Complex Manifolds — Problems 05.12.2025

Let V be a complex vector space with a Hermitian structure

$$\langle\!\langle v, w \rangle\!\rangle = \langle v, w \rangle - i\omega(v, w)\,.$$

The vector space V (as a real vector space) has a natural orientation, thus Hodge $*$ is defined. Consider the real exterior power $\Lambda V^* := \Lambda_{\mathbb{R}} V^*$. Recall that the scalar product in ΛV^* satisfies

$$\langle \alpha, \beta \rangle vol = \alpha \wedge *\beta.$$

We extend the scalar product to a Hermitian form on $(\Lambda_{\mathbb{R}} V^*) \otimes \mathbb{C} = \Lambda_{\mathbb{C}}(V^*_{\mathbb{C}})$ by the formula

$$\langle\!\langle \alpha, \beta \rangle\!\rangle vol = \alpha \wedge *\overline{\beta}.$$

Problem 1 Show that the Hodge star induces an isomorphism

$$* : \Lambda^{p,q} \xrightarrow{\simeq} \Lambda^{n-q,n-p}$$

$$*(dz_I \wedge d\bar{z}_J) = const\, dz_{J^\vee} \wedge d\bar{z}_{I^\vee}$$

where $J^\vee = \{1, 2, \dots, n\} \setminus J$, $I^\vee = \{1, 2, \dots, n\} \setminus I$. Compute the constant.

Remark: Assume that dx_j, dy_j form an othonormal basis of V^*. The Hodge star is defined on the real vector space ΛV^*, it is easily described by values on $dx_I \wedge dy_J$. The goal is to find what happens in the complexificatation, when the star is applied to $dz_I \wedge d\bar{z}_J$.

Problem 2 Show that $\Lambda^{p,q} \perp \Lambda^{p',q'}$ if $(p, q) \neq (p', q')$.

Problem 3 „Linear version of Hard Lefschetz”. Suppose $\dim_{\mathbb{C}} V = n$. Show that that a proper power of the operator L defines isomorphisms between the opposite exterior powers

$$L^k : \Lambda^{n-k} V^* \xrightarrow{\simeq} \Lambda^{n+k} V^*\,.$$

Problem 4 Decompose $\Lambda_{\mathbb{R}} V^*$ for $V = \mathbb{C}^2$ into irreducible representations of $\mathfrak{sl}_2$. For each summand find the the vectors spanning the lowest weight space.

Problem 5 For a hermitian manifold show that

$$L^* = (-1)^k * L * \qquad \text{restricted to } \Lambda^k V^*,$$

$$\partial^* = - * \overline{\partial} *,$$

$$\overline{\partial}^* = - * \partial *$$

are the adjoint operators to $L, \partial, \overline{\partial}$.

Problem 6 Show that the metric associated to the Fubini-Study form ω is given by the following: fix $w \in \mathbb{C}^{n+1} \setminus \{0\}$ and let $[w] \in \mathbb{P}^n$. The scalar product of vectors $\alpha, \beta \in T_{[w]}\mathbb{P}^n$ satisfies

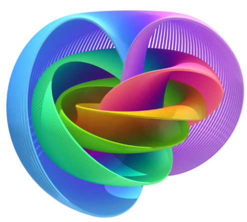
$$\langle \alpha, \beta \rangle = \frac{1}{\pi} \frac{\langle w, w \rangle \langle \tilde{\alpha}, \tilde{\beta} \rangle - \langle \tilde{\alpha}, w \rangle \langle w, \tilde{\beta} \rangle}{\langle w, w \rangle^2},$$

where $\tilde{\alpha}, \tilde{\beta} \in T_w \mathbb{C}^{n+1}$ are lifts of the vectors with respect to the quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

Problem 7 The Hopf fibration is constructed as follows:

Let $S^3 \subset \mathbb{C}^2$ be the unit sphere. The group S^1 of unit complex numbers act on $\mathbb{C}^2$ preserving S^3. The quotient S^3/S^1 is identified with the projective line $\mathbb{P}^1 \simeq S^2$. The quotient map $S^3 \to S^2$ is a fibration with the fiber S^1.

We say that two disjoined circles $S_1, S_2 \subset S^3$ are linked if for any disk $D \subset S^3$, such that $\partial D = S^1$ the intersection $D \cap S_2$ is not empty. Show that any two fibers of the Hopf fibration are linkled.



The picture from Wikipedia

Mini-talk DK: A Hermitian manifold is Kähler if and only if locally there exist a real valued function φ, such that $\omega = i\partial\bar{\partial}\varphi$. The function φ is called the Kähler potential.

References:
1. Shiing-shen Chern - Complex Manifolds without Potential Theory (1979, Springer) **page 59, condition (C)**
2. R. O. Wells Jr. - Differential Analysis on Complex Manifolds-Springer New York Graduate Texts in Mathematics 65, (1980) **Lemma 2.15, page 50**