Complex Manifolds — Problems 23.01.2026

Problem 1 The second attempt to show that

$$c_1(L) = \text{[Zeros of a section]-[Poles of a section]},$$

i.e. the class of the associated divisor.

(1) Let $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and let be s a nonzero section. Show that $c_1(L) =$[Zeros(s)].

(2) Suppose a line bundle L over X is globally generated, and $s \in \Gamma(X; L)$. Show that $c_1(L) =$[Zeros(s)].

(3) Suppose $L = L_1 \otimes L_2^*$ and L_1 and L_2 are globally generated. Show that $c_1(L) =$[Zeros(s)]–[Poles(s)] for a suitably chosen s. Next argue, that in fact s can be arbitrary.

Hint to (3): Suppose f is a global meromorphic function, show that [Zeros(f)]-[Poles(f)]$= 0 \in H^*(X)$.

Problem 2 Let $y \in \mathbb{Q}$ (or you can treat it as a free parameter). For a vector bundle $E \to X$ define a characteristic class

$$td_y(E) = \sum_{p=0}^{\mathrm{rk}(E)} y^p \; td(E) \cdot ch(\Lambda^p E^*) \in H^*(X)\,,$$

(see [Huybrechts 5.1.3-4]).

(1) Show that

$$td_y(E \oplus F) = td_y(E) \cdot td_y(F) \qquad \text{(cup-product in cohomology)}\,.$$

Note that for a Kähler manifold X of dimension n

$$\int_X td_y(TX) = \sum_{p=0}^{n} \sum_{q=0}^{n} (-1)^q y^p h^{p,q}(X)\,.$$

The above is called Hirzebruch χ_y-genus.

(2) What do we get for the speclal value of the parameter $y = -1$?, What is $td_{-1}(E)$?

Problem 3 (1) Compute Hirzebruch χ_y-genus of a smooth hypersurface in $\mathbb{P}^3$ of degree d for $d = 2, 3, 4$.

(2) Apply a corollary from the Weak Lefschetz theorem (TBA):

„For a hypersurface the restriction map $H^k(\mathbb{P}^n) \to H^k(X)$ is an isomprphism for $k < \dim_{\mathbb{C}} X$ and is injective for $k = \dim_{\mathbb{C}} X$."

to compute all Hodge numbers of X as above.

Problem 4 Check that the Hirzebruch-Riemann-Roch gives the right answer for $L = \mathcal{O}_{\mathbb{P}^n}(k)$, $k \geqslant 0$, i.e.

$$\int_{\mathbb{P}^n} td(T\mathbb{P}^n) ch(L) = \dim\left(\Big[\mathbb{C}[z_0, z_1, \ldots, z_n]\Big]_{\text{homogenous part of degree } k}\right)\,.$$

Problem 5 Make sure that you can easily solve all the test-like problems listed in the file `https://www.mimuw.edu.pl/~aweber/ComplexManifolds2025/CoMa26egz.pdf`.

Mini-talk: Assume that X is a connected, simply-connected surface with $\Omega^2_X \simeq \mathcal{O}_X$. It follows $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$ oraz $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 0$. Information about o $h^{1,1}$ is missing.

(1) Applying HRR for $E = \mathcal{O}_X$ compute $c_2(TX)$ (since we know $\chi(X, \mathcal{O}_X)$).

(2) Applying HRR for $E = \Omega^1_X$ compute $\chi(X, \Omega^1_X)$ and deduce what $h^{1,1}$ is.