REPRESENTATION THEORY

Let G be a Lie group.

> **Definition**: A representation of the group G is a Lie group homomorphism $\rho: G \to GL(V)$ where V is a finite dimensional vector space over $\mathbb{R}$ or over $\mathbb{C}$. We speak then of real or complex representations.

The category $\mathrm{Rep}_{\square}(G)$ $\quad \square = \mathbb{R}$ or $\mathbb{C}$

Objects: representations of G

Morphisms: Let $\rho: G \to GL(V)$ and $\rho': G \to GL(W)$ be representations. A morphism is a linear map $\varphi: V \to W$ such that for every $g \in G$ the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

We say that the linear map φ is equivariant.

Definition: Let $\mathfrak{g}$ be a Lie algebra. A representation of $\mathfrak{g}$ is a Lie algebra homomorphism $\rho_*: \mathfrak{g} \to \mathfrak{gl}(V)$ where V is a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$.

We define the category $\mathrm{Rep}_{\square}(\mathfrak{g})$ in a similar way.

A morphism between representations is a linear map $\varphi: V \to W$ such that for every $x \in \mathfrak{g}$ the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho_*(x)"'] & W \arrow[d, "\rho'_*(x)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

Theorem: If G is a Lie group with tangent Lie algebra $\mathfrak{g}$.

(a) There is a functor $\mathrm{Rep}_G \longmapsto \mathrm{Rep}_{\mathfrak{g}}$ taking $\rho \dashrightarrow \rho_*$

(b) If G is connected simply connected then this functor is an equivalence.

In case G is connected but not simply connected consider $\tilde{G} \xrightarrow{p} G$ universal covering. $\ker p$ is discrete & central, so it is enough to determine representations of $\tilde{G}$ trivial on $\ker p$.

From the Lie theory we obtain the following:

Theorem: Let G be a Lie group and $\mathfrak{g}$ it's Lie algebra. There is a functor

$$\mathrm{Rep}_\square(G) \longrightarrow \mathrm{Rep}_\square(\mathfrak{g})$$

which assigns to every representation $\rho: G \to GL(V)$ its derivative $\rho_*: \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. This functor induces identity $\mathrm{Mor}_{\mathrm{Rep}_\square(G)}(\rho, \rho') = \mathrm{Mor}_{\mathrm{Rep}_\square(\mathfrak{g})}(\rho_*, \rho'_*)$

If G is a connected, simply connected group, then this functor is an equivalence of categories. (even more - bijection on objects)

Example: $\mathrm{Ad}: G \to GL(\mathfrak{g}) \rightsquigarrow \mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

OPERATIONS ON REPRESENTATIONS

Consider $\mathrm{Rep}_\square(G)$. Basically all functorial constructions on the category of vector spaces carry over to $\mathrm{Rep}_\square(G)$, and all functorial isomorphisms too.

Suppose $\rho: G \to GL(V)$ and $\rho': G \to GL(W)$ are representations.

Direct sum: $\rho \oplus \rho' : G \longrightarrow GL(V \oplus W)$ $\quad (\rho \oplus \rho')(g) = \rho(g) \oplus \rho'(g)$

Hom $\quad \mathrm{Hom}(\rho, \rho'): G \longrightarrow GL(\mathrm{Hom}(V, W))$

Let $\varphi \in \mathrm{Hom}(V, W)$. Consider the diagram:

$$\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\rho'(g)"] \\
V \arrow[r, "\rho'(g)\varphi\rho(g)^{-1}"] & W
\end{tikzcd}$$

and we set: $\mathrm{Hom}(\rho, \rho')(g)(\varphi) = \rho'(g) \circ \varphi \circ \rho(g^{-1})$

Tensor product: $\rho\otimes\rho' : G \to GL(V\otimes W)$ $\quad \rho\otimes\rho'(g)(v\otimes w) = \rho(g)(v)\otimes\rho'(g)(w)$

Subrepresentation: If $V' \subseteq V$ is a vector subspace which is invariant e.g $\forall_{g\in G}\ \rho(g)(V')\subseteq V'$ Then $\rho_{|V'} : G \to GL(V')$ is a subrepresentation of V.

Quotient representation: If $\rho_{|V'}$ is a subrepresentation of ρ, then

$\rho_{V/V'} : G \to GL(V/V')$ $\quad \rho_{V/V'}(g)$ is the isomorphism of V/V' induced by $\rho(g)$.

The same constructions apply to representations of the Lie algebra $\mathfrak{g}$. One has to be prudent with the definition of the tensor product of representations $\rho_*: \mathfrak{g}\to \mathfrak{gl}(V)$ and $\rho'_*: \mathfrak{g}\to\mathfrak{gl}(W)$.
The tensor product $\rho_*\otimes\rho'_*$ should be the derivative of the tensor product $\rho\otimes\rho'$, e.g $\rho_*\otimes\rho'_* = (\rho\otimes\rho')_* : \mathfrak{g}\to\mathfrak{gl}(V\otimes W)$

Let $X\in\mathfrak{g}$ be a vector and $\gamma: \mathbb{R}\to G$ one parameter subgroup corresponding to it. What should be the value of $(\rho\otimes\rho')_*(X)$?

$$(\rho\otimes\rho')_*(X)(v\otimes w) = \frac{d}{dt}\Big|_0 (\rho\otimes\rho')(\gamma(t))(v\otimes w) = \frac{d}{dt}\Big|_0 \rho(\gamma(t))(v)\otimes\rho'(\gamma(t))(w) =$$

$$= \frac{d}{dt}\Big|_{t=0} \rho(\gamma(t))(v)\otimes w + v\otimes\frac{d}{dt}\rho'(\gamma(t))(w) = \rho_*(X)(v)\otimes w + v\otimes\rho'_*(X)(w)$$

NIEZMIENNIKI:

If $\rho: G\to GL(V)$ is a representation then $V^G = \{v\in V: \forall_{g\in G}\ \rho(g)(v) = v\}$ is a subrepresentation. Analogously $V^{\mathfrak{g}} = \{v\in V\ \forall_{X\in\mathfrak{g}}\ \rho_*(X)(v) = v\}$ is a subrepresentation.
For connected Lie groups obviously $V^G = V^{\mathfrak{g}}$.

Remark: Recalling the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\rho'(g)\varphi\rho(g)^{-1}"] & W \end{tikzcd}$$

we see that $\mathrm{Hom}(\rho,\rho')^G = \mathrm{Hom}_G(V,W)$ where $\mathrm{Hom}_G(V,W)$ denotes equivariant maps between G-spaces V and W.

IRREDUCIBLE REPRESENTATIONS

Definition: A representation $\rho: G \to GL(V)$ is called irreducible iff it is non-zero and does not have a subrepresentation other then 0 or V.

Definition: A representation is called completely reducible or semisimple iff it is isomorphic to a direct sum of irreducible representations.

Notation: For simplicity of notation instead of writing $\rho: G \to GL(V)$ is a representation we will say that V is a representation, The same for direct sums & tensor products. Instead of writing $\rho(g)(v)$ we will simply write $g(v)$.

Examples: ① $\mathbb{R}^n$ with a natural action of $SO(n)$ is an irreducible representation.

② $\mathbb{C}$ with natural action of S^1 is irreducible. But $\mathbb{C}^2$ with action $z(z_1,z_2) = (zz_1, zz_2)$ is completely reducible. Any $V_1 \oplus V_2 = \mathbb{C}^2$ is a direct sum of irreducible subrepresentations.

③ Not every reducible representation is completely reducible. Take matrix $A \in M_{n\times n}(\mathbb{R})$ and assume it has $\lambda \in \mathbb{R}$ as an eigenvalue. Consider representation of a Lie algebra $\mathbb{R}$ on the Lie algebra $\mathbb{R}^n$ given by $t \to tA$. The eigenspace of the value λ is of course a subrepresentation. This representation is completely reducible iff A is diagonalisable. If one wants representation of the group one has to consider $t \to \exp tA$,

Before we investigate when representations of a given group are completely reducible, why this is important and why it is more convenient to consider complex representations.

Theorem (Schur's lemma): Let V and W be irreducible representations of the group G.

(a) If $\varphi : V \to W$ is a G-map (e.g. $\varphi(g(v)) = g(\varphi(v))$) then either $\varphi = 0$ or φ is an isomorphism

(b) If V is a complex representation and $\varphi : V \to V$ is a G-map then $\varphi = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{C}$.

Proof: (a) both $\ker\varphi$ and $\operatorname{im}\varphi$ are subrepresentations of V and W respectively.
Thus $\ker\varphi = V$ and φ is a zero map or $\ker\varphi = \{0\}$ and φ is a monomorphism. Then $\operatorname{im}\varphi \leq W$ is a non-zero subrepresentation, so $\operatorname{im}\varphi = W$.

(b) As V is a vector space over complex numbers there exists $\lambda \in \mathbb{C}$ which is an eigenvalue of φ. Then $\varphi - \lambda \cdot \mathrm{Id} : V \to V$ is still a G map with non-trivial kernel. From (a) we know that $\varphi - \lambda \,\mathrm{Id} = 0$, hence $\varphi = \lambda \cdot \mathrm{Id}$.

∎

The same assertion holds for representations of the Lie algebra $\mathfrak{g}$.
Schur lemma has important consequences:

Corollary: If G is a commutative group then every irreducible complex representation of G is one dimensional.

Proof: If G is commutative, then for every $g \in G$, the map defined by $g : V \to V$ is equivariant, so this map must be of the form $\lambda_g \cdot \mathrm{Id}_V$ for some $\lambda_g \in \mathbb{C}$. But in this case every one dimensional subspace of V is a subrepresentation. Hence $\dim V = 1$

Corollary: Let V be a completely reducible complex representation of G (or of $\mathfrak{g}$). Then
(a) If $V = \bigoplus V_i$, V_i irreducible, pairwise non-isomorphic then every Gmap of V is of the form $\bigoplus \lambda_i \mathrm{Id}_{V_i}$, where $\lambda_i \in \mathbb{C}$
(b) If $V = \bigoplus n_i V_i = \bigoplus \mathbb{C}^{n_i} \otimes V_i$, V_i irreducible, pairwise non-isomorphic then every Gmap of V is of the form $\bigoplus \varphi_i \otimes \mathrm{Id}_{V_i}$ where $\varphi_i : \mathbb{C}^{n_i} \to \mathbb{C}^{n_i}$ is a linear map.

Proof is left as an excersise.

Theorem: If $\bigoplus_i m_i V_i$ is equivalent to $\bigoplus_i n_i V_i$ then $m_i = n_i$.

Pf: Then $\mathrm{Hom}_G(V_j, \bigoplus m_i V_i) \cong \mathrm{Hom}_G(V_j, \bigoplus_i n_i V_i)$
which means that

$$\bigoplus_i m_i \,\mathrm{Hom}_G(V_j, V_i) \cong \bigoplus_i n_i \,\mathrm{Hom}_G(V_j, V_i)$$

If $i \neq j$ then $\mathrm{Hom}_G(V_i, V_j)$ is trivial, so

$$m_j \mathrm{Hom}_G(V_j, V_j) \cong n_j \mathrm{Hom}_G(V_j, V_j)$$

taking dimensions on both sides we get $m_j = n_j$.

Proposition: Assume that G is such that every representation is completely reducible. Let $\{V_i\}_{i\in I}$ be the collection of all irreducible complex representations of the group G. Then

$$\mu : \bigoplus_{i\in I} \left(\mathrm{Hom}_G(V_i, V) \otimes_{\mathbb{C}} V_i\right) \longrightarrow V \quad \text{is an isomorphism}$$

of representations of G (where G acts trivially on $\mathrm{Hom}_G(V_i, V)$) and μ is given by the formule

$$\mu(f_i \otimes v_i) = f_i(v_i)$$

Pf: Remark first that by the Schur lemma the sum on the left is finite. The assertion is trivial for V irreducible and as each representation by assuption is completely reducible the thesis follows.

As the example of a trivial representation shows the invariant subspaces that are irreducible summands are not unique. However certain summands are unique:

Let us go back to the isomorphism μ. Let V_i be an irreducible representation. The image $\mu(\mathrm{Hom}_G(V_i, V) \otimes_{\mathbb{C}} V_i)$ is called V_i isotipical summand of V and is determined uniquely. It is isomorphic to the sum $\underbrace{V_i \oplus \cdots \oplus V_i}_{n_i}$.

We face obvious questions:

1) What type of groups have the property that every representation is completely reducible?
2) How to determine if the given representation is irreducible?
3) How to classify complex irreducible representations up to equivalence?
4) What is the relation between complex and real representations?

Let's start with the first problem. Recall that for a linear map $\varphi \in U(n)$ of $\mathbb{C}^n$ and $V \subseteq \mathbb{C}^n$ invariant i.e $\varphi(V) \subseteq V$ the orthogonal complement $V^{\perp}$ is also invariant. This is the clue.

Df: A complex representation $\rho: G \to GL(V)$ is called unitary if there is a Hermitian positive definite form which is G-invariant i.e for every $g \in G$, $\langle gv, gw\rangle = \langle v, w\rangle$. Using diagrams:

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr] & GL(V) \\ & U(V) \arrow[u, hook] \end{tikzcd}$$

Exactly the same definition for real representations and scalar product.

Positive definite Hermitian form over $\mathbb{C}$ and scalar product over $\mathbb{R}$ will be also called inner product.

Theorem: Every unitary representation is completely reducible.

Proof: Let V be a unitary representation and $W \subset V$ is a subrepresentation, then $W^{\perp}$ is also a subrepresentation and $V = W \oplus W^{\perp}$.

Question: When for a given representation there is an invariant inner product?

Always for finite groups: G finite $\langle \, , \, \rangle$ inner product then

$$\langle v, w\rangle' = \frac{1}{|G|} \sum_{g \in G} \langle gv, gw\rangle$$

For a Lie group the sum must be replaced by integration.

Theorem: If G is compact topological group, then there exists a continuous linear map

$$\int_G : C(G,\mathbb{R}) \longrightarrow \mathbb{R}$$

such that:

(1) it is a positive linear functional

(2) $\int_G 1 = 1$

(3) $\int_{x\in G} f(gx) = \int_{x\in G} f(x)$ $\qquad \int_{x\in G} f(xg) = \int_{x\in G} f(x)$

We can integrate maps taking values in vector spaces. The integral commutes with linear maps.

This theorem is much easier for Lie groups because we have the smooth structure at our disposal, and here is the proof in this case

Proof:

Take $0 \neq \omega \in \Lambda^n \mathfrak{g}^*$, propagate it by right translation to get a diff form $\tilde{\omega}$ G orientable

For compact G, $I = \int_G \tilde{\omega}$ is finite. Take $\omega = \frac{\tilde{\omega}}{I}$, so $\int \omega = 1$

This form is also left invariant. (proof in Kirilov book)

NOW WE ASSUME THAT G IS COMPACT

Proposition: If G is compact then for every representation V of G there exists an invariant inner product on V.

Pf: Take $\langle\, ,\rangle'$ arbitrary inner product on V. Then

$$\langle v, w\rangle = \int_G \langle gv, gw\rangle\, dg$$ is an invariant inner product.

Thm: Every representation of a compact Lie group is completely reducible.

Existence of invariant inner product has one more consequence.

If V is a complex vector space, by $\bar{V}$ we denote a complex vector space with the same addition and with scalar multiplication given by $z \cdot v =: \bar{z}v$

scalar multiplication in V.

An inner product gives us an isomorphism $\bar{V} \simeq V^*$

Proposition; Let $\theta : G \to GL(V)$ be a representation.

Then $f = \int_G \theta \, dg \in Hom(V, V)$ is an idempotent (i.e $f^2 = f$)

and $\operatorname{im} f = V^G$.

Pf: Evaluation is linear (in $\mathbb{R}$), so $f(v) = \int_G \theta(g)(v)$

And $\theta(g')f(v) = \theta(g') \int_G \theta(g)(v) = \int_G \theta(g')(\theta(g)(v)) = \int_G \theta(g'g)(v) = f(V)$

linearity of the integral   left invariance

So $f(v) \in V^G$.

If $v \in V^G$ to $f(v) = \int_G \theta(g)(v) = \int_G v = v$. which proves $f^2 = f$ ∎

CHARACTERS OF REPRESENTATION

So far although we know that every representation is completely reducible we have no means to determine which irreducible summands are its components. So we look at numerical invariants of endomorphisms.

Trace: $Hom(V, V) \cong Hom(V, V^{**}) \cong Hom(V, Hom(V^*, K)) \cong Hom(V \otimes V^*, K)$

take $V \otimes V^* \to K$ $\quad v \otimes \varphi = \varphi(v)$. This is trace, denoted by tr.

Definition. Let $\rho : G \to GL(V)$ be a complex representation. Then its character $\chi_V : G \to \mathbb{C}$ is a function:

$$\chi_V(g) = \operatorname{tr} \rho(g)$$