mathbb{Z}}$.

10.4 Exercise: $\mathfrak{t}_{\mathbb{Z}}/P^{\vee} \simeq \pi_1(G)$.

From Dynkin diagram to Lie algebra

10.5 Dynkin diagram determines the root system: since W acts transitively on Weyl chambers we obtain all roots as images of the simple roots.

∘ Exercise: If the algebra is simple, then W acts transitively on roots of the same length.

10.6 [Kirillov 7.52]. For each simple root $\alpha \in \Pi$ we fix elements $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$.

• The elements e_α for $\alpha \in \Pi$ generate the algebra $\mathfrak{n}_+ = \bigoplus_{\alpha \in \mathcal{R}_+} \mathfrak{g}_\alpha$.

∘ Proof is based on the lemma (Kirillov 7.52) : if $\beta \in \mathfrak{R}_+$ is not a simple root then it can be written as $\alpha + \beta'$ with $\beta' \in \mathcal{R}_+$, $\alpha \in \Pi$.

• Analogously: the elements f_α for $\alpha \in \Pi$ generate the algebra $\mathfrak{n}_+ = \bigoplus_{\alpha \in \mathcal{R}_-} \mathfrak{g}_\alpha$.

10.7 Serre relations The elements h_α, e_α, f_α generate $\mathfrak{g}$ and satisfy the relations

1) $[h_\alpha, h_\beta] = 0$

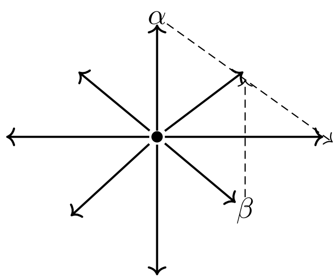
2) $[h_\alpha, e_\beta] = n_{\alpha,\beta} e_\beta, \quad [h_\alpha, f_\beta] = -n_{\alpha,\beta} f_\beta, \quad$ where $n_{\alpha,\beta} = (\alpha^\vee, \beta) = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$

3) $[e_\alpha, f_\beta] = \delta_{\alpha,\beta} h_\alpha$

4) $(ad_{e_\alpha})^{(1-n_{\alpha,\beta})} e_\beta = 0, \qquad \circ\ (ad_{f_\alpha})^{(1-n_{\alpha,\beta})} f_\beta = 0$

(note $n_{\alpha,\beta} < 0$)

10.8 Example Sp_2



$$n_{\alpha,\beta} = 1\,, \quad n_{\beta,\alpha} = 2\,.$$

$$[e_\alpha, e_\beta] \in \mathfrak{g}_{L_1+L_2}\,, \quad [e_\alpha, [e_\alpha, e_\beta]] = 0\,.$$

$$[e_\beta, e_\alpha] \in \mathfrak{g}_{L_1+L_2}\,, \quad [e_\beta, [e_\beta, e_\alpha]] \in \mathfrak{g}_{2L_1}\,, \quad [e_\beta, [e_\beta, [e_\beta, e_\alpha]]] = 0\,.$$

10.9 Proof.

∘ 1) $h_\alpha, h_\beta \in \mathfrak{t}$, which is commutative

∘ 2) $n_{\alpha,\beta}$ is the lowest weight of the representation $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ of $K_\alpha = \exp(lin\{h_\alpha, e_\alpha, f_\alpha\})$

∘ 3) $[e_\alpha, f_\beta] \in \mathfrak{g}_{\alpha-\beta}$. The weight $\alpha - \beta$ is not a root: not positive, nor negative. So $\mathfrak{g}_{\alpha-\beta} = 0$ for $\alpha \neq \beta$.

∘ 4) The vector $e_\beta \in \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ is the vector the highest weight with respect to the action the tripple $(f_\alpha, e_\alpha, -h_\alpha)$. This weight is equal to $-n_{\alpha,\beta}$.

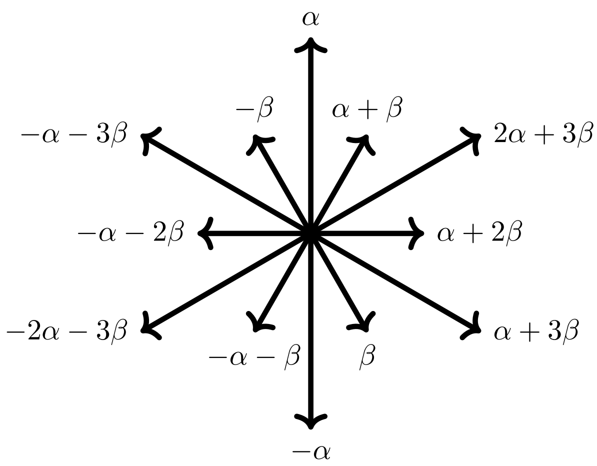
10.10 Theorem: The algebra defined by the above relations is of finite dimension and it is semisimple.

◦ Corollary: Each root system uniquely determines a semisimple algebr.

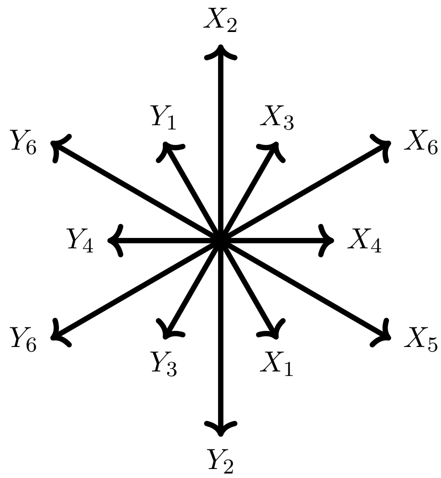
10.11 Example: G_2. [Fulton-Harris, §22] The algebra generated by 6 generators.

◦ assume $n_{\alpha,\beta} = 1$, $n_{\beta,\alpha} = 3$

$|\alpha|^2 = 3|\beta|^2$



Weights



Basis vectors

$$X_1 = e_\beta, \quad X_2 = e_\alpha, \quad X_3 = [X_1, X_2], \quad X_4 = [X_1, X_3], \quad X_5 = [X_1, X_4], \quad X_6 = [X_2, X_5]$$

$$Y_1 = f_\beta, \quad Y_2 = f_\alpha, \quad Y_3 = [Y_1, Y_2], \quad Y_4 = [Y_1, Y_3], \quad Y_5 = [Y_1, Y_4], \quad Y_6 = [Y_2, Y_5]$$

◦ We partially know the actions of
$ad_{e_\alpha} = [X_2, -]$, $ad_{f_\alpha} = [Y_2, -]$, $ad_{h_\alpha} = [h_\alpha, -]$, $ad_{e_\beta} = [X_1, -]$, $ad_{f_\beta} = [Y_1, -]$, $ad_{h_\beta} = [h_\beta, -]$.

◦ We compute the remaining commutators. For example:

$$[Y_1, X_3] \overset{def}{=} [Y_1, [X_1, X_2]] \overset{Leibniz}{=} [[Y_1, X_1], X_2] - [X_1, [Y_1, X_2]] = [-h_\beta, X_2] - [X_1, 0] = -\alpha(h_\beta)X_2 = 3X_2$$

since $\alpha(h_\beta) = n_{\beta,\alpha} = -3$.

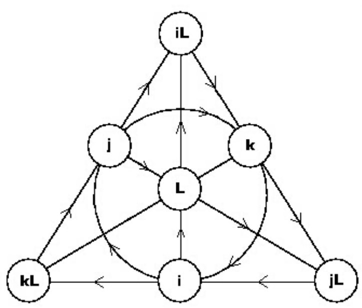
◦ Exercise: check that $[Y_1, X_4] = 4X_3$.

◦ More complicated comutators

$$[X_3, X_4] = [[X_1, X_2], X_4] = [X_1, [X_2, X_4]] - [X_2, [X_1, X_4]] = [X_1, 0] - [X_2, X_5] = -X_6$$

(Here we use the rule $ad_{[X_1,X_2]} = ad_{X_1} ad_{X_2} - ad_{X_2} ad_{X_1}$.)

10.12 G_2 as automorphism of octonions:



see `https://math.ucr.edu/home/baez/octonions/octonions.pdf`

∘ $a \cdot \bar{a} = \bar{a} \cdot a \in \mathbb{R}_+$ is a scalar product.

∘ The imaginary part of the multiplication:

$$im\mathbb{O} \otimes im\mathbb{O} \to im\mathbb{O}$$

defines a 3-tensor $\omega \in (im\mathbb{O})^{\otimes 3}$. Here we use the isomorphism defined by the scalar product $im\mathbb{O} \simeq (im\mathbb{O})^*$.

∘ Claim $\omega \in \wedge^3 im\mathbb{O}$.

10.13 Alternative definition of $(G_2)_{\mathbb{C}}$is a stabilizer of a generic form $\omega \in \wedge^3\mathbb{C}^7$.

10.14 Triality automorphism of $\mathfrak{so}(8)$ and $\mathfrak{g}_2$ as the fixed point set $\mathfrak{so}(8)^{triality}$.
For real Lie algebra $\mathfrak{so}(8)$ the triality was constructed by E. Cartan, *Le principe de dualité et la théorie des groupes simples et semi–simples*, Bulletin sc. Math. (2) 49, 361–374 (1925). The approach presented here is equivalent, to the Cartan's work in the complex case. The triality automorphism given below has an advantage, that the root spaces coincide with the coordinates of the matrix and these coordinates are permuted by $\mathbb{Z}_3$.
Working with the complex coefficients we choose a basis in $\mathbb{C}^8$ (as in [Fu-Ha, §19]) in which the quadratic form is equal to

$$x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5 \,.$$

For real coefficients this means that we deal with $SO(4,4)$. The maximal torus of $SO(4,4)$ consists of the diagonal matrices

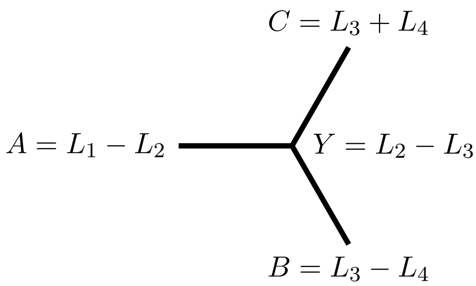
$$diag(e^{t_1}, e^{t_2}, e^{t_3}, e^{t_4}, e^{-t_4}, e^{-t_2}, e^{-t_3}, e^{-t_1})\,.$$

The following weights form the root system of $\mathfrak{so}(4,4)$:

$$\pm L_i \pm L_j \quad \text{for } i \neq j\,,$$

where $L_i(t_1,t_2,t_3,t_4) = t_i$. Choosing the $\mathfrak{n}_+$ as the upper triangular matrices we obtain the Dynkin diagram of simple roots:

D_4 Dynkin diagram:



The triality automorphism rotates the diagram clockwise:

$$(L_1 - L_2) \mapsto (L_3 + L_4) \mapsto (L_3 - L_4) \mapsto (L_1 - L_2)$$

and fixes the root $L_2 - L_3$. In the basis consisting of the weights L_i the triality automorphism is given by the remarkable matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{1}$$

10.15 Triality in $\mathfrak{so}(4,4)$

We have given a formula for triality automorphism acting on the dual $\mathfrak{t}^*$ of the Cartan subalgebra of $\mathfrak{so}(4,4)$. It does not preserve the lattice corresponding to the group $SO(4,4)$ but it preserves the lattice spanned by L_i's and $\frac{1}{2}(L_1 + L_2 + L_3 + L_4)$, which corresponds to $Spin(4,4)$, the cover of $SO(4,4)$. We list below the set of positive roots:

$$\begin{array}{cccccc} L_1 - L_2 & L_1 - L_3 & L_1 - L_4 & L_1 + L_4 & \boxed{L_1 + L_3} & \boxed{L_1 + L_2} \\ & \boxed{L_2 - L_3} & L_2 - L_4 & L_2 + L_4 & L_2 + L_3 & \\ & & L_3 - L_4 & L_3 + L_4 & & \end{array}$$

The boxed roots are fixed by triality. It can be easily seen when we express the roots in the basis of simple roots:

$$\begin{array}{cccccc} A & A+Y & A+B+Y & A+C+Y & \boxed{A+B+C+Y} & \boxed{A+B+C+2Y} \\ & \boxed{Y} & B+Y & C+Y & B+C+Y & \\ & & B & C & & \end{array}$$

We have the fixed roots

$$Y, \quad A+B+C+Y, \quad A+B+C+2Y\,.$$

There are three free orbits:

$$\{A,\ B,\ C\}, \quad \{A+Y,\ B+Y,\ C+Y\}, \quad \{A+B+Y,\ B+C+Y,\ C+A+Y\}.$$

10.16 Explicit computations. From general theory it follows that the triality automorphism of weights lifts to a self-map of the Lie algebra $\mathfrak{so}(4,4)$. But a priori it is not clear that one can find such a lift of order three. Not every lift satisfies $\phi \circ \phi \circ \phi = Id$. The choice of signs is not obvious and demands a careful check. The elements of $\mathfrak{so}(4,4)$ for our quadratic form defined by the matrix with 1's on the antidiagonal are the matrices $(m_{ij})_{1 \le i,j \le 8}$ which are antisymmetric with respect to the reflection in the antidiagonal. Such a matrix is transformed by the triality automorphism to the following one

$$\begin{pmatrix} \bullet & m_{34} & -m_{24} & m_{26} & m_{14} & \boxed{m_{16}} & \boxed{m_{17}} & 0 \\ m_{43} & \bullet & \boxed{m_{23}} & m_{25} & -m_{13} & m_{15} & 0 & \boxed{-m_{17}} \\ -m_{42} & \boxed{m_{32}} & \bullet & m_{35} & m_{12} & 0 & -m_{15} & \boxed{-m_{16}} \\ m_{62} & m_{52} & m_{53} & \bullet & 0 & -m_{12} & m_{13} & -m_{14} \\ m_{41} & -m_{31} & m_{21} & 0 & \bullet & -m_{35} & -m_{25} & -m_{26} \\ \boxed{m_{61}} & m_{51} & 0 & -m_{21} & -m_{53} & \bullet & \boxed{-m_{23}} & m_{24} \\ \boxed{m_{71}} & 0 & -m_{51} & m_{31} & -m_{52} & \boxed{-m_{32}} & \bullet & -m_{34} \\ 0 & \boxed{-m_{71}} & \boxed{-m_{61}} & -m_{41} & -m_{62} & m_{42} & -m_{43} & \bullet \end{pmatrix}$$

The upper half of the diagonal $(t_1, t_2, t_3, t_4) = (m_{11}, m_{22}, m_{33}, m_{44})$ is transformed by the matrix (1) to

$$\frac{1}{2}(m_{11} + m_{22} + m_{33} - m_{44})$$
$$\frac{1}{2}(m_{11} + m_{22} - m_{33} + m_{44})$$
$$\frac{1}{2}(m_{11} - m_{22} + m_{33} + m_{44})$$
$$\frac{1}{2}(m_{11} - m_{22} - m_{33} - m_{44})$$

We remark that this is an unique automorphism with real coefficients extending the self-map of the maximal torus. We would like to stress, that both: the Cartan construction of triality for $\mathfrak{so}(8)$ and the triality for $\mathfrak{so}(4,4)$ presented here works for any ring in which 2 is invertible.

11 16.05 Representations, Verma modules

[Kirillov §8.1]

11.1 Character of a compact group/reductive is determined by its value on the maximal torus.

∘ If $\chi(V_1) = \chi((V_2)$ then $V_1 \simeq V_2$.

11.2 Example: $G = U(n)$ (or $GL_n(\mathbb{C})$)

∘ $V \simeq \mathbb{C}^n$ – the natural representation: $t_1 + t_2 + \cdots + t_n$

∘ The "determinant" $\wedge^n V$: $t_1 t_2 \dots t_n$

∘ The exterior product $\wedge^k V$: $\sigma_k(t_1, t_2, \dots, t_n)$, often denoted e_k, the elementary symmetric function.

∘ The symmetric power $Sym^k(V)$:

$$\sum_{i_1+i_2+\cdots+i_n=k,\ i_j\geq 0} t_1^{i_1} t_2^{i_2} \dots t_n^{i_n}$$

also denoted by h_k (the complete symmetric function).

∘ The dual representation V^*: $t_1^{-1} + t_2^{-1} + \cdots + t_n^{-1}$

11.3 For $G = SU_n$ or $SL_n(\mathbb{C})$

$$\chi(V^*) = \chi(\wedge^{n-1} V)\,.$$

11.4 Fixing the coordinates of T we write characters as Laurent polynomials.

11.5 For $w \in NT/T$ and a representation W

$$\chi(W)(\underline{t}) = \chi(W)(w\underline{t})$$

∘ The Newton polytope of a character is a convex body in $\mathfrak{t}^*$, symmetric with respect to action of the Weyl group.

∘ For $G = SL_3$, Sp_2 we draw pictures on the plane.

11.6 Exercise: $\mathfrak{so}(n) \simeq \wedge^2 V$, $\mathfrak{sp}(n) \simeq Sym^2 V$ as a representations of the corresponding groups.

11.7 [Kirillov Def 8.9 and Th 8.10]

∘ We say that V is a highest weight representation if there exists $v \in V$ such that v generates V as $\mathfrak{g}$ representation and $v \in \bigcap_{X\in\mathfrak{n}_+} ker(X)$.

∘ The condition on v implies that $v \in V_\lambda$, where $\lambda \in \overline{C}_+ \subset \mathfrak{t}^*$, i.e. v is of weght λ, and λ belongs to the closure of the positive Weyl chamber.

∘ Every irreducible representation is a highest weight vector representation. Later we will see that the isomorphism type is determined by λ. Therefore we write V_λ for $\lambda \in \overline{C}_+$.

Theorem: Let $P = \{\lambda \in \mathfrak{t}^* \mid \forall\, \alpha \in \mathcal{R} \quad \lambda(\alpha^\vee) \in \mathbb{Z}\}$.
There is a bijection between irreducible representations of $\mathfrak{g}$ and the lattice points $P \cap \overline{C}_+$
There is a bijection between irreducible representations of G (reductive/compact) and the lattice points $\mathfrak{t}^*_\mathbb{Z} \cap \overline{C}_+$

∘ Such weights are called *dominant*.

11.8 Enveloping algebra of a Lie algebra $U(\mathfrak{g})$: it the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by $XY - YX - [X,Y]$. Let $\mathfrak{g}$ be a reductive Lie algebra (a complexification of a Lie algebra of a compact group). A representation of $\mathfrak{g}$ is the same as $U(\mathfrak{g})$-module.

11.9 Filtration of $T(\mathfrak{g})$ by the length of the tensors induces a filtration of $U(\mathfrak{g})$, denoted by $F_iU(\mathfrak{g})$. Let $Gr_FU(\mathfrak{g}) = \bigoplus_{i=0}^\infty F_iU(\mathfrak{g})/F_{i-1}U(\mathfrak{g})$. Exercise: show that $Gr_FU(\mathfrak{g}) \sim \bigoplus_{i=0}^\infty Sym_i(\mathfrak{g})$ as a graded commutative algebra.

11.10 We will need enveloping algebras of:

- Let $\mathfrak{b}_+ = \mathfrak{t} \oplus \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$. Similarly $\mathfrak{b}_-$.
- Let $\mathfrak{n}_+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$. Similarly $\mathfrak{n}_-$.

11.11 Suppose that $v \in V$ for $\lambda \in \overline{C}_+$, and suppose that $Xv = 0$ for $X \in \mathfrak{g}_\alpha$ with $\alpha \in \mathfrak{R}_+$. Then:

- then $V' = U(\mathfrak{g})v$ is a subrepresentation and $V' = U(\mathfrak{n}_-)v$.
- $\dim V_\lambda = 1$.

Pf: one can replace each monomial in $U(\mathfrak{g})$ be a combination of monomials with increasing (ρ, α).

- $V' = U(\mathfrak{g})v$ is simple.

Pf. If $V' = \bigoplus W_i$ then for some i_0 the projection of v onto W_{i_0} does not vanish. The projection preserves weights, thus $v \in W_{i_0}$. Hence $V' = W_{i_0}$.

11.12 Cor. Any two irreducible representations with v satisfying the assumptions above are isomorphic.
Pf. Consider the direct sum $V \oplus V'$ and the subrepresentatation generated by (v, v') and the projections to V and V'.

11.13 Let $\lambda \in \overline{C}_+$. There is a map $U(\mathfrak{b}_+) \to U(\mathfrak{t})$ which allows to treat $\mathbb{C}_\lambda$ as a $U(\mathfrak{b}_+)$-module. The induced representation $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\lambda$ is called the Verma module. It is of infinite dimension, but irreducible representation of the weight λ is a quotient of M_λ. [Kirillov Lemma 8.13]

11.14 To show that M_λ admits a finite dimensional quotient we study the structure of M_λ. [Kirillov 8.14]

Theorem

- $U(\mathfrak{n}_-) \to M_\lambda$, $u \mapsto u \otimes v$ is an isomorphism of vector spaces.

- M_λ admits the weight decomposition $M_\lambda = \bigoplus M_\lambda[\mu]$, $\dim M_\lambda[\mu] < \infty$, the sum is over $\mu = \lambda - \sum n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$

11.15 [Kirillov, see proof of Theorem 8.18] M_λ has a unique maximal proper submodule $I_\lambda = \bigcup_{\text{proper submodules}} W$.

11.16 Example: Let $\mathfrak{g} = \mathfrak{sl}_2$. Let α be the positive root.
(reminder: $v \in V_\lambda$ is a highest weight vector iff $Xv = 0$) then $XY^{n+1}v = (n+1)(\lambda - n)Y^n v$

- Take $\lambda = k\alpha$ with $2k \in \mathbb{N}$. The submodule $I_\lambda \subset M_\lambda$ is isomorphic to $M_{-\lambda-\alpha}$.
- If $\lambda \neq k\alpha$ with $2k \in \mathbb{N}$, the M_λ is irreducible.

11.17 Theorem [Kirillov 8.23]: If $\lambda \in P \cap \overline{C}_+$ then M_λ / I_λ is of finite dimension.

11.18 A method of constructing finite dimensional representation without Verma modules (for SL_n):

- λ is dominant if $\lambda = \sum_{i=1}^{n-1} \lambda_i L_i$ with $\lambda_1 \geq \lambda_2 \geq \dots \lambda_{n-1} \geq 0$.
- Let $\omega_k = \sum_{i=1}^{k} L_i$; this is the highest weight of $\wedge^k V$ (with $V = \mathbb{C}^n$ the defining representation).

$$\lambda = \sum_{i=1}^{n-1} k_i \omega_i \,, \quad k_i = \lambda_i - \lambda_{i-1}$$

- Consider $W = \bigotimes (\wedge^i V)^{\otimes k_i}$, and there find a vector of the weight λ anihilated by $\mathfrak{n}_+$. It generates V_λ.

12 23.05: Berstein-Gelfand-Gelfand resolution, Weyl character formula

12.1 [Kirillov Th 8.28]. Let v_λ be the highest weight vector of M_λ. For each simple root α_i define

$$v_i = f_i^{n_i+1} v_\lambda \quad n_i = (\lambda, \alpha_i^\vee) \,.$$

It generates a submodule $W_i \simeq M_{\lambda-(n_i+1)\alpha_i}$.

- Claim: The kernel $M_\lambda \to V_\lambda$ is generated by the vectors v_i (with i indexing simple roots):

$$I_\lambda = \sum_i W_i \,.$$

12.2 We define the shifted action of W on $\mathfrak{t}^*$:

$$w.\lambda = w(\lambda + \rho) - \rho \,.$$

- In particular $s_i.\lambda = \lambda - (n_i + 1)\alpha_i$ and particular v_i has weight $s_i.\lambda$.
- [Kirillov: Th 8.30] There is an exact sequence

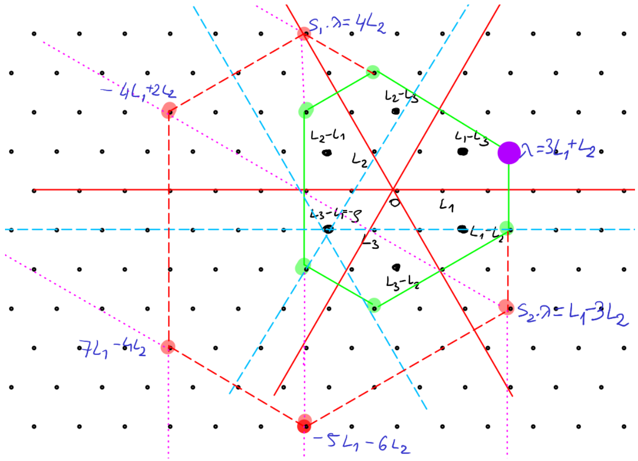
$$0 \to M_{w_0.\lambda} \to \dots \to \bigoplus_{w \in W:\ \ell(w) = k} M_{w.\lambda} \to \dots \to \bigoplus_i M_{s_i.\lambda} \to M_\lambda \to V_\lambda \to 0$$

i.e a $U(\mathfrak{n}_-)$-free resolution of V_λ. Here w_0 is the longest element of W.

12.3 Example: $\mathfrak{sl}_2$: $0 \to M_{-\lambda-2} \to M_\lambda \to V_\lambda \to 0$, i.e. the resolution has length 2.

12.4 For $\mathfrak{sl}_3$ the resolution has length 4 (see Kirillov Ex. 8.32).

∘ If $\lambda = 3L_1 + L_2$ the relevant weights $w.\lambda$ are shown on the picture



The center of the bigger hexagon is at $-\rho = L_3 - L_1 = -2L_1 - L_2$. It is spanned by the vertices $w.\lambda$. The smaller hexagon is spanned by the vertices $w(\lambda)$.

Weyl character formula

12.5 Given a dominant weight $\lambda \in P_+$, what is the character of the representation V_λ? (It is a representation of G with $Lie(G) = \mathfrak{g}$, $\pi_1(G) = 1$.)

12.6 Example: For $\mathfrak{sl}_2$, $\lambda \in P_+ \simeq \mathbb{Z}{\geq}\, 0$ we have

$$V_\lambda = Sym^k(\mathbb{C}^2)\,, \chi(V_\lambda) = \sum_{i+j=\lambda} t_1^i t_2^j$$

with the convention that $t_1 t_2 = 1$ i.e. $t_1 = t$, $t_2 = t^{-1}$

$$\chi(V_\lambda) = \frac{t_1^{\lambda+1} - t_2^{\lambda+1}}{t_1 - t_2}\,.$$

12.7 Character makes sense for infinite dimensional representations of $\mathfrak{g}$ provided that each weight space is of finite dimension. Then the character is a formal series

12.8 Example for $\mathfrak{sl}_2$: (calculus with $|t| > 1$)

$$\chi(M_\lambda) = \sum_{i=0}^{\infty} t^{\lambda-2i} = t^\lambda \sum_{i=0}^{\infty} t^{-2i} = t^\lambda \frac{1}{1-t^{-2}}$$

$$\chi(M_{-\lambda+2}) = t^{-\lambda-2}\frac{1}{1-t^{-2}}$$

$$\chi(V_\lambda) = t^\lambda \frac{1}{1-t^{-2}} - t^{-\lambda-2}\frac{1}{1-t^{-2}} = \frac{t^\lambda - t^{-\lambda-2}}{1-t^{-2}} = \frac{t^{\lambda+1} - t^{-\lambda-1}}{t - t^{-1}}$$

12.9 Convenient notation in general case. For a weight λ let $e^\lambda \in \mathrm{Hom}(T, \mathbb{C}^*)$ be the corresponding character.

∘ Proposition [Kirillov 8.33]:

$$\chi(M_\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \mathcal{R}_+} 1 - e^{-\alpha}}$$

12.10 Theorem [Kirillov 8.34]. From BGG resolution we obtain

$$\chi(V_\lambda) = \sum_{w\in W} \frac{(-1)^{\ell(w)} e^{w.\lambda}}{\prod_{\alpha\in\mathcal{R}_+} 1 - e^{-\alpha}}$$

12.11 Since $e^{w.\lambda} = e^{w(\lambda+\rho)}e^{-\rho} = \frac{e^{w(\lambda+\rho)}}{\prod_{\alpha\in\mathcal{R}_+} e^{\alpha/2}}$ the above formula can be transformed to

$$\chi(V_\lambda) = \frac{\sum_{w\in W}(-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha\in\mathcal{R}_+}(e^{\alpha/2} - e^{-\alpha/2})}$$

12.12 Corollary: if $\lambda = 0$ then $\chi(V_\lambda) = 1$. Hence

$$\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)} = \prod_{\alpha\in\mathcal{R}_+} (e^{\alpha/2} - e^{-\alpha/2})$$

12.13 Let us focus on $G = SL_n(\mathbb{C})$. Fix the notation $e^{L_i} = t_i$. We have $\prod_{i=1}^n t_i = 1$.

- The dominant weight are of the form $\sum_{i=1}^n \lambda_i L_i$, written as a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$.
- $\rho = (n-1, n-2, \dots, 0)$
- The Weyl character formula

$$\chi(V_\lambda) = \sum_{w\in\Sigma_n} \frac{sgn(w) t^{w(\lambda+\rho)} t^{-\rho}}{\prod_{i<j}(1 - t_j/t_i)} = \sum_{w\in\Sigma_n} \frac{sgn(w) t^{w(\lambda+\rho)}}{\prod_{i<j}(t_i - t_j)}$$

12.14 The above formula can be written as

$$\chi(V_\lambda) = \frac{A_{\lambda+\rho}}{A_\rho} = \frac{W_\lambda}{W_0}$$

where

$$A_{\underline{a}} = \det \begin{pmatrix} t_1^{a_1} & t_1^{a_2} & \dots & t_1^{a_n} \\ t_2^{a_1} & t_2^{a_2} & \dots & t_2^{a_n} \\ \vdots & & & \\ t_n^{a_1} & t_n^{a_2} & \dots & t_n^{a_n} \end{pmatrix}$$

$$W_\lambda = A_{\lambda+\rho} = \det \begin{pmatrix} t_1^{\lambda_1+n-1} & t_1^{\lambda_2+n-2} & \dots & t_1^{\lambda_n} \\ t_2^{\lambda_1+n-1} & t_2^{\lambda_2+n-2} & \dots & t_2^{\lambda_n} \\ \vdots & & & \\ t_n^{\lambda_1+n-1} & t_n^{\lambda_2+n-2} & \dots & t_n^{\lambda_n} \end{pmatrix}$$

- The above formula defines the Schur function S_λ. We do not assume that $t_1 t_2 \dots t_n = 1$. We obtain a polynomial (!!) in $t_1, t_2, \dots, t_n$ of degree $|\lambda| := \sum_{i=1}^n \lambda_i$. It is symmetric with respect to permutations of variables.
- Example: for $\lambda = (2, 1, 0, \dots, 0)$

$$S_\lambda = e_2 e_1 - e_3$$

(attention, here $e_i \in \mathbb{Z}[t_1, t_2, \dots, t_n]$ is the elementary symmetric function, not an element of $\mathfrak{g}$. Do not get confused!)

12.15 Easiest representations of SL_n

- Exterior power: $S_{1^k} = S_{1,1,\dots,1,0,\dots,0} = \chi(\wedge^k V)$
- Symmetric power: $S_{k,0,\dots,0} = \chi(Sym^k V)$
- Adjoint representation $S_{21^{n-1}} = S_{2,1,1,\dots,1,0} = \chi(\mathfrak{sl}_n)$.

– for $n = 3$

$$S_{210} = \frac{\begin{vmatrix} t_1^4 & t_1^2 & t_1^0 \\ t_2^4 & t_2^2 & t_2^0 \\ t_3^4 & t_3^2 & t_3^0 \end{vmatrix}}{\begin{vmatrix} t_1^2 & t_1^1 & t_1^0 \\ t_2^2 & t_2^1 & t_2^0 \\ t_3^2 & t_3^1 & t_3^0 \end{vmatrix}} = \frac{(t_1^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_3^2)}{(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)} = (t_1 + t_2)(t_1 + t_3)(t_2 + t_3)\,,$$

which taking into account $t_1 t_2 t_3 = 1$, is equal to

$$2 + \tfrac{t_1}{t_2} + \tfrac{t_1}{t_3} + \tfrac{t_2}{t_1} + \tfrac{t_2}{t_3} + \tfrac{t_3}{t_1} + \tfrac{t_3}{t_2} = \dim \mathfrak{t} + \sum_{\alpha \in \mathcal{R}} t^\alpha\,.$$

∘ In the example 12.4 $\chi(V_\lambda)$ is a sum of monomials corresponding by the dots in the smaller hexagon:

$$\chi(V_\lambda) = S_{310} = \frac{\begin{vmatrix} t_1^5 & t_1^2 & t_1^0 \\ t_2^5 & t_2^2 & t_2^0 \\ t_3^5 & t_3^2 & t_3^0 \end{vmatrix}}{\begin{vmatrix} t_1^2 & t_1^1 & t_1^0 \\ t_2^2 & t_2^1 & t_2^0 \\ t_3^2 & t_3^1 & t_3^0 \end{vmatrix}} = \frac{t_1^5 t_2^2 - t_1^2 t_2^5 - t_1^5 t_3^2 + t_2^5 t_3^2 + t_1^2 t_3^5 - t_2^2 t_3^5}{(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)}\,,$$

$$= t_2 t_1^3 + t_3 t_1^3 + t_2^2 t_1^2 + t_3^2 t_1^2 + 2 t_2 t_3 t_1^2 + t_2^3 t_1 + t_3^3 t_1 + 2 t_2 t_3^2 t_1 + 2 t_2^2 t_3 t_1 + t_2 t_3^3 + t_2^2 t_3^2 + t_2^3 t_3$$

After substitution $t_3 := t_1^{-1} t_2^{-1}$

$$= t_2 t_1^3 + t_2^2 t_1^2 + \frac{t_1^2}{t_2} + t_2^3 t_1 + 2t_1 + 2t_2 + \frac{1}{t_2^2} + \frac{t_2^2}{t_1} + \frac{2}{t_2 t_1} + \frac{1}{t_2^3 t_1^2} + \frac{1}{t_1^2} + \frac{1}{t_2^2 t_1^3}\,.$$

In terms of the elementary symmetric function

$$= e_1^2 e_2 - e_2^2 - e_1 e_3\,.$$

In terms of full symmetric functions

$$= h_3 h_1 - h_4\,.$$

12.16 Representation ring

∘ $R(SU(n)) = \mathbb{Z}[t_1, t_2, \dots t_n]^{\Sigma_n}/(e_n)$

∘ $R(U(n)) = \mathbb{Z}[t_1, t_2, \dots t_n]^{\Sigma_n}[e_n^{-1}]$

Further basic properties of GL_n (or $U(n)$) representations

12.17 * Weyl construction of the representation V_λ [Fulton-Harris, p.233]

12.18 * Admissible fillings of Young diagram correspond to a basis of V_λ. It can serve to compute S_λ not requiring determinant computations. [Fulton-Harris, Exercise A.31]. See also [Fulton, *Young Tableaux*].

12.19 * Formula for S_λ interms of e_k or h_k. [Fulton-Harris, formula (A.5-6) in §A.1]

12.20 * Multiplication in the representation ring $R(GL_n)$ in terms of Young diagrams (Pieri formula) [Fulton-Harris 15.25 and formula (A.7) in §A.1]

13 6.06 Spinors

13.1 Summary of basic information about representations of $SL_n(\mathbb{C})$:

∘ Roots $L_i - L_j$,

• Simple roots $L_1 - L_2$, $L_2 - L_3$,... , $L_{n-1} - L_n$

• Coroots $L_i - L_j$,

• Simple coroots $L_1 - L_2$, $L_2 - L_3$,... , $L_{n-1} - L_n$

• $\mathfrak{t}^*_{\mathbb{Z}} = P$ spanned by L_i

• $\overline{C}_+$ is given by $\sum \lambda_i L_i \in \overline{C}_+$ if and only if $\lambda_i \geq \lambda_{i+1}$ for all $i = 1, 2, \dots, n-1$.

• $\mathbb{C}_+$ spanned (using nonnegative combinations) by the fundamental weights L_1, L_1+L_2, $L_1+L_2+L_3$, $\dots L_1 + L_2 + \cdots + L_n$.

∘ 1) Every irreducible representation of SL_n is given by a certain functorial construction, called the Schur functor $\mathbb{S}_\lambda(\mathbb{C}^n)$

2) Characters of irreducible representations are the Schur functions $S_\lambda(t_1, t_1, \dots, t_n)$

3) One can compute Schur either by a combinatorial algorithm or using polynomial formulas in e_i

4) Multiplication of irreducible representations (or Schur functions) : $\mathbb{S}_\lambda(\mathbb{C}^n) \otimes \mathbb{S}_\mu(\mathbb{C}^n) =?$

– Pieri rule

– Littlewood-Richardson rule.

13.2 Example. $SO(5)$.

• Roots $\alpha = \pm L_i \pm L_j$, $\pm L_i$

• Coroots $\alpha^\vee = \pm L_i \pm L_j$, $2L_i$

• $P = \{xL_1 + yL_2 \mid 2x \in \mathbb{Z},\ 2y \in \mathbb{Z},\ x + y \in \mathbb{Z}\rangle\}$

• Simple roots $L_1 - L_2$, L_2

• Simple coroots $L_1 - L_2$, $2L_2$

• $\overline{C}_+ = \{(x, y) \in \mathbb{R}^2 \mid x > 0,\ x > y\} = Conv(t(0, 1), s(1, 1))$.

• $P \cap \overline{C}_+ = \{mL_1 + nL_2 \mid m, n \in \frac{1}{2}\mathbb{Z},\ m \geq 0,\ n \in [0, m],\ \text{same fractional part}\}$

• The natural representation has the highest weight L_1

• In the second exterior power $\Lambda^2\mathbb{C}^5$ the vector $e_1 \wedge e_2$ has weight: $L_1 + L_2$ but no way to get a half weight

• Missing the representation with the highest weight $\frac{1}{2}(L_1 + L_2)$. TBA **Spinor representation**. But here one can use the isomorphism $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$. Then $ker(\Lambda^2\mathbb{C}^4 \to \mathbb{C})$ is switched to the natural representation of $SO(5)$ and the spinor representation corresponds to the natural representation of $Sp(2)$.

13.3 $SO(4)$

• Roots $\alpha = \pm L_i \pm L_j$,

• Coroots $\alpha^\vee = \pm L_i \pm L_j$,

• $P = \{xL_1 + yL_2 \mid x - y \in \mathbb{Z},\ x + y \in \mathbb{Z}\rangle\}$

• Simple roots $L_1 + L_2$, $L_1 - L_2$,

- Simple coroots $L_1 + L_2$, $L_1 - L_2$,
- $\overline{C}_+ = \{(x,y) \in \mathbb{R}^2 \mid x > |y|\} = Conv(t(1,-1), s(1,1))$.
- $\beta_0 = \frac{1}{2}((L_1 + L_2) + (L_1 - L_2)) = L_1$
- $P \cap \overline{C}_+ = \{mL_1 \pm nL_2 \mid m,n \in \frac{1}{2}\mathbb{Z},\ m \geq n \geq 0,\ \text{same fractional part}\}$
- The natural representation has the highest weight L_1
- In the second exterior power $\Lambda^2\mathbb{C}^4$ the vector $e_1 \wedge e_2$ has weight: $L_1 + L_2$
- Missing half weight representations $\frac{1}{2}(L_1 \pm L_2)$, the spinor representations S_+ and S_-. They do not come from representations of $SO(4)$ since $\frac{1}{2}(L_1 \pm L_2) \notin \Lambda^*$. But they come form the representation of the universal cover $\widetilde{SO}(4) = SU(2) \times SU(2)$ a.k.a. $Spin(4)$.

Construction of the Clifford algebra, see [Brocker-tomDieck §I.6] and my separate notes.

13.4 Let $V \simeq \mathbb{K}^n$ be a vector space with a quadratic form $Q : V \to \mathbb{K}$: $C(Q) = T(V)/(v \otimes v - Q(v))$.
- $\iota : V \hookrightarrow C(V)$ as generators
-

$$\iota(v+w)^2 = \iota(v)^2 + \iota(v)\iota(w) + \iota(w)\iota(v) + \iota(w)^2$$

hence $\iota(v)\iota(w) + \iota(w)\iota(v) = 2\phi(v,w)$, where $\phi(v,w) = \frac{1}{2}(Q(v+w) - Q(v) - Q(w))$ is the associated bilinear form.
- $\mathbb{Z}_2$ gradation $C(Q) = C(Q)^{ev} \oplus C(Q)^{odd}$
- antihomomorphism $x \mapsto t(x) = x^t$, $\iota(v)^t = \iota(v)$ for $v \in V$
- canonical homomorphism $x \mapsto \alpha(x)$, $\alpha(\iota(v)) = -\iota(v)$ for $v \in V$
- $\overline{x} := t\alpha(x) = \alpha t(x)$

13.5 $\Gamma(Q) = \{x \in C(Q)^* \mid \alpha(x)vx^{-1} \in V\}$ acts on V

13.6 Norm map $N : C(Q) \to C(Q)$, $N(x) = x\overline{x}$, for $v \in V$ we have $N(\iota(v)) = -Q(v) \cdot 1$

13.7 $\mathbb{H} \simeq C(Q)^{ev}$ for $V = \mathbb{R}^3$, $Q(v) = -|v|$

13.8 Construction of Pin and Spin groups: $V = \mathbb{R}^n$, $Q(v) := -|v|^2$, $N(v) = |v|^2$.
- ∘ The group $\Gamma(Q)$ for this choice of V and Q will be denoted Γ_n.
- ∘ $ker(\Gamma_n \to Aut(\mathbb{R}^n)) = \mathbb{R}^*$

13.9 For $x \in \Gamma_n$ we have $N(x) \in \mathbb{R}^*$.

Proof. We check that for any $v \in \mathbb{R}^n$ the element $N(\overline{x}) = \overline{x}x = t\alpha(x)x$ acts trivially on $\mathbb{R}^n$.
By the definition of Γ_n we have

$$\alpha(x) \cdot v \cdot x^{-1} \in V.$$

t is constant on V, hence

$$t(x)^{-1} \cdot v \cdot t\alpha(x) = \alpha(x) \cdot v \cdot x^{-1}$$

hence

$$v = t(x) \cdot \alpha(x) \cdot v \cdot (t\alpha(x) \cdot x)^{-1} = \cdots = \alpha(\overline{x}x) \cdot v \cdot (\overline{x}x)^{-1},$$

hence

$$\overline{x}x \in ker(\Gamma_n \to Aut(\mathbb{R}^n)) = \mathbb{R}^*.$$

Remark: this implies $\frac{1}{N(\overline{x})}\overline{x} = x^{-1}$, so $N(x) = N(\overline{x})$.

13.10 $N : \Gamma_n \to \mathbb{R}^*$ is a homomorphism .

13.11 The group Γ_n acts on $\mathbb{R}^n$ via isometries $ker(\rho : \Gamma_n \to O(n)) = \mathbb{R}^*$.
Proof: Claim: $v \in \Gamma_n$ because it defines the reflection in $v^\perp$. $N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x^{-1})$.

13.12 The map $\Gamma_n \to O(n)$ is surjective because the image contains reflections in hyperplanes.

13.13 Definition: $Pin(n) := ker(N : \Gamma_n \to \mathbb{R}^*)$.
∘ The map

$$Pin(n) \to O(n)$$

is a 2-fold cover.
Proof. $ker(N) \cap ker(\rho) = \{x \in \mathbb{R}^* : \ N(x) = 1\} = \{1, -1\}$.

13.14 Definicja $Spin(n) := Pin(n) \cap C(Q)^{ev} = \rho^{-1}(SO(n))$,
∘ ρ is a nontrivial double covering.
Proof: the path $s \mapsto \gamma(s) = \cos(s) + \sin(s)e_1e_2$, $s \in [0, \pi]$ joins 1 with -1, $\rho(\gamma(s))$ =rotation in the plane $lin(e_1e_2)$ by the angle $2s$.

Spin representation

13.15 For $\mathfrak{so}(2n)$ there are two minimal nonintegral highest weight vectors $\alpha = \frac{1}{2}(L_1 + L_2 + \cdots + L_{n-1} \pm L_n)$.
∘ The associated representations is called the spinor representation $S^\pm_{2n} \subset \Lambda^*V$.
∘ Taking the convex hull of $W\cdot\alpha \subset P$ we obtain the polytope spanned by $\frac{1}{2}(\pm L_1 \pm L_2 \pm \cdots \pm L_{n-1} \pm L_n)$ with even or odd number of $-$ depending whether it is S^+_{2n} or S^-_{2n}. Zero lies in the interior but cannot be a weight of $S^\pm_{2n}$. Hence $\dim S^\pm_{2n} = 2^{n-1}$.

13.16 Similarly: S_{2n+1} has the highest weight $\frac{1}{2}(L_1 + L_2 + \cdots + L_n)$, $\dim S_{2n+1} = 2^n$.

13.17 Construction via complexification: $V \otimes \mathbb{C} = W \oplus W^*$. Then $S^+_{2n} = \Lambda^{ev}W$, $S^-_{2n} = \Lambda^{odd}W$ with the action of $C(Q_n)$ given on the generators $w \cdot \xi = w \wedge \xi$ for $w \in W$, $\xi \in \Lambda^*W$ and the derivation $f \cdot \xi = D_{2f}\xi$ for $f \in W^*$.
[Fu-Ha §Lemma 20.9]. Weight of e_I is equal to $\frac{1}{2}(\sum_{i\in I} L_i - \sum_{i\notin I} L_i)$

13.18 For $\mathfrak{so}(2n+1)$ the highest vectors is equal to $\frac{1}{2}(L_1 + L_2 + \cdots + L_n)$. The spinor representation is constrcted via complexification: $V \otimes \mathbb{C} = W \oplus W^* \oplus \mathbb{C}$. The additional unit vector acts as α on $S_{2n+1} = \Lambda^*W$. This action cannot be split into even and odd parts.

13.19 Special cases: by π we mean the natural action of $SO(n)$ on $\mathbb{R}^n$

Group	π	$S^\pm$
$Spin(3) = SU(2)$	$im\mathbb{H}$ $\pm 2L_1$, 0	$\mathbb{C}^2$ $\pm L_1$
$Spin(4)_\mathbb{C} = SL_2(\mathbb{C})^2$	$M(2\times 2)$ $\pm L_1 \pm L_1'$	two copies of $\mathbb{C}^2$ $\pm L_1$ and $\pm L_1'$
$Spin(5)_\mathbb{C} = Sp(2)$	$ker(\omega) \subset \Lambda^2(\mathbb{C}^4)$ $\pm L_1 \pm L_2$, 0	$\mathbb{C}^4$ $\pm L_1$, $\pm L_2$
$Spin(6)_\mathbb{C} = SL_4(\mathbb{C})$	$\Lambda^2(\mathbb{C}^4)$ $L_i + L_j$	$\mathbb{C}^4$ and $(\mathbb{C}^4)^*$ L_i and $-L_i$

13.20 Representation ring:[Brocker-tomDieck, §VI.6]

14 13.06 Homogeneous spaces

See [Fu-Ha, §23.3-4] and Giorgio Ottaviani RATIONAL HOMOGENEOUS VARIETIES
`http://people.dimai.unifi.it/ottaviani/rathomo/rathomo.pdf`

14.1 By rational homogeneous spaces we mean compact complex manifolds containing an open subset $U \simeq \mathbb{C}^n$.

∘ If K is a compact group, H its closed subgroup, then K/H is a compact manifold.

∘ If G is a complex group, H its closed complex subgroup, then K/H is a complex manifold.

∘ Goal: classify all homogeneous manifolds which are both compact and admit a complex structure.

14.2 Basic notion: the orbit of G-action on X

$$G \cdot x = \{g \cdot x \in X \ \mid \ g \in G\} \ \simeq \ G/G_x$$

14.3 Example $\mathbb{CP}^1 = \mathbb{C} \sqcup \{\infty\}$

∘ Action of $GL_2(\mathbb{C})$ (or $SL_2(\mathbb{C})$ by homographies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}$$

∘ It is homogeneous with respect to $SL_2(\mathbb{C})$ action.

∘ $\mathbb{C}^*$ has 3 orbits,

∘ $B = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has 2 orbits.

14.4 Example 1: Complex Grassmannian

$$Gr(k,n) = \{V \subset \mathbb{C}^n \mid V \text{ is a linear subspace } \dim(V) = k\}\,.$$

$$Gr(k,n) \simeq U(n)/U(k) \times U(n-k) \simeq GL_n(\mathbb{C})/P_k\,,$$

$$P_{k,n} = \begin{pmatrix} GL_k(\mathbb{C}) & * \\ 0 & GL_{n-k}(\mathbb{C}) \end{pmatrix}$$

14.5 Exercise: There are $\binom{n}{k}$ orbits of B action. Each contains exactly one torus fixed point, which is a coordinate subspace $lin\{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}$. Each orbit is isomorphic to $\mathbb{C}^\ell$, $\ell = \sum_{j=1}^k (i_j - j)$.

∘ The orbits are called Schubert cells, their closures — the Schubert varieties. The closures have singularities at the boundary in general.

∘ Example:

$$\big\{V \in Grass(2,4) \ \mid \ \underbrace{\dim(V \cap lin\{e_1\}) = 0, \ \dim(V \cap lin\{e_1,e_2\}) = 1\ , \dim(V \cap lin\{e_1,e_2,e_3\}) = 1}_{\text{Schubert conditions}} \ \big\}\,.$$

The closure

$$\big\{V \in Grass(2,4) \ \mid \ \dim(V \cap lin\{e_1,e_2\}) \geq 1 \ \big\}$$

has one singular point $V = lin\{e_2, e_2\}$.

14.6 The manifold of complete flags in $\mathbb{C}^n$

$$F\ell_n \subset \prod_{k=0}^{n} Gr(k,n)$$

$$\{0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n\}$$

14.7 $F\ell_n = GL_n/B$, where B is the group of upper-triangular matrices.

14.8 Bruhat decomposition

$$F\ell_n = \bigsqcup_{w\in\Sigma_n} BwB/B \simeq \bigsqcup_{w\in\Sigma_n} \mathbb{C}^{\ell(w)}$$

where $\ell(w)$ is the length of the permutation $w \in \Sigma_n$

∘ Bruhat decomposition of $GL_n(\mathbb{C})$

$$GL_n = \bigsqcup_{w\in\Sigma_n} BwB$$

$$a = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & \bullet & * \end{pmatrix} \quad \rightsquigarrow \quad ba = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \quad \rightsquigarrow \quad bab' = \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

row operations ↑ column operations →

14.9 For $n = 2$

$$GL_2(\mathbb{C}) \quad = \quad B \ \sqcup \ B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}B$$

$$B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}B = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \mid \ c \neq 0\right\}$$

14.10 Let K be a compact connected Lie group, T_c its maximal torus. Let $G = K_{\mathbb{C}}$ the compexification of K and $T = (T_c)_{\mathbb{C}}$, the complexification of the torus.

∘ Equivalently, G is a reductive group with a fixed Cartan involution, $K = G^\theta$, $T_c = T^\theta$.

∘ $\dim K = r + 2m$, where $r = \dim T_c$.

∘ We fix a decomposition of the roots of $\mathfrak{g}$ into positive and negative roots, $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\mathcal{R}_+} \mathfrak{g}_\alpha$.

∘ Claim: the group B corresponding to $\mathfrak{b}$ is closed.

Proof. We have $\dim \mathfrak{b} = m + r$. We consider the action of G on $Gr(m + r, \mathfrak{g})$ and we examine H – the stabilizer of $\mathfrak{b} \in Gr(k, \mathfrak{g})$. It is a closed subgroup. Its Lie algebra

$$\mathfrak{H} = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{b} \ \ [X, Y] = 0\} = \mathfrak{b}\,.$$

Hence B is the identity component of H, so it is closed. (In fact $B = H$.)

∘ We have $\mathfrak{b} \cap \mathfrak{k} = \mathfrak{t}_c$ and $B \cap K = T_c$. It follows

$$K/T_c \to G/B$$

is injective. Since $\dim_{\mathbb{R}} K/T_c = 2 \dim_{\mathbb{C}} G/B$ we have shown $\boxed{K/T_c \simeq G/B}$ as real manifolds.

14.11 Example:

$$F\ell_n = SL_n(\mathbb{C})/B \simeq SU(n)/T_c = SU(n)\text{Diagonal matrices.}$$

∘ For every flag $V_\bullet$ one can find a unitary basis $\{v_i\}$, such that $V_k = lin\{v_1, v_2, \ldots, v_k\}$. This basis is unique up to scalars $a_k \in \mathbb{C}^*$, $|a_k| = 1$.

14.12 Construction of rational homogeneous spaces via representations

$\circ$ Let $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ be a dominant weight, V_λ the corresponding irreducible representation, $v_\lambda \in V_\lambda$ the highest weight vector. Consider the orbit of $[v_\lambda] \in \mathbb{P}(V_\lambda)$. Since B stabilizes $[v_\lambda]$, so the stabilizer, denoted by P_λ (do not confuse with the lattice $P \subset \mathfrak{t}^*$) contains B. Hence the orbit

$$G \cdot [v_\lambda] \simeq G/P_\lambda$$

is compact. It is a quotient of G/B. It comes with an equivariant embedding into a projective space.

14.13 In general, for any closed complex subgroup $P \subset G$ containing B the quotient G/P is a compact complex manifold.

$\circ$ Let $\mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ be complex Lie algebras. The algebra $\mathfrak{p}$ decomposes with respect to the torus action

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha = \mathfrak{t} \oplus \mathfrak{b} \oplus \bigoplus_{\alpha \in A \cap \mathcal{R}_-} \mathfrak{g}_\alpha \,.$$

We claim that the set $A \cap \mathcal{R}_-$ has the property

- If $\alpha = \sum_{\text{simple roots}} a_i \alpha_i \in A \cap \mathcal{R}_-$ and $a_i \neq 0$ then $-\alpha_i \in A \cap \mathcal{R}_-$,
- If $\alpha, \beta \in A \cap \mathcal{R}_-$ and $\alpha + \beta \in \mathcal{R}$ then $\alpha + \beta \in A \cap \mathcal{R}_-$.

$\circ$ Therefore the algebra $\mathfrak{p}$ is determined by a choice of simple roots. We encode the choice of the subset of simple roots by colouring the nodes of the Dynkin diagram. A black dot means that $-\alpha \notin A \cap \mathcal{R}_-$.

14.14 Let $\mathfrak{g} = \mathfrak{sl}_5(\mathbb{C})$. Some examples of homogeneous spaces (the total number is equal to 2^5).

marked Dynkin	P	G/P	representation
$\circ - \circ - \circ - \circ$	SL_n	pt	$\mathbb{C}$ (trivial)
$\bullet - \circ - \circ - \circ$	$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{pmatrix}$	$Gr(1,5) = \mathbb{P}^4$	$\mathbb{C}^n$
$\circ - \bullet - \circ - \circ$	$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}$	$Gr(2,5)$	$\wedge^2\mathbb{C}^n$
$\circ - \circ - \circ - \bullet$	$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$	$G/P = Gr(4,5) = (\mathbb{P}^4)^\vee \simeq \mathbb{P}^4$	$\wedge^4\mathbb{C}^n$
$\circ - \bullet - \circ - \bullet$	$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$	$G/P \subset Gr(2,5) \times Gr(4,5)$	$V_{2L_1+2L_2+L_3+L_4} \subset \wedge^2\mathbb{C}^n \otimes \wedge^4\mathbb{C}^n$
$\bullet - \circ - \circ - \bullet$	$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$	$G/P \subset \mathbb{P}^4 \times (\mathbb{P}^4)^\vee$	$V_{2L_1+L_2+L_3+L_4} \subset \mathbb{C}^n \otimes \wedge^4\mathbb{C}^n$
$\bullet - \bullet - \bullet - \bullet$	$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$	$F\ell_5$	$V_\rho \subset \bigotimes_{k=1}^{n-1} \wedge^k\mathbb{C}^n$

14.15 Example: $G = Sp_n(\mathbb{C})$, $P =$ the stabilizer of $V_0 = lin\{e_1, e_2, \dots, e_n\}$. We get the Lagrangian grassmannian.

$$LG(n) = \{V \subset \mathbb{C}^n \mid \omega_{|V} \equiv 0, \quad \dim V = n\} \subset Gr(n, 2n)$$

∘ Similarly symplectic isotropic Grasmannians. For $k < n$

$$\{V \subset \mathbb{C}^n \mid \omega_{|V} \equiv 0, \quad \dim V = k\} \subset Gr(k, 2n)$$

(marked the $k-th$ shorter root). For $k = 1$ we get $\mathbb{P}^{2n-1}$ because $\omega_{|line}$ is automatically 0.

14.16 Let $G = SO_m(\mathbb{C})$, the group preserving a quadratic form Q. Let

$$X = \{[v] \in \mathbb{P}(\mathbb{C}^m) \ : \ Q(v) = 0\}\,.$$

This is a quadric. As before we can define for $k \leq m/2$

$$\{V \subset \mathbb{C}^n \mid Q_{|V} \equiv 0, \quad \dim V = k\} \subset Gr(k, m)$$

14.17 Exercise: deduce which nodes are marked in the Dynkin diagrams.

14.18 Bruhat decomposition

$$G/P = \bigsqcup_{w \in W/W_P} B\tilde{w}P/P\,.$$

Here W_P is the Weyl group of the reductive part of P, called Levi subrgroup. The Lie algebra of P is generated by $X \in \mathfrak{g}_\alpha$ for $\pm\alpha \in A \cap \mathcal{R}_-$. For $w \in W/W_P$ the element $\tilde{w} \in G$ denotes a representative of w.