Complex Manifolds - written exam AD 2026

Problem 1 Let $\alpha = dx \wedge dy \in A^2(\mathbb{C})$. Write that form in terms of dz and $d\bar{z}$.

$$dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\,dx \wedge dy$$

so $dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}$

Problem 2 Let $f = \Big((z_1+1)^2 + z_2^2\Big)\Big((z_1-1)^2 + z_2^2\Big) - 1$. What is the degree Weierstrass polynomial $g \in \mathcal{O}_{(\mathbb{C},0)}[z_1]$ (the distinguished variable z_1).

$$f(z_1, 0) = (z_1+1)^2(z_1-1)^2 - 1 = (z_1^2-1)^2 - 1 = z_1^4 - 2z_1^2$$

The order of 0 is 2, thus the Weierstrass polynomial has degree 2.

Problem 3 Let $X_k \subset \mathbb{C}^n$, $k \in \mathbb{N}$ be a family of analytic sets. Prove that $\bigcap_{k\in\mathbb{N}} X_k$ is an analytic set (can be described locally by a finite set of equations).

Let $p \in \mathbb{C}^n$ and $I_k \subset \mathcal{O}_{\mathbb{C}^n, p}$ be the ideal of X at p; $I_k = (f_1, \dots, f_{m_k})$. Then the ideal of $\bigcap_{k\in N} X_k$ is generated by $\bigcup_{k\in N} I_k$. Since the local ring is Noetherian, the above ideal is finitely generated. Thus $\bigcap_{k\in N} X_k$ is described by a finite set of equations.

Problem 4 Let V be a real vector space and let $I \in \mathrm{End}(V)$ be an almost complex structure. Construct a $\mathbb{C}$-linear embedding $(V, I) \longrightarrow (V \otimes \mathbb{C}, 1 \otimes i)$.

$$f(v) = v \otimes 1 - I(v) \otimes i \text{ is } \mathbb{C}\text{-linear.}$$

Indeed: $f(I(v)) = I(v) \otimes 1 - I^2(v) \otimes i =$

$$= v \otimes i - I(v) \otimes i^2$$

$$= i(v \otimes 1 - I(v) \otimes i)$$

It is an embedding since

$$f(v) = v \otimes 1 - I(v) \otimes i = 0 \iff v = 0 \wedge I(v) = 0.$$

Problem 5 Let $\varphi : \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic map. Check that $\varphi^*(\alpha) \in A^{p,q}(U)$ for $\alpha \in A^{p,q}(V)$.

$V \subset \mathbb{C}^m$, $U \subset \mathbb{C}^n$

We check first for $(1,0)$ forms dz_i $i = 1, \dots, m$.

$\varphi = (\varphi_1, \varphi_2, \dots \varphi_m)$

$$\varphi^*(dz_i) = d\varphi_i = \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial z_j} dz_j + \cancel{\sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial \bar{z}_j} d\bar{z}_j} \in A^{10}(U)$$

Since by assumption φ_i is holomorphic, the second sum vanishes. The statement for (p,q) forms follow by taking wedge and conjugation.

Problem 6 Let $X = \mathbb{P}^1$. Let α be a 1–form of the type $(0,1)$. Does there exist a 0–form β, such that $d\beta = \alpha$?

The answer is no (for a general form).

Example: let $f : \mathbb{C} \to \mathbb{C}$ be a function with a compact support. The form $\alpha = f\, d\bar{z}$ extends to a form on $\mathbb{P}^1$ of the type $(0,1)$. In general $d(f\, d\bar{z}) = \frac{\partial}{\partial z} f\, dz \wedge d\bar{z}$ is a non-zero $(1,1)$ form. Therefore α cannot be in the image of d.

Problem 7 Give an example of a number $g \geqslant 0$, such that there does not exist a curve $C \subset \mathbb{P}^2$ of genus $g = \dim H^{1,0}(C)$.

We compute the Euler characteristic of C

$$\chi(C) = \int_C \frac{(1+h)^3}{1+dh} = \int_C (3-d)h = d(3-d)$$

where d is the degree of C, $h = c_1(\mathcal{O}(1)|_C)$.

$$\text{genus} = \frac{2-\chi(C)}{2} = \frac{1}{2}(2-3d+d^2) = \frac{1}{2}(d-1)(d-2).$$

For example genus $= p$ (a prime number) never appears.
$p > 3$

Problem 8 Suppose that M is a compact Kähler manifold of dimension 3 (over $\mathbb{C}$). Is it possible, that the cohomology ring (with complex coefficients) is isomorphic to

$$\mathbb{C}[a,b]/(a^2,b^2), \qquad a \in H^2(M), \qquad b \in H^4(M)\ ?$$

If M is Kähler, then there exists a class $\omega \in H^2(M)$, such that $\omega^3 \neq 0$.

Here $H^2(M)$ is spanned by a and $\forall \lambda \in \mathbb{C}$ $(\lambda a)^2 = 0$, hence $(\lambda a)^3 = 0$.

Problem 9 Let X be a connected compact Kähler manifold, $\dim(X) = 4$. Suppose that its cohomology is generated by the fundamental classes of complex submanifolds and $\dim H^2(X) = \dim H^4(X) = 6$. Compute the signature of X.

$H^{pq}(X) = 0$ for $p \neq q$, thus the Hodge diamond is as follows

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 6 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 6 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 6 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

$$\sigma(X) = 1 - 6 + 6 - 6 + 1 = -4$$

Problem 10 Let X be a smooth surface in $\mathbb{P}^3$ od degree 4. Compute $\chi(X;\mathcal{O}(k))$. Is it a polynomial in k?

By HRR $\chi(X,\mathcal{O}(k)) = \int_X td(TX)\, ch(\mathcal{O}(k))$.

For surfaces

$$td(TX) = \left(1+\tfrac{1}{2}x_1+\tfrac{1}{12}x_1^2\right)\left(1+\tfrac{1}{2}x_2+\tfrac{1}{12}x_2^2\right) =$$

$$= 1+\tfrac{1}{2}(x_1+x_2)+\tfrac{1}{12}(x_1^2+x_2^2+3x_1x_2) = 1+\tfrac{1}{2}c_1+\tfrac{1}{12}(c_1^2+c_2)$$

$$c(TX) = (1+h)^4(1+4h)^{-1} = (1+4h+6h^2)(1-4h+16h^2) = 1+6h^2 \mod h^3$$

$$\chi(X;\mathcal{O}(k)) = \int_X \left(1+\tfrac{1}{2}h^2\right)\left(1+kh+\tfrac{1}{2}(kh)^2\right) = \int_X \tfrac{1}{2}(h^2+k^2h^2) = 2(1+k^2)$$

Problem 11 Let X be a Kähler manifold, L a holomorphic positive line bundle. Prove that $\chi(X;L^{\otimes d})$ is a polynomial in d. Find its degree. Find the leading coefficient in terms of $c_1(L)$.

Suppose $\dim X = n$. By possitivity assumption $\int_X c_1(L)^n > 0$.

Claim: the degree is n, the top coefficient is as below:

$$\chi(X, L^{\otimes d}) = \int_X td(TX)\, ch(L^{\otimes d}) =$$

$$\int_X (1+\dots)\underbrace{\left(1+dc_1(L)+\dots+\tfrac{1}{n!}(d\,c_1(L))^n\right)}_{\text{polynomial in } d} = \frac{d^n}{n!}\overbrace{\int_X c_1(L)^n}^{\neq 0} + \text{lower terms}$$

Problem 12 Let X be quintic (a hypersurface of degree 5) in $\mathbb{P}^3$. Its χ_y-genus is equal to $5-45y+5y^2$ (take it for granted). Write the Hodge diamond $h^{p,q}$.

By weak Lefschets the Hodge diamond is of the following form:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ h^{20} & & h^{11} & & h^{02} \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$1-0+h^{02} = 5 \Rightarrow h^{02} = 4$ $(= h^{20})$

$0-h^{11}+0 = -45 \Rightarrow h^{11} = 45$