Complex Manifolds — Problems 19.12.2025

Problem 1 Let $Q_n \subset \mathbb{P}^{n+1}$ be a quadric. Compute its cohomology and show that $H^{p,q}(Q_n) = 0$ for $p \neq q$.

Hint: Show that $Q^n = \mathbb{C}^n \sqcup cQ_{n-2}$ where $cQ_{n-2} \subset \mathbb{P}^n$ is the projective cone over Q_{n-2}. Construct the filtration leading to a cell decomposition.

(For any variety $M \subset \mathbb{P}^m$ the projective cone $cM \subset \mathbb{P}^{m+1}$ is the variety defined by the same homogenous equations as M but considered in the ring with extra one variable. In general it is a singular variety. The point $[0:0:\cdots:0:1] \in \mathbb{P}^{m+1}$ is almost always a singular point. The exception is when $M \simeq \mathbb{P}^k$ embedded linearly.)

Problem 2 *(Non-algebraic tori)*: Let V be a complex vector space, $A \subset V$ a lattice. Then canonically

$$H^*(V/A;\mathbb{C}) \simeq \Lambda V^*_{\mathbb{C}}$$

$$H^*(V/A)_{\mathbb{Z}} = \Lambda A^\vee\,,$$

$$A^\vee = \{f \in \mathrm{Hom}(V,\mathbb{R}) \mid \forall a \in A \;\; f(a) \in \mathbb{Z}\}$$

Show that if $\dim V > 1$ then for a generic lattice $H^{1,1}(M) \cap H^*(M)_{\mathbb{Z}} = \{0\}$, hence M cannot be embeded into $\mathbb{P}^n$.

Consider the case $V = \mathbb{C}^2$. First find a convenient basis $\{\alpha_j\}_{j=1\ldots 4} \subset H^{1,1}(M) \cap H^*(M,\mathbb{R}) = \Lambda^{1,1}V^* \cap \Lambda^2 V^*$. Fix spanning vectors $\{v_i\}$ of the lattice A. You can assume that $v_1 = \varepsilon_1$, $v_2 = \varepsilon_2$, v_3, v_4 are arbitrary (there are 8 real parameters). For each pair $\{k,\ell\} \subset \{1,2,3,4\}$ consider the corresponding real torus $S^1 \times S^1 \simeq T_{k,\ell} \hookrightarrow V/A$ and compute the integrals $\int_{T_{k,\ell}} \alpha_j$. Show that for a random choice of vectors v_3, v_3 there is no nontrivial combinantion $\alpha = \sum a_j \alpha_j$ for which all the integrals $\int_{T_{k,\ell}} \alpha$ are integers.

Problem 3 Suppose M is a compact Kähler manifold. Let $\alpha \in \Omega^p(M)$ be a global holomorphic form. Show that $\partial\alpha = 0$.

Hint: Apply the equality $\Delta_{\bar\partial} = \frac{1}{2}\Delta_d$.

For a complex line bundle $L \to B$ consider the associated fibration $\mathbb{P}^1 \hookrightarrow \mathbb{P}(L \oplus 1\!\!1_B) \twoheadrightarrow B$. Loosely speaking we add to each fiber of $L \to B$ a point at infinity. Here $1\!\!1_B$ denotes the trivial line bundle.

Problem 4 Let $B = \mathbb{P}(V)$. Construct a map $\mathbb{P}(\mathcal{O}(1) \oplus 1\!\!1_B) \to \mathbb{P}(V \oplus \mathbb{C})$, which is a bijection except one fiber, shrunk to a point.

Problem 5 Let B be a C^∞-manifold, and $L \to B$ a complex line bundle. For convenience assume that B is compact and oriented (although it is not essential). The manifold $\mathbb{P}(L \oplus 1\!\!1_B)$ contains $B_0 = \mathbb{P}(1\!\!1_B)$ which we identify with B. Show using axioms of c_1 that the Poincaré dual class $[B_0] \in H^2(\mathbb{P}(L \oplus 1\!\!1_B))$ restricted to B_0 coincides with $c_1(L)$ under the identification $B_0 = B$.

Hint: Check the equality for $\mathcal{O}(1)$.

Problem 6 Suppose $L \to M$ is a holomorphic line bundle over a complex manifold and let $s : M \to L$ a holomorhic section, which is transverse to the zero section. Let $Z(s) \subset M$ be the zero set. Show that the Poincaré dual class $[Z(s)] \in H^2(M)$ is equal to $c_1(L)$.

No Problem Let V be a complex vector space, $k \in \mathbb{N}$, $n = \dim V$. Construct an isomorphism of vector bundles over Grassmannians

$$\begin{array}{ccc} Q_{k,V^*} & \longrightarrow & S^*_{k,V} \\ \downarrow & & \downarrow \\ Gr_{n-k}(V^*) & \longrightarrow & Gr_k(V)\,. \end{array}$$

(*It is an easy exercise on duality. I suppose we do not have to do it on the blackboard. Do it for yourself.*)

Mini-talk: KO about signature of Kähler manifolds.