Complex Manifolds — Problems 24.10.2025

Old:

Problem 1 Suppose $U \subset \mathbb{C}^n$ is connected and $0 \neq f : U \to \mathbb{C}$ is holomorphic. Show that $U \setminus Z(f)$ open dense and connected.

Problem 2 (*A bit difficult*) Let

$$U \simeq \prod_{i=1}^{n} \mathbb{D} = \{(z_1, z_2, \dots z_n) : |z_i| \leqslant 1\}$$

be the polydisc. Show that if $f : U \to U$ is biholomorphic (i.e. f^{-1} exists and it is holomorphic), $f(0) = 0$, then f is a composition of permutation of variables and rotations, i.e.

$$Aut(U, 0) = (S^1)^n \rtimes \Sigma_n \,.$$

Hint: Suppose $f = (f_1, f_2, \dots, f_n)$. Show (or take for granted) a generalization of the Schwartz Lemma

$$\max_i |f_i(z)| \leqslant \max_i |z_i| \,.$$

Since f is invertible

$$\max_i |f_i(z)| = \max_i |z_i| \,.$$

The polydisc U is a sum of closed sets U_j on which $\max_i |f_i(z)| = |f_j(z)|$, there $\max_i |z_i| = |f_j(z)|$. Find a subset $V \subset U$ on which $|z_k| = |f_j(z)|$ for some k. Composing f with a permutation you can assume that $j = k$. Deduce that $|z_j| = |f_j(z)|$ on the whole U and proceed with the proof of the main claim.

Problem 3 (*very easy from the previous*) Assuming the statement given in Problem 5 deduce that the polydisc is not biholomophicaly equivalent to a ball in $\mathbb{C}^n$.

Hint: Show that the automorphism group of the ball fixing 0 is much bigger, than the automorphism group of the polydisc.

New:

Problem 4 Suppose $U \subset \mathbb{C}^n$ is an open set, let $\mathcal{O}(U)$ be the ring of holomorphic functions on U. Show that if $\mathcal{O}(U)$ is Noetherian, then $U = \emptyset$ or $n = 0$.

Problem 5 Suppose M and N are two compact, connected differentiable manifolds of the same dimension. Let $f : M \to N$ be a smooth injective map. Show that f is a homeomorphism.

Hint: Compute the degree of the map f.
[See e.g. J. W. Milnor, Topology from the Differentiable Viewpoint.]

Problem 6 Show the properties of the Hodge star on a manifold of the dimension n:
(i) $**\alpha = (-1)^{k(n-k)}\alpha$ for a k-form α.
(ii) $\langle \alpha, *\beta \rangle = (-1)^{k(n-k)} \langle *\alpha, \beta \rangle$.

Problem 7 Suppose V is a real vector space with a scalar product. Let $\Lambda \subset V$ be a lattice. Then $M := V/\Lambda$ is diffeomorphic to $(S^1)^{\dim V}$. Consider the Riemannian structure induced from the scalar product in V. Find harmonic representatives of cohomology classes. (Start with $H^1(M)$.)