Lie Groups and Algebras 2024

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6 Lecture 04.04. Examples, complexification

6.1 (Summary) Every compact Lie group G contains a maximal torus $T \simeq (S^1)^r$.

- All maxima tori are conjugate
- $\mathfrak{t} = \mathfrak{g}^T$
- If G is connected, $g \in G$, then there exist $h \in G$, such that $g \in hTh^{-1}$. In other words

$$\bigcup_{h\in G} hTh^{-1} = G\,.$$

- The Weyl group $N(T)/T$ is finite and acts on T.
- If $g, h \in T$ are conjugate in G, then they lie in the same W-orbit.

6.2 Example: $U(n) \subset \mathrm{GL}_n(\mathbb{C})$. The maximal torus consists of diagonal matrices.

- Lie algebra: since $U(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \ : \ \bar{A}^T A = I\}$,

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}_n(\mathbb{C}) \ : \ \bar{A}^T + A = 0\}$$

The maximal torus is of rank r, a distinguished example of a maximal torus of consists of diagonal matrices. The action of the matrix $t = diag(t_1, t_2, \dots, t_n) \in T$ on $\mathfrak{gl}_n(\mathbb{C})$:

$$t\{a_{ij}\}t^{-1} = \{t_i t_j^{-1} a_{ij}\}\,.$$

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{i\neq j} V_{ij}\,.$$

Here $\mathfrak{t}_{\mathbb{C}}$ consists of diagonal matrices with entries $a_{ii} \in \mathbb{C}$, while the Lie algebra of T consists of the matrices with purely imaginary entries. The space V_{ij} consists of matrices having everywhere 0 except the a_{ij}. This is a decomposition of complex T-representations. The associated weights re equal to $t_i - t_j$.

- The Lie algebra $\mathfrak{u}(n)$ decomposes as

$$\mathfrak{t} = \mathfrak{u}(n)^T \oplus \bigoplus_{i<j} V'_{ij}$$

where $V'_{ij} = (V_{ij} \oplus V_{ji}) \cap \mathfrak{u}(n)$

6.3 Examples: $SO(n) \subset \mathrm{GL}_n(\mathbb{R})$. Let $r = \lfloor \frac{n}{2} \rfloor$. An example of a maximal torus consists of 2×2-block diagonal matrices with rotations in each block. There are r blocks, and if $2r < n$, then the S-E corner entry is equal to 1. For example for $n = 3$

$$\begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- Exercise: use elementary linear algebra to show that it indeed is a maximal torus.

- $$\mathfrak{so}(n) = \{A \in \mathfrak{gl}_n(\mathbb{R}) \ : \ A^T + A = 0\}$$

To see decomposition of $\mathfrak{so}(n)$ as T representation it is convenient to pass to the complexofication. Let $SO_n(\mathbb{C})$ be the group preserving the complex 2-linear form defined by the matrix I. Let us focus on the case $n = 2r$. It is convenient to change coordinates, so that the 2-linear form is given by the matrix $Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. The associated quadratic form is equal to

$$\sum_{i=1}^{r} x_i x_{i+r}$$

In new coordinates the complexification of the torus consists of the diagonal matrices

$$diag(t_1, t_2, \ldots, t_r, t_1^{-1}, t_2^{-1}, \ldots, t_r^{-1})$$

Thus

$$\mathfrak{so}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(\mathbb{C}) \ : \ A^T Q + QA = 0\}$$

If $A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, then

$$Z^T + Z = 0\,, \quad Y^T + Y = 0\,, \quad W = X^T\,.$$

- The decomposition of the $\mathfrak{so}(n) \otimes \mathbb{C} = \mathfrak{so}_n(\mathbb{C})$ has the form

$$\mathfrak{t}_{\mathbb{C}} \oplus \bigoplus \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the one dimensional representation of the torus with $\alpha = t_i - t_j$ (with $i \neq j$) or $\pm(t_i + t_j)$.

- Note that $\mathfrak{so}_n(\mathbb{C}) \simeq \wedge^2 \mathbb{C}^n$ as T representations:

$$\wedge^2(\mathbb{C}^{2n}) = \wedge^2(\mathbb{C}^n_+ \oplus \mathbb{C}^n_-) = \underbrace{\wedge^2 \mathbb{C}^n_+}_{weights\ t_i + t_j} \oplus \underbrace{(\mathbb{C}^n_+ \otimes \mathbb{C}^n_-)}_{weights\ t_i - t_j} \oplus \underbrace{\wedge^2 \mathbb{C}^n_-}_{weights\ -(t_i + t_j)}\,.$$

- Corollary: $\mathfrak{so}_n(\mathbb{C}) \simeq \wedge^2 \mathbb{C}^n$ as $SO(n)$ representations.

6.4 Exercise: Analyse the Lie algebra $\mathfrak{sp}(n)$ and show that $\mathfrak{sp}(\mathfrak{n}) \otimes \mathbb{C} \simeq Sym^2 \mathbb{C}^n$ as T-representation.

6.5 General strategy in Lie theory:
1) complexify the Lie algebra and (if possible) find the corresponding complex Lie group.
2) study representations of the complexified Lie algebra
3) derive conclusions of the group itself.

- Complexification of Lie groups: For a given real Lie group there always exist a complex Lie group with the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. The point is to realize this group as a matrix group defined by polynomial identities.

6.6 Definition: Complex linear group is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ defined by polynomial identities.

6.7 Definition: A complex linear group is said to be reductive if the category of its representations is semisimple. This means, that every representation admits a decomposition into a direct sum of simple representations.

6.8 The groups $\mathrm{GL}_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_n(\mathbb{C})$ are reductive.

6.9 Fact: any reductive group has an embedding into $\mathrm{GL}_n(\mathbb{C})$, such that the image is invariant with respect to the Cartan involution: $\Theta : A \mapsto (\overline{A}^T)^{-1}$.

6.10 Another characterization: *the largest connected solvable normal subgroup (the radical) is an algebraic torus* $\simeq (\mathbb{C}^*)^r$.

6.11 Remark: there are no compact connected linear groups of positive dimension (any algebraic subset of $\mathbb{C}^{n^2}$ is finite or noncompact).

6.12 There are equivalences of categories

$$\begin{gathered}\{\text{Complex representations of (real) comapact connected, simplyconnected group } G\}\\ \updownarrow \\ \{\text{Complex representations of } \mathfrak{g}\} \\ \updownarrow \\ \{\text{Complex representations of } \mathfrak{g}_\mathbb{C}\} \\ \updownarrow \\ \{\text{Complex representations of the reductive group } G_\mathbb{C}\}\end{gathered}$$

The last equivalence requires explanation: it is not clear that $G_\mathbb{C}$ is an algebraic group. It will follow from classification.

6.13 For any compact group $G \subset U(n)$ it is clear how to define $G_\mathbb{C}$. Namely since

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus i\,\mathfrak{u}(n) = \mathfrak{u}(n)_\mathbb{C}$$

$$A = \tfrac{A-\overline{A}^T}{2} + \tfrac{A+\overline{A}^T}{2}\,,$$

hence $\mathfrak{g}_\mathbb{C}$ is naturally isomorphic to $\mathfrak{g} + i\,\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ and $G_\mathbb{C}$ is the corresponding subgroup. It is missing to show that $G_\mathbb{C}$ is closed and algebraic.

∘ Example of $SO_n(\mathbb{C})$: suppose $A^T A = I$. Let $B = \Theta(A) = (\overline{A}^T)^{-1}$. The equation $B^T B = I$ reads as

$$((\overline{A}^T)^{-1})^T \cdot (\overline{A}^T)^{-1} = I$$

i.e.

$$\overline{A}^{-1} \cdot (\overline{A}^T)^{-1} = I\,.$$

Hence

$$A^{-1} \cdot (A^T)^{-1} = I\,.$$

and $A^T A = I$.

6.14 The opposite direction of reasoning: Having a reductive group $G_\mathbb{C}$, together with embedding into $\mathrm{GL}_n(\mathbb{C})$, invariant with respect to the Cartan involution, construct the compact group $K := G_\mathbb{C} \cap U(n)$.

6.15 Properties of the Cartan involution $\Theta : G_\mathbb{C} \to G_\mathbb{C}$ and $\theta : \mathfrak{g} \to \mathfrak{g}$, $\theta(A) = -\overline{A}^T$, see [Knapp §1]

- the fixed points is a compact subgroup $K := G_\mathbb{C}^\Theta = G_\mathbb{C} \cap U(n)$
- θ is a homomorphism of Lie algebras

• the Lie algebra $\mathfrak{g}$ decomposes into eigenspaces of θ: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$

∘ $\mathfrak{k} := \mathfrak{g}_1$ is the Lie algebra of K (here $\mathfrak{k}$ is the $\mathfrak{gothic}$ k).

∘ $\mathfrak{p} := \mathfrak{g}_{-1}$ satisfies $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$,

∘ $\mathfrak{p} = i\mathfrak{k}$ hence $\mathfrak{g} \simeq \mathfrak{k} \otimes \mathbb{C}$ as complex Lie algebras.

6.16 For $G_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$ the space $\mathfrak{p}$ consists of the hermitian (or self-adjoint) matrices matrices $A = \overline{A}^T$.

6.17 Corollary: let $\phi, \psi : G \to H$ homomorphism of complex Lie groups, G reductive, connected. If $\phi_{|K} = \psi_{|K}$ then $\phi = \psi$.

6.18 The map $K \times \mathfrak{p} \to G_{\mathbb{C}}$ given by $(g, X) \mapsto g \cdot exp(X)$ is a diffeomorphism.

6.19 Proof of 6.18 for $G_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$: by polar decomposition every invertible matrix A can be written uniquely as $A = Q\,P$, where $Q \in U(n)$ and $P = \theta(P)$ is positive definite. Any positive definite matrix P has logarithm.

• Reminder from linear algebra course: for $P = (A^*A)^{\frac{1}{2}}$, $Q = AP^{-1}$ we check $QQ^* = (AP^{-1})(P^{-1}A^*) = A(A^*A)^{-1}A^* = I$).

7 Lecture 11.04. $SL_2(\mathbb{C})$

7.1 Lie group G comes with the adjoint representation: the action by conjugation of G on G fixes e, hence we get $Ad : G \to Aut(\mathfrak{g})$.

∘ If G is connected, then $ker(Ad) = Z(G)$.

7.2 The differential of Ad, i.e. $ad : \mathfrak{g} \to End(\mathfrak{g})$ is given by the commutator $ad_X(Y) = [X, Y]$.

• We assume that $G \subset \mathrm{GL}_n(\mathbb{C})$ and check the equality for matrices.

∘ First note that $Ad_A(Y) = AYA^{-1}$. Then set $A = e^{tX}$ and differentiate:

$$\tfrac{d}{dt}(e^{tX}Ye^{-tX})_{|t=0} = (Xe^{tX}Ye^{-tX} + e^{tX}Y(-X)e^{-tX})_{|t=0} = XY - YX\,.$$

7.3 A **representations of the Lie algebra** $\mathfrak{g}$ is a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where V is a vector space, or equivalently for any $X, Y \in \mathfrak{g}$ and any $v \in V$

$$\rho(X)\rho(Y)v - \rho(Y)\rho(X)v = \rho([X, Y])v$$

7.4 The kernel of the adjoint representation

$$ker(ad) = \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g}\ [X, Y] = 0\}\,.$$

This is called the center of the Lie algebra and denoted by $Z(\mathfrak{g})$. If $Z(\mathfrak{g}) = 0$, then Ado theorem about embedding of $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ is for free; $\mathfrak{g}$ embeds in $End(\mathfrak{g})$.

Representations of $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{su}(2)$

7.5 Groups $SU(2)$, $SL_2(\mathbb{C})$, $SL_2(\mathbb{R})$ and relations between their representations.

$$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{su}(2) \oplus i\mathfrak{su}(2) = \mathfrak{sl}_2(\mathbb{R})_{\mathbb{C}}$$

7.6 Action of T allows to decompos $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ into weight spaces

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathfrak{t}^* \setminus 0} \mathfrak{g}_\alpha .$$

(α's are called roots.)

- ∘ In the case of $SL_2(\mathbb{C})$

$$\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{t} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2} .$$

7.7 $\mathfrak{sl}_2(\mathbb{C})$ is spanned by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

- ∘ $[X, Y] = H$.
- ∘ $[H, X] = 2X$, i.e. $X \in \mathfrak{g}_2$
- ∘ $[H, Y] = -2Y$, i.e. $Y \in \mathfrak{g}_{-2}$

7.8 Maximal torus $\mathbb{C}^* \hookrightarrow SL_2(\mathbb{C})$

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

decomposes any representation V into weight spaces $V = \bigoplus_{k\in\mathbb{Z}} V_k$. For $v \in V_k$:

- ∘ $Hv = kv$
- ∘ $Xv \in V_{k+2}$
- ∘ $Yv \in V_{k-2}$
- • In general:
- ∘ if $\alpha, \beta \in \mathfrak{t}^*$ and
- ∘ $X \in \mathfrak{g}_\alpha$ (i.e. $\forall H \in \mathfrak{t}\ [H, X] = \alpha(H)X$) ,
- ∘ and $v \in V_\beta$ (i.e. $\forall H \in \mathfrak{t}\ Hv = \beta(H)v$)

then $Xv \in V_{\alpha+\beta}$.

7.9 Examples of representations of $\mathfrak{sl}_2(\mathbb{C})$:

- ∘ Natural representation ("defining representation") $V \simeq \mathbb{C}^2$
- ∘ symmetric powers of the natural representations $Sym^k(V)$

7.10 General construction of the symmetric power:

$$T^k(V) = V \otimes V \otimes \cdots \otimes V \qquad k \text{ times}$$

is a representation of the permutation group Σ_k. Let

$$sym_k = \tfrac{1}{k!} \sum_{\sigma \in \Sigma_k} \sigma \in \mathbb{C}[\Sigma_k] .$$

be the symmetrizing operator: $sym_k \circ sym\pi_k = sym_k$. It acts on $T^k(V)$.

$$Sym^k(V) = T^k(V)^{\Sigma_k} = im(sym_k) = coker(sym_k) .$$

Hence we have two descriptions of $Sym^k(V)$

- ∘ as Σ_k-invariant tensors
- ∘ as $T^k(V)$ modulo the relation $v_1 \otimes v_2 \otimes \cdots \otimes v_k \sim v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)}$

∘

$$Sym^k(V^*) = \text{Polynomial functions on } V \text{ of degree } k$$

• The above construction is natural, hence for any G and a representation V of G we have well defined representation $Sym^k(V)$.

7.11 The algebra $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the subalgebra of differential operators in 2 variables generated by $X = x\frac{\partial}{\partial y}$ and $Y = y\frac{\partial}{\partial x}$, $H = [X, Y] = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. The natural representation: linear forms, $Sym^k(\mathbb{C}^2) \simeq \{k - \text{polynomial forms}\}$.
∘ $x\frac{\partial}{\partial y}(x^k y^\ell) = \ell x^{k+1} y^{\ell-1}$
∘ $y\frac{\partial}{\partial x}(x^k y^\ell) = k x^{k-1} y^{\ell+1}$

7.12 Highest weight vectors in the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ is a vector$v \in V$ such that $Xv = 0$.

7.13 [Fulton-Harris, §11] Theorem: irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ (or $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{sl}_2(\mathbb{Z})$) are isomorphic to $Sym^k(V)$. They are characterized by the weight of the vector $v \in ker(X)$ (highest weight vector), which is a natural number.

7.14 Key Lemma: Suppose $v \in V_m$ is a highest weight vector (i.e$Xv = 0$) then $XY^{n+1}v = (n+1)(m - nY^n v$.
∘ $(n = 0)$ then $XYv = [X, Y]v + YXv = Hv = mv$
∘ $(n = 1)$ then $XY^2v = [X, Y]Yv + YXYv = HYv + Y(mv) = (m - 2 + m)Yv$
∘ $(n = 2)$ then $XY^3v = [X, Y]Y^2v + YXY^2v = HY^2v + Y((m-2+m)v) = (m-4+m-2+m)Yv$
∘ ...

7.15 Corollary: if $\dim V < \infty$ then $m \in \mathbb{N}$ and $Y^{m+1}v = 0$.

7.16 The representations Sym^kV, $k \in \mathbb{N}$ are irreducible, it has the highest vector of the weight k. This is the full list of irreducible representations of $SL_2(\mathbb{C})$ (and $SU(2)$ as well).

7.17 Every complex representation of $\mathfrak{s}l_2(\mathbb{C})$ extends to a representation of $SL_2(\mathbb{C})$:
∘ By polar decomposition $SL_2(\mathbb{C}) = SU(2) \times \mathbb{R}^3 = S^3 \times \mathbb{R}^3$ as topological spaces, hence $\pi_1(SL_2(\mathbb{C})) = \pi_1(S^3) = 1$. So every representation of $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ lifts to $SL_2(\mathbb{C}) \to \mathrm{GL}(V)$.

7.18 Corollary: Every complex/real representation of $\mathfrak{s}l_2(\mathbb{R})$ extends to a representation of $SL_2(\mathbb{R})$.
∘ Proof: complexify.

7.19 If $W = \bigoplus_{n\in\mathbb{N}} Sym^n(V)^{\oplus a_n}$ as a $\mathfrak{s}l_2(\mathbb{C})$-representation, then $a_n = \dim W_n - \dim W_{n+2}$.

7.20 Examples of computations $Sym^k(V) \otimes V = Sym^{k+1}(V) \oplus Sym^{k-1}(V)$.

7.21 The character of the representation $Sym^k(V)$ restricted to the maximal torus is equal to

$$t^{-k} + t^{-k+2} + \cdots + t^k = \sum_{i+j=k} (t^{-1})^i t^j = \frac{t^{k+1} - t^{-k-1}}{t - t^{-1}}$$

Z ćwiczeń:

7.22 Trace form defined for $\mathfrak{gl}_n(\mathbb{C})$:

$$B_0(X,Y) = Tr(XY).$$

7.23 Suppose $\mathfrak{g} \subset \mathfrak{u}(n)$. Then B_0 is nondegenerate on $\mathfrak{g}_\mathbb{C}$ since $B_0(X,\theta(X)) = B_0(X,\theta(X))$ is real and < 0 for $X \neq 0$

7.24 Killing form: $B(X,Y) = Tr(ad_X \circ ad_Y)$. This form is symmetric and G-invariant.

7.25 Killing form is nondegenerate on $\mathfrak{g}/Z(\mathfrak{g})$.
- Beacause this is the form from (7.22) for $G := Ad(G)$.

7.26 If G is compact, then B is nonpositive definite:
- Because one can choose a G-invariant metric in $\mathfrak{g}$, such that $Ad(G) \subset O(\mathfrak{g})$.

8 Lecture 18.04 – Systems of roots

8.1 Rank of the Lie group $r(G) := \dim(T)$, where T is a maximal torus.

8.2 For compact groups:
- $\mathfrak{t}$ is the Lie algebra of the compact maximal torus
- $\mathfrak{t}_\mathbb{C}$ the complexification
- $\mathfrak{t}_\mathbb{Z} = ker(exp : \mathfrak{t} \to T)$
- $\mathfrak{t}^*_\mathbb{Z} = \mathrm{Hom}(\mathfrak{t}_Z, \mathbb{Z}) \subset \mathfrak{t}^*$, here belong the roots of the Lie algebra

8.3 Having chosen $T \subset G$ we decompose

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathfrak{t}^*_\mathbb{Z} \setminus \{0\}} \mathfrak{g}_\alpha \,.$$

into eigenspaces
- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$
- $$\mathcal{R} = \{\alpha \in \mathfrak{t}^* \setminus \{0\} \ : \ \mathfrak{g}_\alpha \neq 0\}$$

is called the root system. We will show:

8.4 More general: for any representation of G

$$V = \bigoplus_{\alpha \in \mathfrak{t}^*} V_\alpha \,.$$

- $\mathfrak{g}_\alpha V_\beta \subset V_{\alpha+\beta}$

8.5 For an invariant 2-linear form on $\mathfrak{g}$: if $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha \perp_\phi \mathfrak{g}_\beta$.
Proof: for $H \in \mathfrak{t}$

$$0 = \phi(Hv_\alpha, v_\beta) + \phi(v_\alpha, Hv_\beta) = \alpha(H)\phi(v_\alpha, v_\beta) + \beta(H)\phi(v_\alpha, v_\beta) = (\alpha - \beta)(H)\phi(v_\alpha, v_\beta)$$

8.6 Suppose $\mathfrak{g}$ is a Lie algebra of a compactgroup. If $\mathfrak{g}_\alpha \neq 0$ then $\mathfrak{g}_{-\alpha} \neq 0$. The invariant scalar product identifies $\mathfrak{t} \simeq \mathfrak{t}^*$. Define $H_\alpha \in \mathfrak{t}$ such that $(H_\alpha, v) = \alpha(v)$ for $v \in \mathfrak{t}$. Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ then

$$[x, y] = (x, y)H_\alpha$$

∘ Proof:

$$([x, y], h) + (y, [x, h]) = 0$$

$$([x, y], h) = (y, [h, x]) = \alpha(h)(x, y) = (H_\alpha, h)(x, y)$$

8.7 Suppose $(x, y) = \frac{2}{(\alpha,\alpha)}$, then $x, y, h_\alpha = 2H_\alpha/(\alpha, \alpha)$ is a basis of $lin(x, y, H_\alpha)$ satisfying the standard relations of $\mathfrak{sl}_2$.

$$[h_\alpha, x] = \alpha(h_\alpha)x = (H_\alpha, h_\alpha)x = \frac{2}{(\alpha, \alpha)}(H_\alpha, H_\alpha)$$

$$[x, y] = (x, y)H_\alpha = h_\alpha$$

∘ Corollary: for every root α we have constructed a copy of $\mathfrak{sl}_2 \in \mathfrak{g}_\mathbb{C}$, hance also a copy of $SL_2(\mathbb{C})$ in G (or $SU(2)$ in the compact group.) We denote such a coppy by $\mathfrak{sl}_2(\mathbb{C})_\alpha$

8.8 Let E be a real vector space with a scalar product. An abstract system of roots is a finite set $\mathcal{R} \subset E$ such that

1. The roots $\mathcal{R}$ span $\mathfrak{t}^*_\mathbb{Z} \setminus \{0\}$
2. If $\alpha \in \mathcal{R}$ then $-\alpha \in \mathcal{R}$
3. If $\beta \in lin\{\alpha\}$, then $\beta = \pm\alpha$
4. The reflections in $\alpha \in \mathcal{R}$ preserve $\mathcal{R}$
5. for $\alpha, \beta \in \mathcal{R}$ the quotient $n_{\alpha,\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

(see Kirillov Def 7.1)

8.9 We fix a G-invariant scalar product in $\mathfrak{g}$, hence we have a W-invariant scalar product in $\mathfrak{t}$ and $\mathfrak{t}^*$. If $Z(G)$ is finite, then the preferred choice is the (minus) Killing form.

8.10 Theorem. Let $\mathcal{R} \subset \mathfrak{t}^*$ be the set of roots of a compact Lie group with $Z(G)$ finite. Then it satisfies the axioms of an abstract system of roots. Moreover
∘ $\dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \mathcal{R}$
∘ For $\alpha, \beta \in \mathcal{R}$ the $\mathfrak{sl}_2(\mathbb{C})$ representation $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\beta+k\alpha}$ is irreducible
∘ For $\alpha, \beta \in \mathcal{R}$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ (not only „$\subset$”)
[see Kirillov Theorem 6.44]

8.11 **Examples** of root systems of rank 2 (from ćwiczenia) $SO(4)$, $Sp(2)$, $SO(5)$, see [FuHa §21]

8.12 Lie group $SU(3)$ (and its complexification $SL_3(\mathbb{C})$)

- $\mathfrak{t}_{\mathbb{C}}$ =diagonal matrices with trace =0,
- the Lie algebra of the compact torus: $diag(t_1, t_2, t_3)$ s.t. $Re(t_i) = 0$, $t_1 + t_2 + t_3 = 0$
- weights $L_i : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$, i.e. $L_i \in \mathfrak{t}^*_{\mathbb{C}}$, $i = 1, 2, 3$

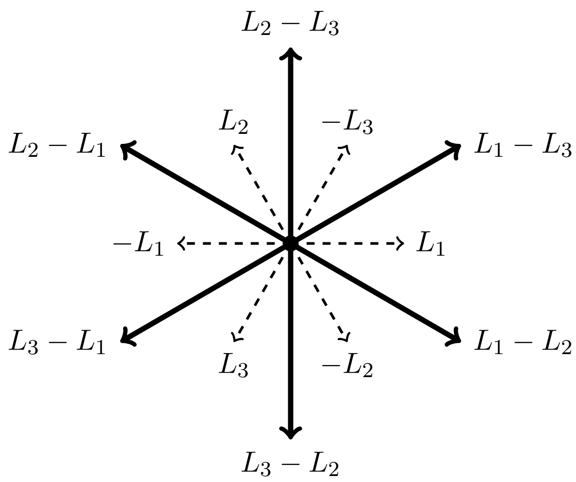
$$diag(t_1, t_2, t_3) \mapsto t_i \, .$$

There is a relation $L_1 + L_2 + L_3$.

- Roots: $\alpha_{i,j} = L_i - L_j$, $i \neq j$
- Cartan numbers

$$n_{\alpha_{1,2},\alpha_{2,3}} = 2\frac{((1,-1,0),(0,1,-1))}{((1,-1,0),(1,-1,0))} = -1$$

$\angle(\alpha_{1,2}, \alpha_{2,3}) = 2\pi/3$



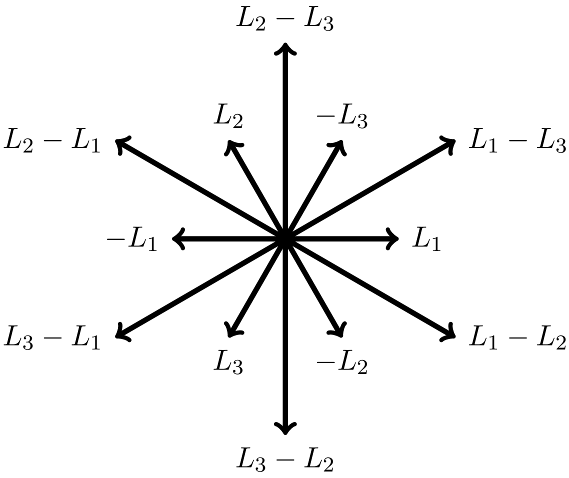
8.13 Examples of root systems of rank 2 (from ćwiczenia) $SO(4)$, $Sp(2)$, $SO(5)$, see see [FuHa §21]

- $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
- $\mathfrak{sp}(2)$: $\mathfrak{t}_{\mathbb{C}} \ni diag(t_1, t_2, -t_1, -t_2) \overset{L_i}{\mapsto} t_i$
- Roots $\pm 2L_i$ or $L_i - L_j$
- For $\alpha \neq \pm\beta$

$$n_{\alpha,\beta} \in \left\{0\ ,\ \pm 2\frac{((1,-1),(2,0))}{((1,-1),(1,-1))} = 2\ ,\ \pm 2\frac{((2,0),(1,-1))}{((2,0),(2,0))} = 1\right\}$$

- $\mathfrak{so}(5)$: $\mathfrak{t}_{\mathbb{C}} \ni diag(t_1, t_2, -t_1, -t_2, 0) \overset{L_i}{\mapsto} t_i$
- Roots $\pm L_i$ or $L_i - L_j$
- $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$

8.14 Exceptional G_2 (it will be later)



8.15 In general there are strong restrictions for $n_{\alpha,\beta}$. The Cartan numbers $n_{\alpha\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ satisfy

- $n_{\alpha\beta} n_{\beta\alpha} = 4\cos^2(\angle(\alpha,\beta))$,
- $n_{\alpha\beta} n_{\beta\alpha} \in \mathbb{Z}$, therefore $n_{\alpha\beta} n_{\beta\alpha} \in \{0,1,2,3\}$ for $\alpha \neq \pm\beta$.

∘ $\angle(\alpha,\beta) \in \{30^o, 45^o, 90^o, 120^o, 135^o, 150^o\}$

8.16 The Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is spanned by $\mathfrak{t}$ (the diagonal matrices of trace 0) and $E_{i,j} \in \mathfrak{g}_{L_i - L_j}$

$$[H, E_{i,j}] = (L_i(H) - L_j(H))E_{i,j}$$

i.e. if $H = diag(t_1, t_2, t_3)$, then

$$[H, E_{i,j}] = (t_i - t_j)E_{i,j}$$

8.17 With this notation we list the basic representations of $SL(3)$

- The defining representation $V = \mathbb{C}^3$

The weights L_1, L_2, L_3. The corresponding eigenvectors e_1, e_2, e_3

$$E_{1,2}e_1 = 0,\ E_{1,3}e_1 = 0,\ E_{2,3}e_1 = 0$$

The *highest weight vector* $e_1 \in \bigcap_{i>j} ker(E_{i,j})$ generates whole representation.

- The second exterior power $\wedge^2 V \simeq V^*$.

The weights: $L_1 + L_2 = -L_3$, $L_1 + L_3 = -L_2$, $L_2 + L_3 = -L_1$. The corresponding eigenvectors $e_1 \wedge e_2, e_2 \wedge e_3, e_2 \wedge e_3$. The action of $E_{i,j}$ for $i < j$

$$E_{12} : e_1 \wedge e_2 \mapsto 0 \wedge e_2 + e_1 \wedge e_1 = 0$$

$$E_{13} : e_1 \wedge e_2 \mapsto 0 \wedge e_2 + e_1 \wedge 0 = 0$$

$$E_{23} : e_1 \wedge e_2 \mapsto 0 \wedge e_2 + e_1 \wedge 0 = 0$$

The *highest weight vector* $e_1 \wedge e_2 \in \bigcap_{i>j} ker(E_{i,j})$ generates whole representation.

- $Sym^2(V) = lin\{e_1^2, e_2^2, e_3^2, e_1e_2, e_1e_3, e_2e_3\}$. The *highest weight vector* $e_1^2 \in \bigcap_{i>j} ker(E_{i,j})$ generates whole representation. Equivalently the *highest weight vector* $e_3^* \in \bigcap_{i>j} ker(E_{i,j})$ generates V^*.
- Some examples of representations of $\mathfrak{sl}_3(\mathbb{C})$. In paricular $Sym^2(V) \otimes (V)^*$, see Fulton-Harris §12-13.

Claim: every irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ is isomorphic to a subrepresentation of $Sym^\bullet(V) \otimes Sym^\bullet(\wedge^2 V)$. For a representation with the highest weight vector of the weight $(a+b)L_1 + bL_2$ take the representation generated by $v = (e_1)^a \otimes (e_1 \wedge e_2)^b$. The remaining vectors are obtained by application of the operators E_{21}, E_{31}, E_{32} given by the action of elementary matrices.

Proofs of properties of the root system 8.8

8.18 Theorem: Any compact connected Lie group of rank 1 is isomorphic to $SU(2)$ or $SO(3)$ or S^1. Proof: Let $n = \dim(G)$. G acts on $S^{n-1} \subset \mathfrak{g}$ via Ad. The action Ad fixes $= \mathfrak{t}$. Therefore $G/T \to S^{n-1}$ is a covering, so it has to be a homeomorphism. We get a fibration $S^1 = T \to G \to G/T = S^{n-1}$. If $n > 3$ the $\pi_1(T) \to \pi_1(G)$ is a monomorphism. The group G contains a subgroup H isomorphic to $SU(2)$ or $SO(3)$ with the Lie algebra $\mathfrak{t} \oplus \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{-\alpha_0}$, where α_0 the longest root. There is an element $g \in N(T) \subset H$ such that $Ad(g)_{|\mathfrak{t}} = -Id : \mathfrak{t} \to \mathfrak{t}$, so $gtg^{-1} = t^{-1}$. But in G the conjugation by g is homotopic to Id. Contradiction. Hence $n \leq 3$. $\square$

8.19 Proof of 8.8.1. More general we have $\bigcap_{\alpha \in R} ker(\alpha) = T(Z(G))$.

8.20 Proof of 8.8.2. $\mathfrak{g}$ is a real representation of T. The summands of the decomposition of $\mathfrak{g}_{\mathbb{C}}$ come in pairs.

8.21 Proof of 8.8.3. For any root α let

$$\mathfrak{k}_\alpha = \mathfrak{t} \oplus \bigoplus_{\beta \text{ proportional to } \alpha} \mathfrak{g}_\beta$$

Let K generated by $exp(\mathfrak{k}_\alpha)$; the roots of the closure have the same kernel as α, hence K closed. By 8.18 $\dim K = 3$.

8.22 Proof of 8.8.4. The action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SU(2)_\alpha$ is the reflection in $ker(h_\alpha)$ (denoted by s_α). It preserves the root system.

8.23 Proof of 8.8.5. The vector space $W = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$ is a representation of $\mathfrak{sl}_2(\mathbb{C})_\alpha$. The number $n_{\alpha,\beta}$ is the weight of $v \in \mathfrak{g}_\beta$. So it is an integer

$$[h_\alpha, v] = \frac{2}{(\alpha,\alpha)}[H_\alpha, v] = \frac{2}{(\alpha,\alpha)}\beta(H_\alpha)v = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}v$$

9 Lecture 25.04. Positive and simple roots, Weyl group

9.1 We have defined the Weyl group as NT/T. It acts on $\mathfrak{t}^*$ preserving roots. For a root α we have a copy of $SU(2)$ or $SU(2)/\langle \pm I \rangle$ in G, dnoted by K_α.

∘ $N(T_{K_\alpha}) \subset N(T)$ and the nontrivial element of W_{K_α} acts as a reflection in α.

∘ We will identify the Weyl group $N(T)/T$ as the group of isometries of $\mathfrak{t}^*$ generated by s_α. For the moment we consider abstract root systems and abstract Weyl group generated by the reflections. If we deal with a Lie algebra of a compact group, then the elements s_α are realized as the effect of the action of elements from $N(T)$.

9.2 For the root systems of rank 2 (ie. $\dim E = 2$) we have

∘ $A_1 \cup A_1$ realized as the root system of $SU(2) \times SU(2)$

- A_2 realized by $SU(3)$
- B_2, also called C_2 realized by $SO(5)$ or $Sp(2)$
- G_2 given by 8.14

Positive and simple roots [Kirillov 7.4]

9.3 Dividing E into two half–spaces we decompose $\mathcal{R} = \mathcal{R}_+ \sqcup \mathcal{R}_-$.
- The division is given by the sign (α, ρ), where ρ is a generic vector of $\mathbf{t}^*$.

9.4 A positive root is simple if cannot be written as a sum of two positive roots. Every positive roots can be written as a sum of simple roots.

9.5 For two simple roots $(\alpha, \beta) \leq 0$. [Kirillov, Lem. 7.11 and 7.14]
- The proof follows from the analysis of root systems of rank 2.

9.6 The set of all simple roots form a basis.
- Obviously it spans
- If $v = \sum a_i \alpha_i = \sum b_j \beta_j$ with $a_i, b_j \geq 0$, then $||v|| = 0$
- $$||v||^2 = \sum_{i,j} a_i b_j (\alpha_i, \beta_j) \leq 0$$
- On the other hand $(v, \rho) > 0$. Contradiction.

9.7 Dynkin diagram:
- vertices = simple roots denoted in Kirillov by Π
- edges:

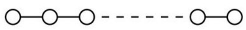
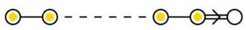
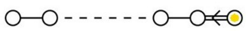
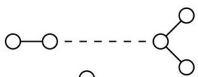
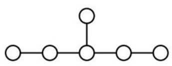
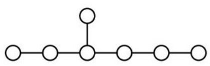
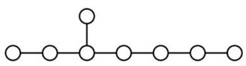
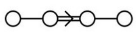
  no edge if $n_{\alpha,\beta} = 0$

  α —— β if $n_{\alpha\beta} n_{\beta\alpha} = -1$

  $\alpha \Leftarrow \beta$ if $n_{\alpha\beta} n_{\beta\alpha} = -2$, $|\alpha| < |\beta|$

  $\alpha \Lleftarrow \beta$ if $n_{\alpha\beta} n_{\beta\alpha} = -3$, $|\alpha| < |\beta|$

9.8 All possible irreducible (i.e. connected) Dynkin diagrams. The longer roots are in colour:

$SU(n+1)$	$SL_n(\mathbb{C})$	A_n	
$SO(2n+1)$	$SO_{2n+1}(\mathbb{C})$	B_n	
$Sp(n)$	$Sp_n(\mathbb{C})$	C_n	
$SO(2n)$	$SO_{2n}(\mathbb{C})$	D_n	
		E_6	
		E_7	
		E_8	
		F_4	
$Aut(\mathbb{O})$		G_2	

9.9 Having chosen division into positive and negative roots one redefine the functional defining the split:

$$\rho = \tfrac{1}{2} \sum_{\alpha\in\mathcal{R}_+} \alpha\,.$$

∘ Theorem: $\rho \in P$.

9.10 Weyl Chambers = connected components of $E \setminus \bigcup_{\alpha\in\mathcal{R}} \mathcal{H}_\alpha$, where

$$\mathcal{H}_\alpha = \{\lambda \in E \ : \ (\alpha, \lambda) = 0\}$$

∘ The positive chamber:

$$C_+ = \{\lambda \in E \ : \ \forall_{\alpha\in\mathcal{R}_+} \quad (\alpha, \lambda) > 0\}$$

∘ The chamber C_+ has exactly $n = \dim E$ walls corresponding to simple roots.
∘ Aplying reflections in walls one can transform C_+ to any other chamber.

9.11 Weyl group = the group generated by the reflections s_α

$$W = \langle s_\alpha \mid \alpha \in \mathcal{R}\rangle\,.$$

9.12 Theorem:
∘ 1) W acts transitively on the set of chambers
∘ 2) W is generated by the reflections in simple roots
∘ 3) W acts freely on the set of chambers
• 1) and 2) is easy by a geometric argument

9.13 Suppose $C = w(C_+)$ let. Define the length $\ell(w)$

$$\ell(w) = |\{\alpha \in \mathcal{R}_+ \mid w(\alpha) \in \mathcal{R}_-\}|\,.$$

(Number of walls separating C from C_0.)

9.14 Theorem: If $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ is a shortest presentation of $w \in W$, then $k = \ell(w)$.
∘ From above follows 9.12.3. That is the stabilizer of C_+ consists of elements of w of length 0, i.e. it consists only of the identity. (Exercise 7.3 in Kirillov.)

9.15 Topological proof of 9.12.3 with the root system of a compact Lie group: if $g \in NT$ preserves the chamber C_+, one may assume $g(X) = X$ for some $X \in C_+$. The group topologically generated by $\exp(tX)$ and g is abelian, $\simeq torus \times \mathbb{Z}_n$ can be topologically generated by one element, so it is contained in a maximal torus. This torus has to be T (*). Hence $[g] = 1 \in N(T)/T$.
∘ (*) The centralizer of the torus $\exp(tX)$ has the Lie algebra equal to

$$\mathfrak{t} \oplus \bigoplus_{\alpha:\ \alpha(X)=0} \mathfrak{g}_\alpha\,.$$

9.16 Corollary: Since $W_{top} := N(T)/T$ acts freely and $W_{alg} := \langle s_\alpha : \alpha \in \mathcal{R}\rangle$ acts transitively, thus $W_{top} = W_{alg}$, i.e. two notions of the Weyl group coincide. The Weyl group acts freely and transitively on the set of Weyl chambers.

9.17 The vertices Π of the Dynkin diagram may be treated as generators of W, the number of the edges between α and β, i.e. $n_{\alpha\beta}n_{\beta\alpha}$ encodes the angle $\angle(\alpha, \beta)$. Hence the order the corresponding rotation $s_\alpha s_\beta$.

- First of all $s_\alpha^2 = 1$
- no edge: s_α, s_β commute $\iff (s_\alpha s_\beta)^2 = 1$
- one edge $(s_\alpha s_\beta)^3 = 1$ (equivalently the braid relation $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$)
- double edge $(s_\alpha s_\beta)^4 = 1$
- triple edge $(s_\alpha s_\beta)^6 = 1$

9.18 Theorem [not so obvious]: These are the relations defining W.

∘ This is an example of a Coxeter group.