Complex Manifolds — Problems 31.10.2025 → 07.11.2025

Problem 1 Suppose V is a real vector space with a scalar product. Let $\Lambda \subset V$ be a lattice. Then $M := V/\Lambda$ is diffeomorphic to $(S^1)^{\dim V}$. Consider the Riemannian structure induced from the scalar product in V. Find harmonic representatives of cohomology classes. (Start with $H^1(M)$, $\dim V = 2$.)

Problem 2 Let V be a real vector space and $I \in \mathrm{End}(V)$ be a complex structure (i.e. an endomorphism satisfying $I^2 = -id$). Let

$$\mathbf{I} : \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) .$$

$$\mathbf{I}(f) = f \circ I .$$

which has the eigenspaces

$$(V^*)^{1,0} = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \ : \ \forall v \in V \quad f(Iv) = if(v)\}, \qquad (\mathbb{C}\text{-linear forms})$$
$$(V^*)^{0,1} = \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \ : \ \forall v \in V \quad f(Iv) = -if(v)\}, \qquad (\text{antilinear forms}).$$

For any $f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ decompose f into the eigenvectors.

For $f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ let $f_{\mathbb{C}}$ denote the $\mathbb{C}$-linear extension to $V_{\mathbb{C}}$. Let $V^{1,0} \subset V_{\mathbb{C}}$ (resp. $V^{0,1}$) be the eigenspace of I with the eigenvalue i (resp. $-i$). Show that if $f \in (V^*)^{0,1}$, then $f_{\mathbb{C}}(V^{1,0}) = 0$. Show that $(V^*)^{1,0} \to (V^{1,0})^*$, $f \mapsto (f_{\mathbb{C}})_{|V^{1,0}}$ is an isomorphism.

Problem 3 Check the signs in the proof of the equality $\langle d\alpha, \beta \rangle_M = \langle \alpha, d^*\beta \rangle_M$.

Problem 4 Write the solution of the heat equation $\frac{d}{dt}\alpha(t) = -\Delta\alpha(t)$, on $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ in terms of the Fourier expansion of the initial condition

$$\alpha(0) = \Big(a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)\Big)dx \in A^1(S^1)$$

Problem 5 Let $U, V \subset \mathbb{C}^n$ and let $\varphi : U \to V$ be a biholomorphism. Check that

$$\varphi^*(A^{p,q}(V)) = A^{p,q}(U) .$$

Problem 6 Let Δ be the Laplasian acting on the space of the differential forms on a compact Riemannian manifold. Take for granted that the solution of the heat equation exists with arbitrary initial condition $\alpha(0)$. The following exercises are (almost) purely algebraic.

- a) Check that if $\alpha(0)$ is closed, then for $t > 0$ the forms $\alpha(t)$ are closed and

$$[\alpha(0)] = [\alpha(t)] \in H^*(M) .$$

- b) Show that $\alpha_H := \lim_{t\to\infty} \alpha(t)$ exists and is a harmonic form. The map

$$\alpha_0 \to \lim_{t\to 0} \alpha(t)$$

is the orthogonal projection $V \to ker(\Delta)$. (*At least check it for eigenvectors of* Δ.)

- c) Let $G(\alpha) = \int_0^{\infty}(\alpha(t) - \alpha_H)dt$ be the Green operator. Check that

$$\alpha = \alpha_H + \Delta G(\alpha) .$$

(*At least check it for eigenvectors of* Δ.)

See [D. Arapura, Algebraic Geometry over the Complex Numbers] §8.3.