ONE PARAMETER SUBGROUPS & EXP MAP

We will now construct chart around 1 which is functorial:

We start with an easy example: Let us take the torus $S^1 \times S^1$, its tangent space $T_1(S^1 \times S^1) = \mathbb{R}^2$ and the vector $(x_0, y_0) \in \mathbb{R}^2$. Consider the line in $\mathbb{R}^2$ determined by this vector i.e a homomorphism $t \to (tx_0, ty_0)$. There is an obvious way to "wind" this line on the torus by means of an exp map. We get a map $\mathbb{R} \to S^1 \times S^1$, $t \longmapsto (e^{2\pi i t x_0}, e^{2\pi i t y_0})$. This map is a homomorphism of a Lie group $\mathbb{R}$ with addition to the Lie group $S^1 \times S^1$ determined by the vector $(x_0, y_0) \in T_1(S^1 \times S^1)$. Now generalise.

Definition: A one parameter subgroup of a Lie group G is a homomorphism $\theta: \mathbb{R} \to G$, where $\mathbb{R}$ is a real line with addition.

More examples:

$$G = GL(3, \mathbb{R}) \qquad \theta(t) = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

Theorem: There exists a 1-1 correspondence between one-parameter subgroups and vectors of $T_1 G$. This correspondence is given by:
$\theta: \mathbb{R} \to G$ one parameter subgroup then $\theta \longmapsto \theta_*(1) \in T_1(G)$
where $\theta_*: T_0\mathbb{R} \to T_1 G$

We need some lemmas from diff equations:

reminder: Given a smooth vector field v defined on a neighbourhood U of $0 \in \mathbb{R}^n$.
Consider the equations for a function $f: \mathbb{R}^1 \to \mathbb{R}^n$: $(f_t)_*(1) = v(f(t))$, $f(0) = 0$ $(f_t)_*$ differential at point t.
Then there exists $\varepsilon > 0$ for which the equations have a solution in $(-\varepsilon, \varepsilon)$ and this solution is both smooth and unique.

Moreover we have the following corollary:

Lemma. Let X be a smooth manifold, $V: M \to TM$ a smooth vector field and $\theta, \varphi: [a,b] \to X$ smooth functions satisfying $(\theta_*)_t(1) = V(\theta(t))$, $(\varphi_*)_t(1) = V(\varphi(t))$ and $\theta(a) = \varphi(a)$. Then $\theta(t) = \varphi(t)$ for $t \in [a,b]$

Proof: Let $c = \sup\{d \in [a,b] : \varphi(t)=\theta(t) \text{ for } a \le t \le d\}$. Of course $\varphi(c)=\theta(c)$
If $c<b$ then take local coordinates at $\varphi(c)=\theta(c)$ and from previous lemma we get that $\theta(t)=\varphi(t)$ in some $(c-\varepsilon, c+\varepsilon)$ contradicting the definition of c. Hence $c=b$

Proof of the Theorem: The map one par. subgroups $\to$ v. tangent at 1 is $1-1$.

Suppose $\theta : \mathbb{R} \to G$ one par subgroup & $v = \theta_*(1) \in T_1(G)$
Extend both $1 \in \mathbb{R}$ and v to left invariant vector fields on $\mathbb{R}$ and G respectively. We get: 1 for every $t \in \mathbb{R}$ in the first case and $TL_g(v) = v(g) \in T_g G$, in the second case.
We have: a commutative diagram;

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\theta"] \arrow[d, "L_t"'] & G \arrow[d, "L_{\theta(t)}"] \\ \mathbb{R} \arrow[r, "\theta"] & G \end{tikzcd}$$

because θ is a homomorphism and $\theta(t+x) = \theta(t)\cdot\theta(x)$

And on tangent spaces

$$\begin{tikzcd} T_0\mathbb{R} \arrow[r, "\theta_*"] \arrow[d, "(L_t)_*"'] & T_1 G \arrow[d, "(L_{\theta(t)})_*"] \\ T_t\mathbb{R} \arrow[r, "(\theta_*)_t"] & T_{\theta(t)} G \end{tikzcd}$$

$1 \longrightarrow v$

$$(\theta_*)_t(1) = (L_{\theta(t)})_*(v) = v(\theta(t))$$

Every one parameter sbgp fullfills this equations so by previous lemma such one-parameter subgroup is unique.
(from the lemma uniqueness for positive reals, for negative from the fact that this is homomorphism)

Now we have to construct one parameter subgroup for given $v \in T_1(G)$. Extend $v \in T_1(G)$ to left invariant vector field $v(\cdot) : G \to TG$. Consider the equation

$$(\theta_*)_t(1) = v(\theta(t))$$
$$\theta(0) = 1_G$$

There is a unique solution for $(-\varepsilon, \varepsilon)$, from the lemma.

Must show that this is homomorphism where it makes sense and that it can be extended to one-parameter subgroup.

We start by showing that:

$$\theta(s)\theta(t) = \theta(s+t) \text{ for } |s|<\tfrac{1}{2}\varepsilon \quad |t|<\tfrac{1}{2}\varepsilon$$

Fix $s \in \mathbb{R}$ and consider two functions $t \longmapsto \theta(s)\theta(t)$ and $t \to \theta(s+t)$

Both functions are solutions to the equations;

$$(\varphi_*)_t(1) = v(\theta(s+t)) \quad \varphi(0)=\theta(s)$$

so they are equal on $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ because the vector field is left invariant.

Now we construct extension of θ to the whole $\mathbb{R}$ which will be our one parameter subgroup.

For $t \in \mathbb{R}$ choose $n \in \mathbb{N}$ such that $|\frac{t}{n}|<\frac{\varepsilon}{2}$ and define $\psi(t)=(\theta(\frac{t}{n}))^n$.

Must show that this does not depend on choice of n.

If $|\frac{t}{n}|<\varepsilon/2$ and $|\frac{t}{m}|<\frac{\varepsilon}{2}$ then $\theta(\frac{t}{n\cdot m})^n = \theta(\frac{t}{m})$ by previous assertion.

$$\theta(\tfrac{t}{m})^m = (\theta(\tfrac{t}{n\cdot m})^n)^m = \theta(\tfrac{t}{n\cdot m})^{n\cdot m} = \theta(\tfrac{t}{n})^n$$

Now ψ is a one parameter subgroup: $|\frac{s}{n}|<\frac{\varepsilon}{2}$ $|\frac{t}{n}|<\frac{\varepsilon}{2}$

$$\psi(s+t) = \theta(\tfrac{s+t}{n})^n = \theta(\tfrac{s}{n})^n \cdot \theta(\tfrac{t}{n})^n = \psi(s)\cdot\psi(t). \quad \blacksquare$$

Definition of the exponential map,

$$\exp : T_1 G \longrightarrow G$$

For $v \in T_1G$ let θ_v be a one parameter subgroup determined by v,

$$\exp(tv) = \theta_v(t), \text{ so } \exp(v) = \theta_v(1)$$

$\boxed{\text{Theorem: The map } \exp: T_1 G \longrightarrow G \text{ is smooth.}}$

Pf: This follows from the fact that the solution of a differential equation depends smoothly on smooth change of initial data. Rigorous proof is left to the reader.

Proposition: The derivative of exp at 0 is the identity,

$$id = (\exp_*)_0 : T_1(G) \longrightarrow T_1(G)$$

Pf: $\exp(tv) = \theta_v(t)$ by definition & the derivative by definition is equal to v.

The theorem on local invertibility gives us:

Theorem: The function $\exp: T_1(G) \to G$ is a diffeomorphism on a neighbourhood of 0.

So exp provides a chart around 1. This chart is very useful because of its functoriality.

Theorem: exp is natural, ie the diagram

$$\begin{tikzcd} T_1 G \arrow[r, "Tf"] \arrow[d, "\exp"'] & T_1 H \arrow[d, "\exp"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

is commutative

Proof: $f \circ \exp v = f(\theta_v(1)) = (f \circ \theta_v)(1)$

$f \circ \theta_v : \mathbb{R} \to H$ is a one parameter subgroup in H

Corresponding to a vector $Tf(v)$, so $f \circ \theta_v = \theta_{Tf(v)}$

$f \circ \theta_v(1) = \theta_{Tf(v)}(1) = \exp(Tf(v))$ ∎

Recall that in a topological connected group, any nbhd of identity generates the whole group. Thus we have the following:

Theorem: If G is a connected Lie group, a homomorphism $f : G \to H$ is determined by the induced homomorphism on tangent spaces: $(Tf)_1 : T_1 G \to T_1 H$.

Proof: We have a diagram

$$\begin{tikzcd} T_1 G \arrow[r, "Tf"] \arrow[d, "\exp"] & T_1 H \arrow[d, "\exp"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

It follows that f is determined by Tf on the image of exp.
But as exp is a <u>local diffeo</u> then f is determined by Tf on a nbhd of the identity. As this generates G, the statement follows.

Corollary: If for a homomorphism $f : G \to H$, $Tf : T_1(G) \to T_1(H)$ is an epimorphism, then f is an epimorphism.

Pf: From the diagram it follows that f is an epimorphism onto a nbhd of $1 \in H$

We know that the group structure on G induces the structure of Lie algebra on the tangent space $T_1 G$. We have also constructed a map $\exp: T_1 G \longrightarrow G$ which provides a chart around 0 which is functorial.

Before we proceed let us make few remarks.

① The inverse of exp where it is defined in the nbhd of $1 \in G$ will be denoted by log (of course!)

② Let us take the Taylor expansion that was used to define the Lie bracket:

$$\log(\exp v \cdot \exp w) = v + w + \alpha(v, w) + \dots$$

we have $\log(\exp v \cdot \exp v) = \log(\exp 2v) = 2v = v + v + \alpha(v,v) + \dots$ for every v close enough to 0. Therefore $\alpha(v,v) = 0$ and as α is bilinear it means that α is skew symmetric. Hence $[v, w] = \alpha(v, w) - \alpha(w, v) = 2\alpha(v, w)$ and we may write the Taylor expansion:

$$\log(\exp v \cdot \exp w) = v + w + \tfrac{1}{2}[v, w] + \dots$$

Using functoriality of exp we are now ready to prove the following

Theorem: If $f: G \longrightarrow H$ is a homomorphism of Lie groups then the linear map $f_*: T_1 G \longrightarrow T_1 H$ is a homomorphism of Lie algebras.

Proof: We have the commutative diagram:

$$\begin{tikzcd} T_1(G) \arrow[r, "f_*"] \arrow[d, shift left, "\exp"] & T_1(H) \arrow[d, shift left, "\exp"] \\ G \arrow[r, "f"] \arrow[u, shift left, "\log"] & H \arrow[u, shift left, "\log"] \end{tikzcd}$$

Must prove $f_*[v,w] = [f_* v, f_* w]$. In the nbhd of $0 \in T_1 G$

$$\log(\exp v \cdot \exp w) = v + w + \tfrac{1}{2}[v,w] + \dots$$

applying f_*:

$$f_* \log(\exp v \cdot \exp w) = f_* v + f_* w + \tfrac{1}{2} f_*[v,w] + \dots$$

On the other hand

$$\log(\exp f_*(v) \cdot \exp(f_* w)) = f_* v + f_*(w) + \tfrac{1}{2}[f_* v, f_* w] + \dots$$

But $\exp f_* = f \exp$, so

$$\log(\exp(f_* v) \cdot \exp(f_* w)) = \log(f\exp(v) \cdot f\exp(w)) = \log f(\exp v \cdot \exp w)$$

because f is a homomorphism

But again $\log f = f_* \log$ and it follows that:

$$f_* v + f_* w + \tfrac{1}{2} f_*[v,w] + \dots = f_* v + f_* w + \tfrac{1}{2}[f_* v, f_* w] + \dots$$

So $f_*[v,w] = [f_* v, f_* w]$ ∎