Complex Manifolds — Problems 17.10.2025

Problem 1 Let $C \subset \mathbb{P}^2$ be a smooth complex curve defined by a homogeneous polynomial $f \in \mathbb{C}[z_0, z_1, z_2]$ of degree d. What is the genus of that curve? (Give an elementary proof, not using e.g. the adjunction formula).

Hint: Assume that $p = [0 : 0 : 1] \notin C$ and consider the projection from

$$\mathbb{P}^2 - \{p\} \supset C \xrightarrow{\pi} \mathbb{P}^1$$

$$[x_0 : x_1 : x_2] \mapsto [x_0; x_1] \,.$$

Considering a general case assume that π has only nondegenerate singularites, i.e. locally π (in local coordinates) is of the form $z \mapsto z^2$. Using Bézout theorem show that there is $d(d-1)$ singular points. Compute the Euler characteristic of C deleting and adding singular points.

Problem 2 Suppose $n > 1$, $f : \mathbb{C}^n \to \mathbb{C}$ holomorphic. Show that if the zero set $Z(f)$ is compact, then it is empty.

Problem 3 Find the Weierstrass polynomial for $f(z_1, z_2) = z_1^3 z_2 + z_1 z_2 + z_1^2 z_2^2 + z_2^2 + z_1 z_2^3$ at 0.

Problem 4 Suppose $U \subset \mathbb{C}^n$ is connected and $0 \neq f : U \to \mathbb{C}$ is holomorphic. Show that $U \setminus Z(f)$ open dense and connected.

Problem 5 (*A bit difficult*) Let

$$U \simeq \prod_{i=1}^{n} \mathbb{D} = \{(z_1, z_2, \dots z_n) : |z_i| \leqslant 1\}$$

be the polydisc. Show that if $f : U \to U$ is biholomorphic (i.e. f^{-1} exists and it is holomorphic), $f(0) = 0$, then f is a composition of permutation of variables and rotations, i.e.

$$Aut(U, 0) = (S^1)^n \rtimes \Sigma_n \,.$$

Hint: Suppose $f = (f_1, f_2, \dots, f_n)$. Show (or take for granted) a generalization of the Schwartz Lemma

$$\max_i |f_i(z)| \leqslant \max_i |z_i| \,.$$

Since f is invertible

$$\max_i |f_i(z)| = \max_i |z_i| \,.$$

The polydisc U is a sum of closed sets U_j on which $\max_i |f_i(z)| = |f_j(z)|$, there $\max_i |z_i| = |f_j(z)|$. Find a subset $V \subset U$ on which $|z_k| = |f_j(z)|$ for some k. Composing f with a permutation you can assume that $j = k$. Deduce that $|z_j| = |f_j(z)|$ on the whole U and proceed with the proof of the main claim.

Problem 6 Assuming the statement given in Problem 5 deduce that the polydisc is not biholomophicaly equivalent to a ball in $\mathbb{C}^n$.

Hint: Show that the automorphism group of the ball fixing 0 is much bigger, than the automorphism group of the polydisc.