Recall for exp the already proved property:

$$\boxed{\underbrace{\exp v \exp w \exp(-v) \exp(-w)}_{\text{commutator of elements } \exp(v)\, \exp(w)} = \exp([v,w] + \text{terms of degree} \geq 3)}$$

We will change the notation. A Lie algebra of a Lie group G will be denoted by gothic letter $\mathfrak{g}$.
Let us sum up what we have up to now:

$$\boxed{\begin{array}{l}\text{Connected}\\ \text{simply connected}\\ \text{Lie groups}\end{array}} \longrightarrow \boxed{\begin{array}{l}\text{Lie}\\ \text{algebras}\\ \text{(finite dimensional)}\\ \text{over } \mathbb{R}\end{array}}$$

$$G \longrightarrow \mathfrak{g}$$

We know that it is a functor and we know that this functor is injective on sets on morphisms. Our goal is to prove that this is an equivalence of categories. So what lacks is:

- If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of Lie groups G and H respectively and $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras then there exists a Lie group homomorphism $f : G \to H$ such that $\varphi = f_*$.

•• Every Lie algebra is isomorphic to a Lie algebra of a Lie group.

Let us clear one doubt: locally around 1 of the group, where the Taylor expansion is convergent the multiplication rule is determined by the whole series, not just by first and second order terms. The answer lies in the following theorem called Baker-Campbell-Hausdorff formula

Theorem: If G is a connected group and $\mathfrak{g}$ its Lie algebra, then in a nbhd of 0, small enough we have:

$$\exp v \cdot \exp w = \exp\left(v + w + \sum_{n=2}^{\infty} \mu_n(v, w) \cdot\right) \text{ where}$$

μ_n are Lie polynomials in variables v and w.

$\mu_2 = \frac{1}{2}[v, w]$, $\mu_3 = \frac{1}{12}\left([v,[v,w]] + [w\,[w,v]]\right)$

Let us go back to functoriality and consider automorphisms o the group G:

An inner automorphism defined by an element $g \in G$ also defines an automorphism of Lie algebras $Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g}$.

This automorphism is an automorphism of Lie algebras, hence

$$\forall_{v, w \in \mathfrak{g}} \ \forall_{g \in G} \quad Ad_g[v, w] = [Ad_g v, Ad_g w]$$

We will need a lemma

Lemma: If $\mathfrak{g}$ is a Lie algebra of G, then for $v, w \in \mathfrak{g}$

$$\exp v \exp w \exp(v)^{-1} = \exp\left(w + [v, w] + \dots\right)$$

Proof: $(\exp v)^{-1} = \exp(-v)$

$$\exp v \exp w \exp(-v) = \exp v \exp(w - v + \tfrac{1}{2}[w, -v] + \dots) = \exp v \exp(w - v + \tfrac{1}{2}[v, w] + \dots$$

$$= \exp\left(v + w - v + \tfrac{1}{2}[v, w] + \tfrac{1}{2}[v, w - v + \tfrac{1}{2}[w, -v]] + \dots\right) =$$

$$= \exp(w + [v, w] + \tfrac{1}{2}\underbrace{[v, -v]}_{0} + \underbrace{\tfrac{1}{2}[v, [w - v]] + \dots}_{\text{higher order}} =$$

$$= \exp(w + [v, w] + \dots)$$

Ad defines an action of G on $\mathfrak{g}$ by linear maps, so Lie group homom.

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$$

taking its derivative and denoting it by ad we get a linear map and Lie algebra homomorphism.

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

We have:

Lemma: (a) $\forall_{v \in \mathfrak{g}}$ $\quad \mathrm{Ad}_{(\exp v)} = \exp(\mathrm{ad}_v)$

(b) $\forall_{v \in \mathfrak{g}} \; \forall_{w \in \mathfrak{g}}$ $\quad \mathrm{ad}_v(w) = [v, w]$

Proof: Part (a) follows from naturality of exp.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp"] \\
G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g})
\end{tikzcd}$$

(b) In the previous diagram for $v \in \mathfrak{g}$ consider $\exp \mathrm{ad}_v \in GL(\mathfrak{g})$ and evaluate it on $w \in \mathfrak{g}$. We get

$$\exp(\mathrm{ad}_v)(w) = \left(1 + \mathrm{ad}_v + \frac{\mathrm{ad}_v^2}{2} + \dots\right)(w) = \left(w + \mathrm{ad}_v(w) + \overbrace{\dots}^{\text{terms} \geq 2}\right)$$

this is equal to $(\mathrm{Ad}_{\exp(v)})(w)$, so let us look at $\mathrm{Ad}_{\exp v}$

$\exp v \in G$. We have the commutative diagram:

$$\begin{tikzcd} w \in \mathfrak{g} \arrow[r, "Ad_{\exp v}"] \arrow[d, "\exp"] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\exp v \cdot (\exp v)^{-1}"] & G \end{tikzcd}$$

$$\exp Ad_{\exp v}(w) = \exp v \cdot \exp w \cdot (\exp v)^{-1} = \exp(w + [v, w] + \dots)$$

but exp is 1-1 close to 0, so in the nbhd of 0 we have

$$Ad_{\exp_v}(w) = w + [v, w] + \dots$$

both maps are linear, so the equality holds everywhere.

Thus we have:

$$(\exp ad_v)(w) = w + ad_v[w] + \dots =$$

$$= v + [v, w] + \dots$$

this proves the theorem.

From the formulla

$$\exp v \exp w \exp v^{-1} \exp w^{-1} = \exp([v,w] + \text{higher terms})$$

we infer that if G is commutative then the Lie bracket is equal to 0.

Proposition: If G is commutative then $\mathfrak{g}$ is a trivial Lie algebra

We will prove the converse, which can be regarded as B-C-H formula for the special case of abelian groups.

Theorem. If G is a connected Lie group with trivial Lie algebra $\mathfrak{g}$, then G is commutative.

The theorem follows from the proposition:

Proposition: If for every $v, w \in \mathfrak{g}$ $[v,w]=0$, then
$\exp v \exp w = \exp(v+w) = \exp w \cdot \exp v$

Proof: Recall that $\mathrm{ad}_v(w) = [v,w]$. Also

$$\mathrm{Ad}\, \exp(v) = \exp(\mathrm{ad}_v),$$ so for every $v \in \mathfrak{g}$

$$\mathrm{Ad}(\exp v) = \mathrm{Id}$$ for every $v \in \mathfrak{g}$.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad_{\exp v}"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\exp v \cdot \exp v^{-1}"] & G \end{tikzcd}$$

So for every $w \in \mathfrak{g}$

$$\exp v \exp w (\exp v)^{-1} = \exp w$$

Which means that $\exp v \cdot \exp w = \exp w \cdot \exp v$

Now we prove that $\exp(v+w) = \exp \cdot \exp w$

Let $\theta(t) = \exp tv \cdot \exp tw$. This is one parameter subgroup:

$$\theta(t+s) = \exp(t+s)v \, \exp(t+s)w = \exp tv \exp sv \exp tw \exp sw =$$

$$= \exp tv \exp tw \exp sv \exp sw = \theta(t) \cdot \theta(s)$$

Taking the differential at $t=0$ we get $v+w$, hence by uniqueness of one parameter subgroups $\exp tv \cdot \exp tw = \exp(t(v+w))$ ∎

On the basis of the above theorem we can prove:

Theorem: If G is an n-dimensional connected abelian group, then $G \cong \underbrace{S^1 \times .. \times S^1}_{k} \times \mathbb{R}^{n-k}$.

Proof: Consider $\exp : \mathfrak{g} \to G$. From the previous theorem we know that exp is homomorphism of Lie groups, which is an epimorphism since some nbhd of $1 \in G$ is contained in the image. Homomorphism exp is a covering (prove!) so $\ker \exp$ is a discrete subgroup of $\mathfrak{g} = \mathbb{R}^n$. Such a discrete subgroup is a lattice i.e. is generated by linearly independent vectors $\alpha_1 \dots \alpha_k$. Coset space is $\mathbb{R}^n / \langle \alpha_1 .. \alpha_k \rangle = (S^1)^k \times \mathbb{R}^{n-k}$ which is a manifold and a Lie group G.

we will use exponential map to prove Cartan's theorem.

Theorem: If G is a connected Lie group and $H \leq G$ a subgroup, which is a closed subset then H is a submanifold. (i.e., H is a closed Lie subgroup)

(proof after Adam's book)

Recall that a Lie subgroup is an immersion: $H \overset{i}{\hookrightarrow} G$ which is a homomorphism of Lie subgroups. Induced map $i_*: \mathfrak{h} \longrightarrow \mathfrak{g}$ is a monomorphism, since it is a monomorphism in the nbhd of 0. So $i_* \mathfrak{h} \leq \mathfrak{g}$ is a Lie subalgebra.

For clarity of the proof we start with two technical lemmas:

Lemma: G connected Lie group, $\mathfrak{g} = V \oplus W$ as a vector space. Then

$\varphi: \mathfrak{g} \longrightarrow G$ $\quad f(v, w) = \exp v \cdot \exp w \quad v \in V, w \in W$ is a diffeomorphism of some nbhd of 0 to some nbhd of 1.

Pf.: A map

$$\begin{array}{ccccc} \mathfrak{g} & \longrightarrow & G \times G & \xrightarrow{\mu} & G \\ (v, w) & \longrightarrow & (\exp v, \exp w) & \longrightarrow & \exp v \cdot \exp w \end{array}$$

is smooth as a composition of smooth maps. Its differential at 0 restricted to each subspace V & W is identity, so it is identity, thus diffeo on small nbhd.

Lemma: Let $H \leq G$ be a subgroup which is a closed subset in G. Fix a scalar product on $\mathfrak{g}$. Suppose we are given a sequence $v_n \in \mathfrak{g}$, $n \in \mathbb{N}$ such that $\exp v_n \in H$, $v_n \longrightarrow 0$ and $\frac{v_n}{|v_n|} \rightarrow v \in \mathfrak{g}$.
Then $\exp(tv) \in H$ for every $t \in \mathbb{R}$.

Pf: Let $t \in \mathbb{R}$ be arbitrary. As $|v_n| \longrightarrow 0$, then one can choose integers $m_n \in \mathbb{Z}, n \in \mathbb{N}$ such that $m_n |v_n| \longrightarrow t$. Then $m_n v_n = m_n |v_n| \cdot \frac{v_n}{|v_n|} \rightarrow tv$. Then $\exp(m_n v_n) \longrightarrow \exp(tv)$.
But $\exp(m_n v_n) = (\exp v_n)^{m_n} \in H$. So $\exp(tv) \in H$ because H is a closed subset.

Proof of the theorem:

Step 1. Candidate for the tangent space to H at 1:

Let $W = \{ v \in \mathfrak{g} : \forall_{t \in \mathbb{R}} \exp tv \in H \}$

We must prove that W is a linear subspace of $\mathfrak{g}$. Subset W is by definition closed under scalar multiplication, we must prove it is closed under additio

For $v, w \in W$ we must show that $\exp(v+w) \in H$. Suppose $v+w \neq 0$ (for $v=-w$ clear)

For small $t \in \mathbb{R}$ $\exp(tv+tw)$ can be approximated by $\exp tv \cdot \exp tw$ as we have the formula

$$\exp tv \cdot \exp tw = \exp(tv + tw + \text{terms of order} \geq 2)$$

As H is a subroup $\exp tv \cdot \exp tw \in H$. Let $f : (-\varepsilon, \varepsilon) \longrightarrow \mathfrak{g}$ be given by

$f(t) = \log \exp(tv + tw + \text{terms of order} \geq 2)$, $\frac{f(t)}{t} \xrightarrow[t \to 0]{} v+w$.

Let $v_n = f(\frac{1}{n})$ for $n \geq N_0$. Then: $v_n \xrightarrow[n \to \infty]{} 0$, $\exp v_n \in H$, $\frac{v_n}{|v_n|} = \frac{f(\frac{1}{n})}{\frac{1}{n}} \cdot \frac{\frac{1}{n}}{|f(\frac{1}{n})|} \longrightarrow \frac{v+w}{|v+w|}$

By the lemma, $\frac{v+w}{|v+w|} \in W$, so $v+w \in W$

Step 2: Construct a map around $1 \in H$. Take W' such that $W \oplus W' = \mathfrak{g}$, and consider

$\varphi : W \oplus W' \longrightarrow G$ $\varphi(w, w') = \exp w \cdot \exp w'$. By the lemma this is a diffeomorphism in some nbhd of 0. To show that this is a good chart for H as a submanifold of G. For this we must show that $\varphi(U \cap W)$ is a nbhd of $1 \in H$ for sufficiently small $U \ni 0$. By definition of W, $\varphi(U \cap W) \subset H$. Suppose on the contrary that there exists a sequence $(w_n, w_n') \longrightarrow 0$, $w_n' \neq 0$ such that $\varphi(w_n, w_n') \in H$. As $\exp w_n \in H$, also $\exp w_n' \in H$.

By compactness of the sphere in W' we can choose a subsequence w_{n_k}' such that $\frac{w_{n_k}'}{|w_{n_k}'|} \longrightarrow w' \in W'$. From the lemma it follows that $\exp tw \in H$ for every $t \in \mathbb{R}$ which means that $w' \in W$. Contradiction.

Step 3: To get a neighbourhood around any other point of H use left translation.

Corollary: If $f: G \to H$ is a homomorphism of Lie groups then $\ker f \leq G$ is a closed Lie subgroup.

Remark: We will devote next exercise session to Lie group actions, proving that if $H \leq G$ is a closed Lie subgroup then G/H is a manifold so if H is normal it is a quotient Lie group.

Corollary: If G, H are Lie groups and $f: G \to H$ is a group homomorphism which is a continuous map then $f: G \to H$ is a Lie group homomorphism (i.e. f is smooth).

Pf: Let $\Gamma_f \subset G \times H$ be a graph of f. It is a subgroup which is a closed subset, hence Γ_f is a submanifold and $p: \Gamma_f \hookrightarrow G \times H \xrightarrow{\pi_G} G$ is a smooth map. Map p is bijective and its differential at $(1_G, 1_H)$ is an isomorphism. Hence p is a local diffeomorphism around $(1_G, 1_H)$. By translation it is local diffeomorphism everywhere, hence global diffeomorphism. So $f = \pi_H p^{-1}$ is smooth.

Corollary: All subgroups of $GL(n, \mathbb{R})$ considered so far ($O(n)$, $SO(n)$, $U(n)$ etc) are obviously closed, hence they are submanifolds and closed Lie subgroups.

Quotient constructions:

Theorem: If $H \leq G$ is a closed subgroup, then the space G/H is a smooth manifold. The tangent space to $1H$ is $\mathfrak{g}/\mathfrak{h}$.

Thm: If $H \trianglelefteq G$ is a closed normal subgroup then G/H is a smooth manifold (ie a Lie group) and $\pi: G \to G/H$ is a group homomorphism with kernel H.

Thm: If $I \trianglelefteq \mathfrak{g}$ is an ideal then the quotient vector space $\mathfrak{g}/I$ is a Lie algebra with bracket

$$[v+I, w+I] = [v,w] + I$$

and a homomorphism $\mathfrak{g} \longrightarrow \mathfrak{g}/I$ with kernel I.

THEOREMS ON HOMOMORPHISMS

Let $f: G \to H$ be a homomorphism of Lie groups with $\mathfrak{g}$ and $\mathfrak{h}$ as Lie algebras. Then the following diagram is commutative

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\pi"] & \operatorname{im} f \arrow[r, hook, "i"] & H \\ G/\ker f \arrow[ur, "1-1"'] & & \end{tikzcd}$$

If $\operatorname{im} f$ is a closed subgroup, then $G/\ker f$ is isomorphic to $\operatorname{im} f$.

If $\varphi: \mathfrak{g} \longrightarrow \mathfrak{h}$ homomorphism of Lie algebras then $\ker\varphi \triangleleft \mathfrak{g}$ is an ideal. and the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d] & \operatorname{im}\varphi \subset \mathfrak{h} \\ \mathfrak{g}/\ker\varphi \arrow[ur, "\text{isomorphism}"'] & \end{tikzcd}$$

for Lie group homomorphism:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f_*"] \arrow[d] & \operatorname{im} f_* \subset \mathfrak{h} \\ \mathfrak{g}/T_1(\ker f) \arrow[ur, "\text{isomorphism}"'] & \end{tikzcd}$$