Complex Manifolds — Problems 12.12.2025

Problem 1 Make step in a proof of the Hodge relation $[L^*, \partial] = i\bar{\partial}^*$ for the flat metric: Consider the summand L_k (multiplication by $\frac{i}{2}dz_k \wedge d\bar{z}_k$) of L and ∂_k (a summand of ∂) as in the §8.1 in my notes. Compute $[L_k^*, \partial_k]\alpha$ for $\alpha = f dz_A \wedge d\bar{z}_B$. The case $k \in A$, $k \in B$ is in my notes. Try other case, e.g. $k \notin A$, $k \in B$.

Problem 2 A complex submanifold $X \subset M$ (or in general a closed analytic set) of a Kähler submanifold defines a cohomology class by Poincaré duality: let $\dim_{\mathbb{C}} M = n$, $\dim_{\mathbb{C}} X = k$. Define a functional on $H^{2k}(M)$: $\alpha \mapsto \int_X \alpha$. Thus X defines an element $[X] \in H^{2k}(M)^* \simeq H^{2(n-k)}(M)$. Show that $[X] \in H^{n-k,n-k}(M)$, i.e. this functional vanishes on classes of the types different from (k,k).

Problem 3 Suppose that M has a filtration

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_m = M$$

consisting of closed analytic subsets, such that $M_k \setminus M_{k-1} \simeq \mathbb{C}^{d_k}$ for $k = 0, \ldots, m$ and some numbers d_k. Show that $H^{p,q}(M) = 0$ for $p \neq q$.

Problem 4 Let $Q_n \subset \mathbb{P}^{n+1}$ be a quadric. Compute its cohomology and show that $H^{p,q}(Q_n) = 0$ for $p \neq q$.

Hint: Construct the filtration as above for smooth quadrics $Q_n \subset \mathbb{P}^{n+1}$ for $n = 1, 2, 3, 4$. Show in general that $Q^n = \mathbb{C}^n \sqcup cQ_{n-2}$ where $cQ_{n-2} \subset \mathbb{P}^n$ is the projective cone over Q_{n-2}.

(For any variety $M \subset \mathbb{P}^m$ the projective cone $cM \subset \mathbb{P}^{m+1}$ is the variety defined by the same homogenous equations as M but considered in the ring with extra one variable. In general it is a singular variety. The point $[0:0:\cdots:0:1] \in \mathbb{P}^{m+1}$ is almost always a singular point. The exception is when $M \simeq \mathbb{P}^k$ embedded linearly.)

Problem 5 **Complex tori**: Let V be a complex vector space, $A \subset V$ a lattice. Then canonically

$$H^*(V/A; \mathbb{C}) \simeq \Lambda V^*_{\mathbb{C}}$$

$$H^*(V/A)_{\mathbb{Z}} = \Lambda A^{\vee},$$

$$A^{\vee} = \{f \in \mathrm{Hom}(V, \mathbb{R}) \mid \forall a \in A \quad f(a) \in \mathbb{Z}\}$$

Show that if $\dim V > 1$ then for a generic lattice $H^{1,1}(M) \cap H^*(M)_{\mathbb{Z}} = \{0\}$, hence M cannot be embeded into $\mathbb{P}^n$.

Problem 6 a) Compute $H^q(\mathbb{P}^n; \Omega^p_{\mathbb{P}^n})$.

b) From the Mayer-Vietoris exact sequence for $\mathbb{P}^1 = \mathbb{C} \cup_{\mathbb{C}^*} \mathbb{C}$ compute $H^q(\mathbb{C}^*, \Omega^p)$.

(Assume that you know that for the analytic cohomology $H^1(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) = 0$. For more details see [Arapura §16.3]. Both $\mathbb{C}$ and $\mathbb{C}^*$ are Stein manifolds, so by Cartan's theorem B [Arapura 16.3.3] their higher cohomology with coefficients in any coherent sheaf vanishes.)