Proposition; Let $\theta : G \to GL(V)$ be a representation.
Then $f = \int_G \theta \, dg \in \mathrm{Hom}(V,V)$ is an idempotent (i.e $f^2 = f$)
and $\mathrm{im} f = V^G$.

Pf: Evaluation is linear (in $\mathbb{R}$), so $f(v) = \int_G \theta(g)(v)$

And $\theta(g')f(v) = \theta(g') \int_G \theta(g)(v) = \int_G \theta(g')(\theta(g)(v)) = \int_G \theta(g'g)(v) = f(V)$

linearity of the integral left invariance

So $f(v) \in V^G$.
If $v \in V^G$ to $f(v) = \int_G \theta(g)(v) = \int_G v = v$. which proves $f^2 = f$ ■

CHARACTERS OF REPRESENTATIONS

So far although we know that every representation is completely reducible we have no means to determine which irreducible summands are its components. So we look at numerical invariants of endomorphisms.

Trace: $\mathrm{Hom}(V,V) \cong \mathrm{Hom}(V, V^{**}) \cong \mathrm{Hom}(V, \mathrm{Hom}(V^*, K)) \cong \mathrm{Hom}(V \otimes V^*, K)$

take $V \otimes V^* \to K$ $\quad v \otimes \varphi = \varphi(v)$. This is trace, denoted by tr.

Definition. Let $\rho : G \to GL(V)$ be a complex representation. Then its character $\chi_V : G \to \mathbb{C}$ is a function:

$$\chi_V(g) = \mathrm{tr}\, \rho(g)$$

Properties of characters:

(1) Character depends only on an G-isomorphism class of V.

(2) $\forall_{g,g'}$ $\chi_V(g' g g'^{-1}) = \chi_V(g)$ — we say that character is a <u>class function</u>

(3) $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$

(4) $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$

(5) $\chi_{V^*}(g) = \chi_V(g^{-1})$

$\rho: G \to V$ $\quad \rho^*: G \to V^*$

$\rho^*(g) = (g^{-1})^*$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & \mathbb{C} \arrow[d, "id"] \\ V \arrow[r, "g(\varphi)"] & \mathbb{C} \end{tikzcd}$$

(6) $\chi_{tV}(g) = \overline{\chi_V(g)}$

(7) $\chi_V(e) = \dim V$

Remark: The definition of character of the representation does not require compactness and properties (1) → (7) hold, as they follow from properties of trace.

Assume the representation is complex and group is compact

Proposition: For compact group G and complex representations,

(1) $\chi_V(g^{-1}) = \chi_{V^*}(g) = \chi_{tV}(g) = \overline{\chi_V(g)}$

(2) $\int \chi_V(g)\,dg = \dim_{\mathbb{C}} V^G$

Pf: (1) is clear as for G compact the invariant inner product gives isomorphism $V^* \cong tV$.

(2) as trace is linear $\int tr\, \rho(g)\,dg = tr \int \rho(g)\,dg = tr\, I$ where I is idempotent with $\operatorname{im} I = V^G$. So $tr\, I = \dim_{\mathbb{C}} \operatorname{im} I$

On the vector space $C(G)$ of cont. functions $G \longrightarrow \mathbb{C}$ one can define an inner product (= positive Hermitian form)

$$\langle f, g\rangle = \int \bar{f} g$$

ORTHOGONALITY OF CHARACTERS

Theorem: Let V, W be representations of G. Then

(1) $\langle \chi_V, \chi_W\rangle = \int_G \bar{\chi}_V \chi_W = \dim_{\mathbb{C}} \mathrm{Hom}_G(V, W)$

(2) If V, W are irreducible, then

$$\langle \chi_V, \chi_W\rangle = \begin{cases} 0 & \text{if } V \text{ and } W \text{ are not isomorphic} \\ 1 & \text{if } V \cong W \end{cases}$$

Proof: Remark that $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$

Recall that for vector spaces V W przekształcenie $\phi: V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$ zadane wzorem $(\phi(\varphi \otimes w))(v) = \varphi(v) \cdot w$ jest funktorialnym izomorfizmem a więc przenosi się na reprezentacje.

Thus

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = \int \chi_{\mathrm{Hom}(V, W)}\, dg =$$

$$= \int \chi_{V^* \otimes W}\, dg = \int \chi_{V^*} \cdot \chi_W\, dg = \int \bar{\chi}_V(g) \chi_W(g)\, dg = \langle \chi_V, \chi_W\rangle$$

Druga część twierdzenia wynika z lematu Schura.

Corollary. If $V = \oplus n_i V_i$ then $n_i = \langle \chi_V, \chi_{V_i} \rangle$

Corollary: Representations V and W are isomorphic iff their characters are equal.

Corollary: Characters of irreducible representations are linearly independent.

Definition: Let G be a compact group. The representation ring of the group G is a free abelian group generated by irreducible representations with tensor product as multiplication.

Denote $Cl(G) \subseteq C(G)$ subring of class functions i.e continuous and constant on conjugacy classes.

Assigning character to a representation gives a homomorphism of rings:

$$\mu : R(G) \longrightarrow Cl(G)$$

Because characters of irreducible representations are linearly independent, then the homomorphism

$$\mu : R(G) \longrightarrow Cl(G)$$

is a monomorphism.

What is the image?

Theorem (Peter-Weyl): Every continuous function $G \to \mathbb{C}$ (G compact topological) can be uniformly approximated by functions of the form $tr(\varphi\theta(g))$ where $\theta: G \to GL(V)$ is a representation and $\varphi \in Hom_{\mathbb{C}}(V, V)$.

Every continuous class function $G \to \mathbb{C}$ (G compact topological) can be uniformly approximated by functions of the form $tr(\varphi\theta(g))$ where $\theta: G \to GL(V)$ is a representation and $\varphi \in Hom_G(V, V)$. That amounts to approximating class functions by linear combinations $\sum \lambda_i \chi_i$ $\lambda_i \in \mathbb{C}$ χ_i character of irreducible complex representation.

The proof of this theorem can be found in Adam's book. We will draw one important conclusion.

Theorem: If G is a compact Lie group, then there exists $n \in \mathbb{N}$ such that G is isomorphic to a $\overset{\text{closed Lie}}{}$ subgroup of $U(n)$.
In a different formulation there exists $i: G \to U(n)$ monomorphism - this monomorphism is a certain representation which is called faithful.

Proof: Step 1: For every element $g_0 \in G$ $g_0 \neq 1$, there exists a representation $\rho_{g_0}: G \to GL(n_{g_0}, \mathbb{C})$ such that $\rho_{g_0}(g_0) \neq 1$.

Take a continuous function $f_{g_0}: G \to \mathbb{C}$ such that $f_{g_0}(1) = 0$ and $f_{g_0}(g_0) = 1$. By P-W theorem there exists $\varphi_{g_0} \in Hom(\mathbb{C}^{n_{g_0}}, \mathbb{C}^{n_{g_0}})$ and a representation $\theta_{g_0}: G \to GL(n_{g_0}, \mathbb{C})$ such that $|f_{g_0}(g) - Tr(\varphi_{g_0}\theta_{g_0}(g))| < \frac{1}{3}$. Taking $g = 1$ we have $|Tr(\varphi_{g_0})| < \frac{1}{3}$ and for $g = g_0$ $|1 - Tr(\varphi_{g_0}\theta_{g_0}(g_0))| < \frac{1}{3}$, so $\theta_{g_0}(g_0)$ must be different from the identity.
Let V_{g_0} be a nbhd of g_0 such that $\theta_{g_0}(g) \neq 1$ for every $g \in V_{g_0}$

Step 2: Let V be a nbhd of $1 \in G$ such that V does not contain any nontrivial subgroup (such a nbhd clearly exists as exp is a diffeo on some nbhd of 1). From compactness of G

$G = V \cup V_{g_1} \cup \dots \cup V_{g_n}$ for some $g_1, \dots, g_n$.

Take $\theta = \theta_{g_1} \oplus \dots \oplus \theta_{g_n}$. Thus $\ker \theta \subset V$ which means that $\ker \theta = \{1\}$.

We get monomorphism $\theta : G \longrightarrow GL(n)$ for some $n \in \mathbb{N}$ and as G is compact, $\theta(G)$ is closed, hence it is a closed Lie subgroup of $GL(n)$. Choosing an invariant inner product we get $\theta : G \hookrightarrow U(n)$ ∎

Compactness assumption in the above theorem is crucial.

Example: The (non-compact) Heisenberg group is not a matrix group i.e. it is not isomorphic to a closed Lie subgroup of any $GL(n, \mathbb{C})$ for any n.

Recall that the Heisenberg group is a quotient N/Z, where N are 3×3 upper triangular matrices with 1 on the diagonal and the normal subgroup Z:

$$Z = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad n \in \mathbb{Z} \right\} \leq N$$

We finish by computing the representation ring $R(T)$:
T - torus

We know that every such representation is one dimensional and can be assummed to be unitary. Hence it boils down to finding all homomorphisms $T^n \longrightarrow S^1$.

If one looks at T^n as $\mathbb{R}^n/\mathbb{Z}^n$ then every homomorphism $T^n \to S^1$ lifts to a homomorphism $\mathbb{R}^n \to \mathbb{R}$ preserving the integral lattice. This is $(x_1, x_2, \dots, x_n) \longmapsto m_1x_1 + \dots + m_nx_n \mod 1 \quad m_i \in \mathbb{Z}$

Thus the irreducible complex representations of T^n are

$$(x_1 \dots x_n) \longrightarrow \exp 2\pi i(m_1x_1 + \dots + m_nx_n).$$

If we denote by ρ_i the representation:

$$(x_1, \dots, x_n) \longrightarrow \exp 2\pi i x_i$$

Then $(x_1, \dots, x_n) \longrightarrow \exp 2\pi i(m_1x_1 + \dots + m_nx_n)$ is nothing but $\rho_1^{m_1} \dots \rho_n^{m_n}$

Thus $R(T)$ is the ring of finite Laurent series of variables $\rho_1 \dots \rho_n$. We are speaking of complex representations.

STRUCTURE OF COMPACT CONNECTED LIE GROUPS

We want to study compact Lie groups. As a general case contains the whole theory of finite groups we restrict our attention to connected compact Lie groups.
Throughout this lecture G denotes compact connected Lie group.

Proposition: A compact connected abelian Lie group of dimension n is isomorphic to a torus $(S^1)^n$

Proof: We know that $\mathbb{R}^n$ is the Lie group with trivial Lie algebra. Any Lie group with trivial Lie algebra is quotient of $\mathbb{R}^n$ by a discrete subgroup (in general lying in the centre – here $Z(\mathbb{R}^n) = \mathbb{R}^n$). So if the quotient is compact it must be torus.

Definition: A topological group is called cyclic iff there exists an element $g \in G$ such that $\overline{\langle g \rangle} = G$. Element g is called a generator.
Of course cyclic group is abelian.
Proposition: Torus is a cyclic group. Moreover the set of generators is dense in G.

If G is abelian Lie group and $G/G_0 \cong \mathbb{Z}_m$ then G is cyclic.

Proposition: If T is a torus then the group of automorphisms of T is discrete.

Proof: Any automorphism of T lifts to an automorphism of $\mathbb{R}^n$ ($n = \dim T$) preserving the integral lattice, so it is isomorphic to $GL(n, \mathbb{Z})$ which is a discrete group.

Definition: A subgroup $T \leq G$ isomorphic to a torus is called a <u>maximal torus</u> if it is maximal with respect to inclusion.

Proposition: In a group G every torus is contained in a maximal one.

Proof: This is clear: every strictly ascending sequence of tori must stabilize for dimensional reasons.

Proposition: Suppose T is a maximal torus in G and $A \leq G$ is a connected abelian subgroup such that $T \subseteq A$. Then $T = A$

Proof: We have $T \subset A \subset \bar{A}$, $\bar{A}$ is abelian connected and compact, hence $\bar{A}$ is a torus. Thus $T = A = \bar{A}$.

$\blacksquare$

Can we determine if the torus is maximal just looking at its tangent space?

Proposition: Let $H \leq G$ be a subgroup. Consider the action of H on G by inner automorphisms and derivative of this action e.g $Ad : H \to GL(\mathfrak{g})$. Then the fixed point subspace $\mathfrak{g}^H = T_1(C_G(H))$.

Proof: If $v \in T_1(C_G(H))$ then $\forall_{t \in \mathbb{R}}$ $\exp tv \in C_G(H)$ and $\forall_{h \in H}$ $h \exp tv\, h^{-1} = \exp tv$. We have

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad(h)"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "h \cdot h^{-1}"] & G \end{tikzcd}$$

so $Ad(h)(v) = id$ and $v \in \mathfrak{g}^H$. Now let $w \in \mathfrak{g}^H$, from the diagram we know that for every $t \in \mathbb{R}$ $h \exp tw\, h^{-1} = \exp tw$ hence $\exp tw \in C_G(H)$ and thus $w \in T_1(C_G(H))$.

Proposition: $T \leq G$ a torus. T is maximal iff $\mathfrak{g}^T = T_1(T)$.

Proof: It is clear that $T_1(T) \subseteq \mathfrak{g}^T$. Suppose $v \in \mathfrak{g}^T \setminus T_1(T)$.

Then elements of a one parameter subgroup $\exp tv$, $t \in \mathbb{R}$ commute with T and the group $\langle T \cup \exp tv ; t \in \mathbb{R} \rangle$ is abelian and connected which contradicts the maximality of T.

Proposition: If $T \leq G$ is a maximal torus then N^T/T is a finite group.

Proof: $\mathfrak{g}^T = T_1(T) = T_1(C_G(T))$. It follows that the identity component of $C_G(T)$ is equal to T and as $C_G(T)$ is compact $C_G(T)/T$ is finite. The group $N_G(T)/C_G(T)$ (it is evident that $C_G(T) \trianglelefteq N_G(T)$) is a subgroup of $Aut(T)$ which is discrete. Thus $N_G(T)/C_G(T)$ is finite.

$$N_G(T)/T = {}^{N_G(T)/C_G(T)}\Big/_{C_G(T)/T}$$ is therefore finite.

∎

We are now ready to prove the main theorem on maximal tori. Before that examples:

Example: • Let $G = U(n)$. Let $T = \left\{ \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix} \quad |z_i| = 1 \right\}$ – it is a maximal torus.
If $A \in U(n)$ then we know that A is diagonalizable, so there exists $B \in U(n)$ such that $BAB^{-1} \in T$, i.e $A \in B^{-1}TB$.

• $G = SO(3)$ $\quad T = \begin{bmatrix} \cos\alpha & \sin\alpha & \\ -\sin\alpha & \cos\alpha & \\ & & 1 \end{bmatrix}$ is a maximal torus. We know, that every isometry is a rotation in a plane perpendicular to some line, which boils down to saying that for every $A \in SO(3)$ there exists $B \in SO(3)$ such that $BAB^{-1} \in T$.

This generalises to the theorem:

> **Theorem:** If G is a connected compact group and $T \leq G$ a maximal torus, then for every $g \in G$ there exists $x \in G$ such that $g \in xTx^{-1}$.

Corollary: If $T' \leq G$ is a torus and $T \leq G$ a maximal torus then $T' \leq xTx^{-1}$ for some $x \in G$. If $T', T \leq G$ are maximal tori then they are conjugate.

Proof: Relies on the fact that torus is a cyclic group and it suffices to know that the generator of T' is contained in some conjugate of the maximal torus.

The formulation of the theorem resembles the formulation of Sylow's theorem. Once the existence of Sylow p-subgroup P in G was proven, to prove that every p-subgroup P' is contained in some conjugate of P we had to prove that the action of P' on G/P has fixed points. This was done counting points and using the fact that the member of points and number of fixed points are the same modulo p.

Here, similarly to prove that there exists $x \in G$ such that $g \in xTx^{-1}$ is equivalent to proving that the map $G/T \xrightarrow{\cdot g} G/T \quad xT \to gxT$ has a fixed point. This is clear: $gxT = xT \iff x^{-1}gx \in T \iff g \in xTx^{-1}$

To determine if the map has fixed points we have the following

Lefshetz fixed point theorem: Suppose that X is a finite CW complex (ex. compact manifold or finite simplicial complex) and $f: X \to X$ a map. The Lefshetz number of f is:

$$L(f) = \sum_{i=0}^{\infty} (-1)^i \operatorname{Tr}(f_i)_*$$

Where $(f_i)_* : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})$.

If f does not have a fixed point then $\Lambda(f) = 0$

Lefshetz number is an invariant of a homotopy class of $f: X \to X$.
If $f = id_X$ then $\Lambda(f) = \chi(X) = \sum_{i=0}^{\infty} \operatorname{rank} H_i(X, \mathbb{Q})$ is the Euler characteristic.

Side remark: If G is a compact Lie group, Then $\chi(G) \neq 0$ if and only if G is finite.
Pf. If G is finite then $\chi(G) = |G|$. If not then the multiplication by an element different from 1 belonging to the component of identity has no fixed points, so the Lefshetz number is 0. But this map is homotopic to the identity so the Lefshetz number is the Euler characteristic.

Similarly we prove that

Proposition: If a torus T acts on a finite CW complex X without fixed points, then $\chi(X)=0$

Pf: The set of fixed points X^T is equal to the set of fixed points of the generator of T, But this map is homotopic to the identity and the Lefshetz number is 0 and is the Euler characteristic.

One may look at the Euler characteristic as an analogue of counting points. It has a similar additivity property: If $X = Y \cup Z$ Y, Z subcomplexes then

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

The analogue of the theorem for p-group actions on finite sets is the following theorem:

Theorem: Suppose M is a compact smooth manifold on which the torus T acts smoothly. Then $\chi(M) = \chi(M^T)$

Sketch of proof: Take M^T. There is a closed tubular nbhd of M^T which is invariant under T. Call it U. Let $W = M \setminus \operatorname{int} U$.

$\chi(M) = \chi(W) + \chi(U) - \chi(U \cap W)$

From previous $\chi(W) = \chi(U \cap W) = 0$ as on these sets the action is fixed point free. The tubular nbhd is homotopy equivalent to M^T, so $\chi(U) = \chi(M^T)$.

Proof of the theorem: We must prove that $G/T \xrightarrow{\cdot g} G/T$ has a fixed point. The map is homotopic to the identity so the Lefshetz number is equal to the Euler characteristic $\chi(G/T)$. So we must prove that $\chi(G/T) \neq 0$.

But the maximal torus T also acts on G/T, and $\chi(G/T) = \chi((G/T)^T) =$
$= \chi(NT/T) \neq 0$

∎

Definition: The rank of a compact connected Lie group G is the dimension of the maximal torus of G and denoted $\operatorname{rk} G$.

Assume G compact connected, $T \leq G$ maximal torus.

Corollary: If $H \leq G$ is a cyclic group, then for some $x \in G$, $H \leq xTx^{-1}$.

Pf: If $H = \overline{\langle g \rangle}$ then $g \in xTx^{-1}$ hence $\overline{\langle g \rangle} \leq xTx^{-1}$.

$\boxed{\text{Corollary: } C_G(T) = T}$

Pf: Let $g \in C_G(T)$. Let $\overline{\langle g \cup T \rangle} = H$. Then $H/T \leq C_G(T)/T \leq N_G(T)/T < \infty$

So H is cyclic, and hence H is contained in some maximal torus T'.

But $T \subseteq H \subseteq T'$ thus $T = H = T'$ and $g \in T$.

Corollary: Every maximal torus of G contains $Z(G)$ the center of G.

Pf: For arbitrary T, $Z(G) \subseteq C_G(T) = T$

Definition: Let T be a maximal torus. Then the finite group $NT/T \subset \mathrm{Aut}\, T$ and is called a Weyl group, denoted W. The isomorphism class of this group is independent of the choice of T.

Proposition: Let $t, t' \in T$ be elements conjugate in G. Then there exists $w \in W$ such that $w(t) = t'$

Proof: Let $g \in G$ be such that $gtg^{-1} = t'$. We have $T \subset C_G(t)$, $T \subset C_G(t')$ and also $gTg^{-1} \subset C_G(t')$, which is an easy computation. So there is $x \in C_G(t')$ such that $xgTg^{-1}x^{-1} = T$ so $xg \in W$ and $(xg)t(g^{-1}x^{-1}) = xt'x^{-1} = t'$

▨

So we see that conjugacy classes of elements are described by action of the Weyl group on maximal torus of G.

Let us look closer at centralisers of elements:

Corollary: For arbitrary $g \in G$ the identity component of the centralizer $C_G(g)_1$ is the union of all maximal tori containing g.

Pf. Every maximal torus containing g is a subgroup of $C_G(g)_1$, so $\mathrm{rk}\, C(g)_1 = \mathrm{rk}\, G$. For all $x \in C_G(g)_1$ there exists a maximal torus T of $C_G(g)_1$ such that $x \in T$. From the theorem on maximal torus and $C_G(g)_1$ there exists $y \in C_G(g)_1$ for which $ygy^{-1} = g \in T$. So $x, g \in T$. From equivalence $\mathrm{rk}\, C(g)_1 = \mathrm{rk}\, G$, T is a maximal torus of G.

Based on considerations above we introduce the following definition:

Definition: For all elements $g \in G$:

g is regular $\Longleftrightarrow$ g lies in a unique maximal torus $\Longleftrightarrow$
$\Longleftrightarrow \dim C_G(g) = \mathrm{rk}\, G$

g is singular $\Longleftrightarrow$ g lies in more than one maximal torus $\Longleftrightarrow$
$\Longleftrightarrow \dim C_G(g) > \mathrm{rk}\, G$