Complex Manifolds — Problems 21.11 → 28.11.2025

Recollection: For a complex-valued function f on $\mathbb{C}$ with compact support we have defined the convolution $f * \frac{1}{u}$ by the formula

$$(f * \tfrac{1}{w})(z) = \tfrac{1}{2\pi i} \int_{\mathbb{C}} \tfrac{f(w)}{w-z} dw \wedge d\bar{w}$$

and we have shown that

$$\tfrac{d}{d\bar{z}}(f * \tfrac{1}{w}) = f\,.$$

Problem 1 *Modification of the convolution:* Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk, f a C^∞ function on $\mathbb{C}$. Let

$$\mathcal{S}(f) = \tfrac{1}{2\pi i} \int_{\mathbb{D}} \tfrac{f(w)}{w-z} dw \wedge d\bar{w}\,.$$

Show that

$$\tfrac{d}{d\bar{z}}\mathcal{S}(f) = f\,.$$

Problem 2 Let f be a complex valued function on $\mathbb{C}^n$. Let

$$\mathcal{S}_k(f)(z) = \tfrac{1}{2\pi i} \int_{\mathbb{D}} \tfrac{f(z_1,\dots,z_{k-1},w,z_{k+1},\dots,z_n)}{w-z} dw \wedge d\bar{w}\,.$$

Define a linear operator

$$H_k : A^{0,q}(\mathbb{C}^n) \to A^{0,q-1}(\mathbb{C}^n)$$

satisfying

$$H_k(f\, d\bar{z}_k \wedge d\bar{z}_A) = \mathcal{S}_k(f) d\bar{z}_A \qquad \text{and} \qquad H_k(f\, d\bar{z}_A) = 0$$

provided that the multiindex $A \subset \{1,\dots,n\}$ does not contain k. For a given $\overline{\partial}$-closed form $\alpha \in A^{0,q}_c(\mathbb{C}^n)$ construct the sequence of forms α_k for $k = 0,1,\dots,n$ by induction

$$\alpha_0 = \alpha\,, \qquad \alpha_{k+1} = \alpha_k - \overline{\partial} H_k(\alpha_k)\,.$$

Show that

a) α_k has no $d\bar{z}_j$ for $j \leqslant k$, when written in the basis $d\bar{z}_A$,

b) the coefficients of α_k are holomorphic with respect to the variables $j \leqslant k$.

Deduce $\overline{\partial}$-Poincaré Lemma.

Problem 3 Show from definition of the Dolbeault cohomology that $H^1(\mathbb{C}^*, \mathcal{O}_{\mathbb{C}^*}) = 0$.

Hint: Consider the covering $\exp : \mathbb{C} \to \mathbb{C}^*$ *and cyclic functions on* $\mathbb{C}$.

Problem 4 Define a 2-tensor $\varphi : T\mathbb{P}^n \otimes T\mathbb{P}^n \to \mathbb{R}$ by the formula

$$\varphi(v,w) = -\omega(iv,w)$$

where ω is the Fubini-Study form [Huybrechts, Examples 3.1.9]. Show that

– $\varphi(v,w)$ is symmetric and positively defined (i.e. it is a Riemanian metric)

– $\varphi(v,w) = \varphi(iv,iw)$ i.e. it is I-invariant.

Hint for the proof of positivity: Show that ω *is* $U(n+1)$*-invariant and check for* $n = 1$.

Problem 5 Show that for any complex submanifold $M \subset \mathbb{P}^n$, $\dim_{\mathbb{C}} M = k$

$$\int_M \omega^k > 0\,.$$

Problem 6 Consider $\mathbb{P}^n$ with the Riemanian metric construted in Problem 4. Let M be a smooth manifold and $f : (0,1) \times M \to \mathbb{P}^n$. Suppose that for $t \in (0,1)$ the image $M_t = f(\{t\} \times M$ is a complex submanifold. Show that the volume $vol(M_t)$ does not depend on t.

Hint: Is there any relation between the volume form on M_t *and the* ω^k*?*
