Complex Manifolds — Problems 10.10.2025

Problem 1 Let $f : \mathbb{C} \to \mathbb{C}$ be a C^∞-function. Prove the formula

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}\,.$$

Problem 2 Suppose f is holomorphic defined on a neighbourhoodof the disc $\{z \in \mathbb{C}{:}|z| \leqslant \varepsilon\}$ and $f(z) \neq 0$ if $|z| = \varepsilon$. Show that for $\ell \geqslant 0$ we have

$$\frac{1}{2\pi i}\int_{S_\varepsilon} \frac{f'(\xi)}{f(\xi)}\xi^\ell d\xi = \sum_{|\alpha|<\varepsilon, f(\alpha)=0} \alpha^\ell.$$

Problem 3 Prove that $SL_2(\mathbb{R})$ does not contain a discrete group Γ isomorphic to $\mathbb{Z}^2$.

Problem 4 Let $C \subset \mathbb{P}^2$ be a smooth complex curve defined by a homogeneous polynomial $f \in \mathbb{C}[z_0, z_1, z_2]$ of degree d. What is the genus of that curve? (Give an elementary proof, not using e.g. the adjunction formula).

Problem 5 For $I \subset \{1, 2, \ldots, n\}$ let $\mathbb{C}^I \subset \mathbb{C}^n$ be the corresponding coordinate subspace. Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$ as a set is the union of charts:

$$Gr(k, n) = \bigcup_I U_I\,,$$

where the indices I run through all subsets of $\{1, 2, \ldots, n\}$, $|I| = k$, and $U_I \simeq \mathrm{Hom}(\mathbb{C}^I, \mathbb{C}^{I^\vee})$. Here $I^\vee = \{1, 2, \ldots, n\} \setminus I$. A linear map $\mathbb{C}^I \to \mathbb{C}^{I^\vee}$ is identified with its graph in $\mathbb{C}^n$. Show that the transition functions between charts are holomorphic.

Problem 6 Construct an embedding $Gr(2, 4) \hookrightarrow \mathbb{P}^5$ and describe the image by a polynomial equation.

A short presentation:

Let Λ be a lattice in $\mathbb{C}$. Let

$$\wp(z) = z^{-2} + \sum_{w\in\Lambda\setminus\{0\}} \left((z-w)^{-2} - w^{-2}\right)$$

be the Weierstrass function. (Argue that the sum is convergent and differentiable term by term.) Show that $\wp(z)$ is Λ-periodic. Moreover

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3\,,$$

where

$$g_2 = 60\sum_{w\in\Lambda\setminus\{0\}} w^{-4}\,,$$

$$g_3 = 140\sum_{w\in\Lambda\setminus\{0\}} w^{-6}\,.$$

Show that

$$z \mapsto [\wp(z) : \wp'(z) : 1]$$

defines a continuous map $\mathbb{C}/\Lambda \to \mathbb{P}^2$, an embedding. The image is given by the equation

$$Q(x, y, z) = y^2z - 4x^3 + g_2x + g_3 = 0\,.$$

(see *Kirwan, Complex Algebraic Curves, Chapter 5*).