Serre's GAGA
(Géometrie Algébrique et Géométrie Analytique)

1 *Throughout, let X be a separated scheme of finite type over $\mathbb{C}$ (in practice, an algebraic variety, often projective, i.e. a set defined in $\mathbb{P}^n$ by homogeneous polynomials). The analytification X^{an} is defined for such X and depends only on its structure as a $\mathbb{C}$-scheme.*

2 For an algebraic manifold X we define its *analytification* X^{an}.

• As a set, we identify

$$X^{an} \cong X(\mathbb{C}),$$

the set of complex points of X. Thus X^{an} and X have the same $\mathbb{C}$-valued points, but endowed with different topologies.

• While X is endowed with the Zariski topology, X^{an} carries the analytic topology, obtained by gluing open subsets of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ in local embeddings. The identity map

$$\iota : X^{an} \longrightarrow X$$

is continuous, since every Zariski open subset is open in the analytic topology (Serre, §5, Lemma 1).

3 Both spaces are *ringed spaces*. We have the sheaves $\mathcal{O}_X$ of regular functions and $\mathscr{H}_X$ of holomorphic functions, whose stalks are local rings. There exists a natural morphism of sheaves of rings

$$\theta_X : \iota^{-1}\mathcal{O}_X \longrightarrow \mathscr{H}_X,$$

so that ι extends to a morphism of ringed spaces.

• The map θ_X is an injective local homomorphism; each stalk $\theta_{X,x}$ is flat, and the induced map on $\mathfrak{m}$-adic completions of stalks is an isomorphism. This means $\mathscr{H}_X$ is a flat $\iota^{-1}\mathcal{O}_X$-module and θ_X is compatible with completions (Serre, §6, Prop. 4).

4 For an algebraic sheaf $\mathcal{F}$ on X, its analytification is defined by

$$\mathcal{F}^{an} = \mathscr{H}_X \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1}\mathcal{F}.$$

In particular, $\mathcal{O}_X^{an} = \mathscr{H}_X$ (Serre, §9, Prop. 10).

• Because $\mathscr{H}_X$ is flat over $\iota^{-1}\mathcal{O}_X$, the analytification functor $(-)^{an}$ is *exact* on coherent sheaves.

5 Definition. Let $(X, \mathscr{R}_X)$ be a ringed space. A sheaf of $\mathscr{R}_X$-modules $\mathcal{F}$ is called *coherent* if

(1) Locally there exists a surjective morphism $(\mathscr{R}_X^N)_{|U} \to \mathcal{F}_{|U}$ for some N (i.e. $\mathcal{F}$ is locally finitely generated).

(2) For any morphism $(\mathscr{R}_X^M)_{|U} \to \mathcal{F}_{|U}$, the kernel is locally finitely generated.

• We denote by $\mathrm{Coh}(X)$ the category of coherent $\mathscr{R}_X$-modules.

• For analytic spaces this condition (2) is necessary, since the rings $\mathscr{R}_X(U) = \mathscr{H}_X(U)$ are not Noetherian in general.

6 Oka's Theorem. If X^{an} is a complex analytic space (in particular, a complex manifold), then $\mathscr{H}_X$ is coherent (Oka–Cartan theorem; Serre, §3, Prop. 1).

7 The analytification functor extends to

$$(-)^{an} : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(X^{an}), \qquad (-)^{an} : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X^{an}),$$

and preserves exact sequences of coherent sheaves.

8 Serre's Fundamental Theorem (GAGA, §12, Th. 1). If X is projective and $\mathcal{F}$ is a coherent algebraic sheaf, the natural comparison map

$$H^*(X;\mathcal{F}) \xrightarrow{\sim} H^*(X^{an};\mathcal{F}^{an})$$

is an isomorphism. Hence algebraic and analytic cohomology agree for projective varieties.

• Properness (or projectivity) is essential: for non-proper varieties, the comparison of cohomology or categories of coherent sheaves may fail.

9 Relative version. If $f : X \to Y$ is a *projective* morphism of algebraic varieties (i.e. there is a factorisation $X \hookrightarrow Y \times \mathbb{P}^N \twoheadrightarrow Y$) , then f induces a functor

$$f_*^{an} : \mathrm{Coh}(X^{an}) \longrightarrow \mathrm{Coh}(Y^{an}),$$

and for all $k \geq 0$,

$$(f_*\mathcal{F})^{an} = f_*^{an}\mathcal{F}^{an}, \qquad (R^k f_*\mathcal{F})^{an} = R^k f_*^{an}\mathcal{F}^{an}.$$

For $Y = \mathrm{pt}$, this reduces to the cohomology comparison above.

10 Equivalence of Categories (Serre, §12, Th. 2–3). If X is projective, the functor

$$(-)^{an} : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X^{an})$$

is an *equivalence of categories*, meaning:

(i) The analytification functor is *fully faithful*:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{H}_X}(\mathcal{F}^{an},\mathcal{G}^{an})$$

for all coherent algebraic sheaves $\mathcal{F},\mathcal{G}$.

(ii) The functor is *essentially surjective*: for every coherent analytic sheaf G on X^{an}, there exists an algebraic coherent sheaf $\mathcal{F}$ on X such that $G \simeq \mathcal{F}^{an}$.

11 Outline of Proof. The argument is reduced to $X = \mathbb{P}^n$. By analyzing long exact cohomology sequences, it suffices to verify the equality $H^*(\mathbb{P}^n;\mathcal{O}(m)) = H^*(\mathbb{P}^{n,an};\mathscr{H}(m))$.

• The standard cohomology of line bundles on projective space is

$$H^q(\mathbb{P}^n,\mathcal{O}(m)) \cong \begin{cases} \mathrm{Sym}^m(\mathbb{C}^{n+1})^\vee & \text{if } q=0,\ m\geq 0, \\ \mathrm{Sym}^{-m-n-1}(\mathbb{C}^{n+1}) & \text{if } q=n,\ m \leq -n-1, \\ 0 & \text{otherwise,} \end{cases}$$

and the same holds analytically via Dolbeault or Hodge theory, giving the desired isomorphism.

• For (i), one uses the identity of internal Hom-sheaves:

$$(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))^{an} = \underline{\mathrm{Hom}}_{\mathscr{H}_X}(\mathcal{F}^{an},\mathcal{G}^{an}),$$

and the principle

$$\mathrm{Hom}_Y(F,G) = H^0(Y;\underline{\mathrm{Hom}}_Y(F,G)),$$

together with the cohomology comparison.

• For (ii), let F be a coherent analytic sheaf on $X = \mathbb{P}^n$. There exists m_0 such that for $m \geq m_0$ the twisted sheaf

$$F(m) = F \otimes_{\mathscr{H}_X} \mathscr{H}_X(m)$$

is globally generated. Equivalently, for every $x \in X$, the evaluation map

$$H^0(X;F(m)) \longrightarrow F(m)_x$$

is surjective (Serre, §16, Lemma 8). Thus $F(m)$ is the cokernel of a morphism $\mathscr{H}_X^\ell \to \mathscr{H}_X^k$, hence algebraic by (i). Twisting back by $\mathscr{H}_X(-m)$ shows F itself is algebraic (Serre, §17).

12 Examples.

(a) For $X = \mathbb{P}^1$, the analytification X^{an} is the Riemann sphere $\mathbb{C} \cup \{\infty\}$. We have $H^0(\mathbb{P}^1; \mathcal{O}(m)) = H^0(\mathbb{P}^{1,an}; \mathscr{H}(m))$, both equal to the space of homogeneous polynomials of degree m. Thus analytic and algebraic cohomology agree explicitly.

(b) If $C \subset \mathbb{P}^n$ is a smooth projective curve, its analytification C^{an} is a compact Riemann surface. GAGA identifies algebraic vector bundles on C with holomorphic vector bundles on C^{an} and ensures

$$H^1(C; \mathcal{O}_C) \simeq H^1(C^{an}; \mathscr{H}_C),$$

so algebraic and analytic deformations coincide.

(c) ♠ For affine varieties X, both the algebraic and analytic theories have vanishing higher cohomology: by Serre's affineness theorem $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and any quasi-coherent $\mathcal{F}$; analytically, X^{an} is Stein, so by Cartan's Theorem B, $H^i(X^{an}, \mathcal{G}) = 0$ for $i > 0$ and any coherent analytic $\mathcal{G}$. However, the equivalence of categories in GAGA generally fails when X is not proper.

13 Corollary (Chow's Theorem). Every closed analytic subvariety of $\mathbb{P}^n(\mathbb{C})$ is algebraic; equivalently, it is the zero locus of finitely many homogeneous polynomials. Indeed, the analytic ideal sheaf is coherent and thus algebraizable by GAGA.

Historical Remark. Serre's 1956 paper *"Géometrie Algébrique et Géométrie Analytique"* established a precise bridge between algebraic and analytic geometry over $\mathbb{C}$. For projective varieties, analytification preserves not only cohomology but also the entire category of coherent sheaves. This result provided a prototype for modern comparisons between algebraic and analytic (or topological) categories and deeply influenced Grothendieck's theory of schemes and coherence.
See also Hartshorne II.4, III.12.