We want to prove the theorem:

The functor from the category of simply connected Lie groups to the category of Lie algebras assigning to a Lie group

$$G \rightsquigarrow \mathfrak{g}$$

$$f: G \to H \rightsquigarrow f_*: \mathfrak{g} \to \mathfrak{h}$$

is an equivalence of categories.

So far we know that the functor is well defined and it is a monomorphism on the set of morphisms. So we have to prove

1) for every homomorphism $\varphi: \mathfrak{g} \to \mathfrak{h}$ of Lie algebras there exists a homomorphism of Lie groups $f: G \to H$ such that $\varphi = f_*$

2) for every Lie algebra L there exists a simply connected group G such that $L \cong \mathfrak{g}$.

We start with 1)

Theorem: Let G and H be simply connected Lie groups, and $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras respectively. Then for every $\varphi: \mathfrak{g} \to \mathfrak{h}$ morphism of Lie algebras there exists a Lie group homomorphism $f: G \to H$ such that $\varphi = f_*$.

Proof: Strategy (following Segal)

Step 1. We will define a map $f : G \to H$ using paths in G and a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$. We will show that the definition is correct i.e does not depend on the path chosen. This is where one connectedness comes in, and the fact that φ respects the Lie algebra structure.

Step 2. Show that the map defined in Step 1 is a group homomorphism.

Step 3. Show that f is smooth and $f_* = \varphi$.

Step 1:

(a) Let $g \in G$. Let $\gamma : [0,1] \to G$ be a smooth path such that $\gamma(0) = 1$ and $\gamma(1) = g$.

Let $\tilde{\gamma} : [0,1] \to \mathfrak{g}$ be a path defined in the following way:

$\tilde{\gamma}(t) = (R_{\gamma(t)^{-1}})_* \gamma_*(t)$ where $\gamma_*(t) \in T_{\gamma(t)} G$ is the derivative of γ taken at the point t. $(R_{\gamma(t)^{-1}})_* : T_{\gamma(t)}(G) \to T_1(G) = \mathfrak{g}$

Path $\tilde{\gamma}$ will be called velocity path of the path γ.

Consider a path $\varphi \circ \tilde{\gamma} : [0,1] \to \mathfrak{h}$ and name it $\tilde{\xi}$. Now from the differential equations we know that there exists a unique path $\xi : [0,1] \to H$ such that

$$\xi(0) = 1 \qquad \xi_*(t) = (R_{\xi(t)})_* \tilde{\xi}(t)$$

Define $f(g) = \xi(1)$.

We have to prove that $f(g)$ does not depend on the path chosen from 1 to g,

assume for a moment that this is done and move to Step 2 and Step 3.

Step 2: $f: G \to H$ is a group homomorphism, i.e $f(g_1 g_2) = f(g_1) f(g_2)$ for $g_1, g_2 \in G$.

Let $\gamma_1 : [0,1] \to G$ be a smooth path from 1 to g_1, and $\gamma_2 : [0,1] \to G$ from 1 to g_2.
Define a path $\gamma : [0,1] \to G$

$$\gamma(t) = \begin{cases} \gamma_2(t) & 0 \le t \le \frac{1}{2} \\ \gamma_1(2t-1)\, g_2 & \frac{1}{2} \le t \le 1 \end{cases}$$

To assure smoothness of γ assume that the tangent vector at $1 \in [0,1]$ of the path γ_2 is equal to the tangent vector at $0 \in [0,1]$ of the path γ_1. So γ is the path from 1 to $g_1 g_2$.

The velocity of γ is given by the formula:

$$\tilde{\gamma}(t) = \begin{cases} 2\,\tilde{\gamma}_2(2t) & 0 \le t \le \frac{1}{2} \\ 2\,\tilde{\gamma}_1(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

If ξ_1, ξ_2 are paths in H corresponding to γ_1, γ_2 respectively, then

$$\xi(t) = \begin{cases} \xi_2(2t) & 0 \le t \le \frac{1}{2} \\ \xi_1(2t-1)\,\xi_2(1) & \frac{1}{2} \le t \le 1 \end{cases}$$

is a path corresponding to γ - which is an easy computation.

Thus $f(g_1 g_2) = \xi(1) = \xi_1(1) \cdot \xi_2(1) = f(g_1) \cdot f(g_2)$

Step 3 From the construction of f, $f(\exp v) = \exp(\varphi(v))$ for every $v \in \mathfrak{g}$.

The diagram:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

(the path in $\mathfrak{g}$ corresponding to one-parameter subgroup is constant, hence it yields one-parameter subgroup in H)

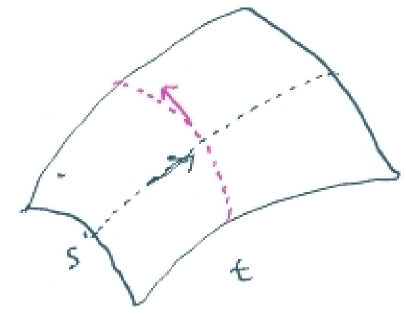
is commutative. This shows that f is smooth in some nbhd of $1 \in G$ and that $f_* = \varphi$.

The homomorphism f is smooth at any point $g_0 \in G$ because $f(g_0 g) = f(g_0) f(g)$ and f is smooth for g in some nbhd of 1.

We have to finish step 1. To deal with the homotopy we will need the following

Lemma: Let $h: I^2 \to G$ be a smooth map, where G is a Lie group.
Define two paths in $\mathfrak{g}$:

for every $s \in I$ path $\tilde{\gamma}(t,s)$ given by the formula



$$\frac{\partial h(t,s)}{\partial t} = \left(R_{h(t,s)}\right)_* \tilde{\gamma}(t,s)$$ (these are velocity paths of paths $h(\cdot,s)$)

and analogously for every $t \in I$ a path $\tilde{\eta}(t,s)$ in $\mathfrak{g}$

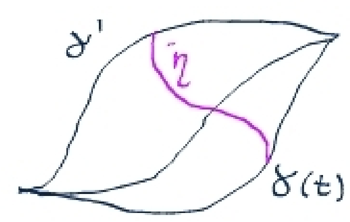
$$\frac{\partial h(t,s)}{\partial s} = \left(R_{h(t,s)}\right)_* \tilde{\eta}(t,s)$$ (these are velocity paths of paths $h(t,\cdot)$)

Then

$$\boxed{\frac{\partial \tilde{\eta}(t,s)}{\partial t} - \frac{\partial \tilde{\gamma}(t,s)}{\partial s} = \left[\tilde{\gamma}(t,s), \tilde{\eta}(t,s)\right]}$$

(These are called Maurer-Cartan equations)

We will assume that the lemma holds and finish the proof of Step 1.
For $g \in G$ assume that γ and γ' are two paths in G from 1 to g.
Let ξ and ξ' be corresponding paths in H. Must prove $\xi(1) = \xi'(1)$.
Let $h: I^2 \to G$ be a homotopy from γ to γ' rel $\{0,1\}$.
Given h, find $\tilde{\gamma}(t,s)$ and $\tilde{\eta}(t,s)$ as in the lemma. Because homotopy is rel $\{0,1\}$ we get $\forall_s\ \tilde{\eta}(0,s) = \tilde{\eta}(1,s) = 0$



We want to define homotopy $k(t,s)$ in H
as the family of solutions for curves in t with s fixed

$$\frac{\partial k(t,s)}{\partial t} = \left(R_{k(t,s)}\right)_* \varphi\left(\tilde{\gamma}(t,s)\right) \qquad k(0,s) = 1$$

We have $k(t,0) = \xi$ $k(t,1) = \xi'$ and $k(\cdot,\cdot)$ is smooth as solutions depend smoothly on initial data.
This homotopy gives the family of paths in H in variable s for every fixed t. It is enough to prove that this is exactly what we would get if we applied our procedure to paths $h(t,\cdot)$ for t fixed. It is enough because our procedure applied to a constant path yields a constant path and thus $\xi(1) = \xi'(1)$.

evaluation of higher terms vanishes

Fix t. For $k(t,s)$ given its velocity path $\tilde{\tau}(t,s)$ on $\mathfrak{k}$ is given by the formula

$$\frac{\partial k(t,s)}{\partial s} = \left(R_{k(t,s)}\right)_* \tilde{\tau}(t,s)$$

The question is: Is $\tilde{\tau}(t,s) = \varphi(\tilde{\eta}(t,s))$?

By the lemma applied to homotopy $k(t,s)$ we get equality in $\mathfrak{k}$

$$\frac{\partial \tilde{\tau}(t,s)}{\partial t} = \frac{\partial \varphi(\tilde{\gamma}(t,s))}{\partial s} + [\varphi(\tilde{\gamma}(t,s)), \tilde{\tau}(t,s)]$$

The lemma applied to homotopy $h(t,s)$ yields equality in $\mathfrak{g}$

$$\frac{\partial \tilde{\eta}(t,s)}{\partial t} = \frac{\partial \tilde{\gamma}(t,s)}{\partial s} + [\tilde{\gamma}(t,s), \tilde{\eta}(t,s)]$$

Applying to this equality Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{k}$ we get

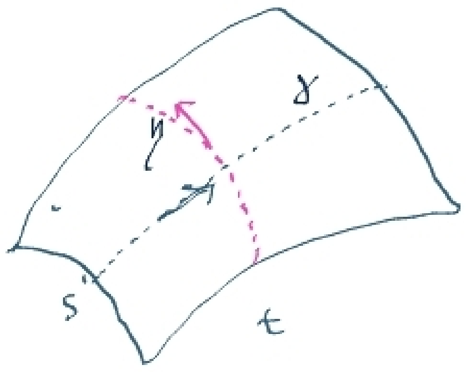
$$\frac{\partial \varphi\tilde{\eta}(t,s)}{\partial t} = \frac{\partial \varphi\tilde{\gamma}(t,s)}{\partial s} + [\varphi\tilde{\gamma}(t,s), \varphi\tilde{\eta}(t,s)]$$

It follows that $\tilde{\tau}(t,s)$ and $\varphi\tilde{\eta}(t,s)$ both satisfy the same differential equation (with respect to t) with the same initial conditions $\tilde{\tau}(0,s) = \varphi\tilde{\eta}(0,s) = 0$, so they are the same.
Therefore $\tilde{\tau}(1,s) = \varphi\tilde{\eta}(1,s) = 0$ and so $\eta(1,s)$ is a constant map.
This finishes the proof that $f : G \to H$ is well defined.

To complete the theorem on realiseability of homomorphism of Lie algebras by homomorphism of Lie groups we still must prove the following

Lemma: Let $h: I^2 \to G$ be a smooth map, where G is a Lie group.

Define two paths in $\mathfrak{g}$:



for every $s \in I$ path $\tilde{\gamma}(t,s)$ given by the formula

$$\frac{\partial h(t,s)}{\partial t} = \left(R_{h(t,s)}\right)_* \tilde{\gamma}(t,s)$$

(these are velocity paths of paths $h(\cdot,s)$)

and analogously for every $t \in I$ a path $\tilde{\eta}(t,s)$ in $\mathfrak{g}$

$$\frac{\partial h(t,s)}{\partial s} = \left(R_{h(t,s)}\right)_* \tilde{\eta}(t,s)$$

(these are velocity paths of paths $h(t,\cdot)$)

Then

$$\boxed{\frac{\partial \tilde{\eta}(t,s)}{\partial t} - \frac{\partial \tilde{\gamma}(t,s)}{\partial s} = \left[\tilde{\gamma}(t,s), \tilde{\eta}(t,s)\right]}$$

(There are called Maurer-Cartan equations)

Proof: For $v \in \mathfrak{g}$ in the small nbhd of 0 and $g \in G$ in the small nbhd of 1

$$\exp\left((R_g)_* v\right) = \exp\left(v + \alpha(v, \log g) + \dots\right)$$

If instead of homotopy h we would consider homotopy $h \cdot g$ curves $\tilde{\gamma}$ and $\tilde{\eta}$ would be the same. Thus to prove M-C equation for point (t_0, s_0) we may assume $h(t_0, s_0) = 1$.

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From the previous proposition (taking log on both sides) in the nbhd of (t_0, s_0)

$$\frac{\partial h(t,s)}{\partial t} = \tilde{\gamma}(t,s) + \alpha(\tilde{\gamma}(t,s), \log h(t,s)) + \dots$$

$$\frac{\partial h(t,s)}{\partial s} = \tilde{\eta}(t,s) + \alpha(\tilde{\eta}(t,s), \log h(t,s) + \dots$$

↑ terms of order ≥ 2 in $(t-t_0, s-s_0)$

Differentiate both at (t_0, s_0)

$$\left.\frac{\partial h(t,s)}{\partial t\, \partial s}\right|_{(t_0,s_0)} = \left.\frac{\partial \tilde{\gamma}(t,s)}{\partial s}\right|_{(t_0,s_0)} + \alpha(\tilde{\gamma}(t_0,s_0), \tilde{\eta}(t_0,s_0))$$

(higher terms vanish)

$$\left.\frac{\partial h(t,s)}{\partial s\, \partial t}\right|_{(t_0,s_0)} = \left.\frac{\partial \tilde{\eta}(t,s)}{\partial t}\right|_{(t_0,s_0)} + \alpha(\tilde{\eta}(t_0,s_0), \tilde{\gamma}(t_0,s_0))$$

$$\left.\frac{\partial \tilde{\eta}(t,s)}{\partial t}\right|_{(t_0 s_0)} - \left.\frac{\partial \tilde{\gamma}(t,s)}{\partial s}\right|_{t_0 s_0} = -\alpha(\tilde{\eta}(t_0 s_0), \tilde{\gamma}(t_0 s_0)) + \alpha(\tilde{\gamma}(t_0,s_0), \tilde{\eta}(t_0,s_0)) =$$

$$= [\tilde{\gamma}(t_0,s_0), \tilde{\eta}(t_0,s_0)]$$

Now we have to find a group with a given Lie algebra. We first prove that if $\mathfrak{g}$ is a Lie algebra of a group G and $\mathfrak{h} \leq \mathfrak{g}$ is a subalgebra, then there exists a subgroup $H \leq G$ whose Lie algebra is $\mathfrak{h}$.

This is obviously true for $\mathbb{R}^n$ – simply connected abelian group. So... abelianize!. But to do so we must assure that the commutant is a closed subgroup. The following is true:

Definition: Let $\mathfrak{g}$ be a Lie algebra. The **derived algebra** of $\mathfrak{g}$ is a subalgebra generated by all brackets $[v,w]$, $v,w \in \mathfrak{g}$.
We denote it $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$.
The **derived algebra** is an ideal and the smallest one such that the quotient algebra is commutative.

Theorem: For a connected group G its commutator $G' = [G,G]$ is a Lie subgroup with the tangent algebra $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$.
If G is simply connected then $[G,G]$ is a closed subgroup.

Proof of this theorem at the end of these notes.

MALCEV CLOSURES

G connected, $\mathfrak{g}$ its Lie algebra. How to find subalgebras of $\mathfrak{g}$ corresponding to closed subgroups.

Definition: Let $\mathfrak{g}$ be a tangent algebra of a Lie group G, and $\mathfrak{h} \subset \mathfrak{g}$ its subalgebra. Then there exists the smallest closed Lie subgroup $H \subseteq G$ such that its tangent algebra $\mathfrak{h}^M$ contains $\mathfrak{h}$. The subalgebra $\mathfrak{h}^M$ is called the Malcev closure of $\mathfrak{h}$.

The natural thing is to take intersection of all closed Lie subgroups of G whose Lie algebra contain $\mathfrak{h}$. This works because the following Lemma holds:

not necc. connected

Theorem: Let $\{H_i\}_{i\in I}$ be a family of closed subgroups of G. Then $\bigcap_{i\in I} H_i$ is a closed Lie subgroup and if $\{\mathfrak{h}_i\}_{i\in I}$ are subalgebras corresponding to subgroups H_i, $i\in I$ then $\bigcap_{i\in I}\mathfrak{h}_i$ is the subalgebra corresponding to $\bigcap_{i\in I} H_i$.

Proof: $\bigcap_{i\in I} H_i$ is a closed subgroup, so by Cartan's theorem is a closed Lie subgroup. Question is what is its Lie algebra? First note that if I is finite, then the theorem holds. This is true for H, H' when we apply theorem on counter image of a homomorphism to the inclusion $i: H \hookrightarrow G$ and $H' \subset G$. Thus it holds for a finite family of subgroups.
Consider $\bigcap_{i\in I}\mathfrak{h}_i$. This is the intersection of a family of vector subspaces of a finite dimensional vector space, so there exist $i_1, \dots, i_k$ such that $\bigcap_{i\in I}\mathfrak{h}_i = \mathfrak{h}_{i_1}\cap\dots\cap\mathfrak{h}_{i_k}$.

Let $H_{i_1},\dots,H_{i_k}$ be closed Lie subgroups corresponding to subalgebras $\mathfrak{h}_{i_1},\dots,\mathfrak{h}_{i_k}$. From the finite case we know that $\mathfrak{h} = \mathfrak{h}_{i_1}\cap\dots\cap\mathfrak{h}_{i_k}$ is the tangent Lie algebra to $H = H_{i_1}\cap\dots\cap H_{i_k}$. For arbitrary $i\in I$, subalgebra $\mathfrak{h}\cap\mathfrak{h}_i = \mathfrak{h}$ is tangent to $H\cap H_i$. So connected component of the identity H_1, $H_1 \subset H\cap H_i \subset H$. As this is true for arbitrary i, we obtain $H_1 \subset \bigcap_{i\in I} H_i \subset H$. Therefore the tangent space to the identity of $\bigcap_{i\in I} H_i$ is $\bigcap_{i\in I}\mathfrak{h}_i$.

The relation of Malcev closure $\mathfrak{h}^M$ to $\mathfrak{h}$ is given by the following theorem:

Theorem: If $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h}^M$ are as above, then $\mathfrak{h}' = [\mathfrak{h},\mathfrak{h}] = \mathfrak{h}^{M'} = [\mathfrak{h}^M,\mathfrak{h}^M]$.

Proof: Consider the homomorphism $Ad : G \longrightarrow GL(\mathfrak{g})$:

Let $H_1 = \{ g \in G : (Ad_g - id)(\mathfrak{h}) \subset [\mathfrak{h}, \mathfrak{h}] \} \leq G$

This is a closed subgroup of G with the tangent algebra $\mathfrak{h}_1 = \{ v \in \mathfrak{g} : ad_v \mathfrak{h} \subset [\mathfrak{h}, \mathfrak{h}] \}$. But $ad_v \mathfrak{h} = [v, \mathfrak{h}]$. This means that $\mathfrak{h} \subset \mathfrak{h}_1$

and as H_1 is closed $\mathfrak{h}^M \subset \mathfrak{h}_1$ and $[\mathfrak{h}^M, \mathfrak{h}] \subset [\mathfrak{h}, \mathfrak{h}]$

Let
$H_2 = \{ g \in G : (Ad_g - id)(\mathfrak{h}^M) \subset [\mathfrak{h}, \mathfrak{h}] \} \leq G$

this is a closed subgroup with tangent algebra:

$$\mathfrak{h}_2 = \{ v \in \mathfrak{g} : ad_v(\mathfrak{h}^M) \subset [\mathfrak{h}, \mathfrak{h}] \}$$

From the previous assertion $\mathfrak{h} \subset \mathfrak{h}_2$. But H_2 is closed, so $\mathfrak{h}^M \subset \mathfrak{h}_2$ and $[\mathfrak{h}^M, \mathfrak{h}^M] \subset [\mathfrak{h}, \mathfrak{h}]$.
As $\mathfrak{h} \subset \mathfrak{h}^M$ the inclusion $[\mathfrak{h}, \mathfrak{h}] \subset [\mathfrak{h}^M, \mathfrak{h}^M]$ is obvious. ∎

We are now ready to prove the theorem:

Proof: Let G be a connected lie group, $\mathfrak{g}$ it's Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra. Let $\mathfrak{h}^M$ be the Malcev closure of $\mathfrak{h}$ and $F \leq G$ a closed subgroup of G with Lie algebra $\mathfrak{h}^M$.
Let $p : \tilde{F} \to F$ be its universal cover. Then $[\tilde{F}, \tilde{F}] \leq \tilde{F}$ is a closed subgroup of $\tilde{F}$. Consider $\tilde{F}/[\tilde{F}, \tilde{F}]$. It is an abelian Lie group, which is simply connected. The simple connectedness comes from the exact sequence of the fibration $[\tilde{F}, \tilde{F}] \to \tilde{F} \to \tilde{F}/[\tilde{F}, \tilde{F}]$.

$$\dots \pi_1([\tilde{F}, \tilde{F}]) \to \underset{1}{\underset{\|}{\pi_1(\tilde{F})}} \to \pi_1(\tilde{F}/[\tilde{F}, \tilde{F}]) \to \pi_0(\tilde{F}/[\tilde{F}, \tilde{F}]) \xrightarrow{\cong} \pi_0(\tilde{F}) \to$$

so $\pi_1(\tilde{F}/[\tilde{F}, \tilde{F}])$ is trivial. Thus $\tilde{F}/[\tilde{F}, \tilde{F}] \cong \mathbb{R}^m$ for some $m \in \mathbb{N}$.

The Lie algebra of $\tilde{F}/[\tilde{F}, \tilde{F}]$ is $\mathfrak{h}^M/[\mathfrak{h}^M, \mathfrak{h}^M] = \mathfrak{h}^M/[\mathfrak{h}, \mathfrak{h}]$.

$\mathfrak{h}/[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}^M/[\mathfrak{h},\mathfrak{h}]$ is a subalgebra of the Lie algebra of $\mathbb{R}^M$.

There exists a subgroup $H \leq \mathbb{R}^M$ corresponding to $\mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$. Let $\tilde{H} \leq \tilde{F}$ be its counter image in $\tilde{F}$. The image $p: \tilde{H} \to p(\tilde{H}) \leq F \hookrightarrow G$
$H = p(\tilde{H}) \xhookrightarrow{i} G$ is the subgroup of G with the Lie algebra $\mathfrak{h}$.

The theorem of Ado, with difficult proof completes the proof of equivalence of categories of simply connected Lie groups and Lie algebras.

Theorem (Ado) Every finite dimensional Lie algebra is a subalgebra of the algebra of square matrices.

Corollary: Every finite dimensional Lie algebra is a Lie algebra of some Lie group, which is a subroup of a linear group $GL(n,\mathbb{R})$ for some $n \in \mathbb{N}$

Remark: This is not true that every Lie group is a subgroup of a linear group – we will show that the Heisenberg group is not a matrix group.

All groups with a given Lie algebra are obtained in the following way: Find simply connected Lie group with a given Lie algebra and find its center. Any group with the given Lie algebra is a quotient of the simply connected one by the discrete subgroup of the center.

Theorem: For a connected group G its commutator $G' = [G,G]$ is a Lie subgroup with the tangent algebra $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$.
If G is simply connected then $[G,G]$ is a closed subgroup.

Proof. The general case follows from the simply connected case in the following way:
Let G be connected group and $p: \tilde{G} \to G$ its universal covering.
$p_*: \tilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism. $[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}]$ is a Lie algebra of $[\tilde{G},\tilde{G}]$, hence $p_*[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}] = [\mathfrak{g},\mathfrak{g}]$ is a Lie algebra of $p[\tilde{G},\tilde{G}] = [G,G]$.

Assume G is simply connected. Consider quotient algebra $\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}$ – it is abelian, so it is Lie algebra of $\mathbb{R}^n$ for some $n = \dim \mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}$.
The map $\mathfrak{g} \to \mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}$ is the differential of some group homomorphism (which is an epimorphism) $G \xrightarrow{f} \mathbb{R}^n$ (by simple connectedness of G). We will prove that $[G,G]$ is equal to a closed subgroup $\ker f$, whose Lie algebra is $[\mathfrak{g},\mathfrak{g}]$. Of course $[G,G] \subseteq \ker f$.
As G and $G/\ker f$ are simply connected $\ker f$ must be connected.
To prove that $\ker f \subset [G,G]$ it is enough to prove that $[G,G]$ contains the nbhd of the unit of $\ker f$.
Let $v_1, \dots, v_k$ be the basis of $[\mathfrak{g},\mathfrak{g}]$ consisting of brackets.
Take paths: $\omega_i : (-\varepsilon, \varepsilon) \to \ker f$ such that: $\omega_i(0) = 1$, $\frac{d\omega_i}{dt}(0) = v_i$, $\forall_t\ \omega_i(t)$ is a commutant. Consider $\varphi : (-\varepsilon,\varepsilon)^k \to \ker f$, $\varphi(t_1,\dots,t_k) = \omega_1(t_1)\cdots\omega_k(t_k)$
Image of φ lies in $[G,G]$. $\frac{d\varphi}{dt}(0)$ is an isomorphism, so it is a diffeomorphism on a small nbhd of 0. This means that $[G,G]$ contains a nbhd of 0.

Such paths exist:
Let $v, w \in \ker f$. Let $\tau : (-\varepsilon,\varepsilon) \to \ker f$ be such that $\frac{d\tau}{dt}\Big|_0 = v$, $\tau(0) = 1$
Similarly $\eta : (-\varepsilon,\varepsilon) \to \ker f$ $\frac{d\eta}{dt}\Big|_0 = w$ $\eta(0) = 1$

Define $\omega : (-\varepsilon,\varepsilon) \to \ker f$
$$\omega(t) = \begin{cases} [\tau(\sqrt{t}), \eta(\sqrt{t})] & t \in \langle 0, \varepsilon) \\ [\tau(\sqrt{-t}), \eta(\sqrt{-t})] & t \in (-\varepsilon, 0\rangle \end{cases}$$

and verify that $\frac{d\omega}{dt}\Big|_0 = [v,w]$, $\omega(0) = 1$. $\square$

Corollary: If G is a connected Lie group such that $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ then the group G is perfect i.e $G' = [G,G] = G$.

Example: $SL(n,\mathbb{R})$. Lie algebra consist of matrices A s. that $\operatorname{tr} A = 0$. $[A,B] = AB - BA$. Basis of the Lie algebra are elementary matrices E_{ij} $i \neq j$ and matrices $E_{ii} - E_{jj}$ for $i \neq j$.
Note that $[E_{ii} - E_{jj}, E_{ij}] = 2E_{ij}$ and $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$.