Complex Manifolds 2025/26

Lecture summary. **This is not a replacement for a textbook.**

`aweber at mimuw.edu.pl`

http://www.mimuw.edu.pl/∼aweber/ComplexManifolds2025

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The main reference: Daniel Huybrechts, Complex Geometry, An introduction. (Springer 2005)

Also:
Donu Arapura, Algebraic Geometry over the Complex Numbers (Universitext)
Claire Voisin, Hodge Theory and Complex Algebraic (Cambridge Studies in Advanced Mathematics)

1 Introduction and Preliminaries

1.1 Differential manifolds

- Differential forms
- Stokes theorem
- DeRham cohomology

http://www.mimuw.edu.pl/∼aweber/ComplexManifolds2025/rz22deRhamKZ.pdf

Analytic functions

1.2 Definition: A C^∞-function $f = u + iv : \mathbb{C} \to \mathbb{C}$ is holomorphic iff

$$\frac{\partial}{\partial x}u = \frac{\partial}{\partial y}v\,, \qquad \frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}v\,,$$

i.e. the differential of f at any point is $\mathbb{C}$-linear.

1.3 Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and complex differential

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

1.4 A C^∞-function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic iff $\frac{\partial}{\partial \bar z} f = 0$

1.5 Differentials $dz = dx + idy$ and $d\bar z = dx - idy$.

- For any C^∞ function on $f : \mathbb{C} \to \mathbb{C}$ the differential can be formally written in variables z and $\bar z$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\,.$$

Variables $z,\ \bar z$ are much more convenient than decomposing $f = u + iv$ and using variables x and y

- If f is holomorphic, then $\frac{\partial f}{\partial z}$ is denoted by f'.

1.6 Recollection of theorems for complex analytic functions in one variable

- series expansion
- Cauchy integration formula (for f of the class C^1 it follows from the Stokes theorem)
- maximum principle
- identity principle
- Liouville theorem
- Riemann extension theorem

1.7 Suppose f is holomorphic on a punctured disk $D \setminus z_0$. Residue $res_{z_0}(f) = \frac{1}{2\pi i}\int_{\partial D} f\,dz$.

• residue is an invariant of the differential form $f(z)dz$. When we change variable $z = g(w)$ then the form $f(z)dz$ should be replaced by

$$g^*(f(z)dz) = f(g(w))g'(u)dw\,,$$

i.e. $res_{z_0} f = res_{g^{-1}(z_0)}(f \circ g)g'$. Thus in the coordinate-free situation we should talk about the residues of 1-forms.

1.8 Residue theorem: Let S be compact complex curve (i.e. a Riemann surface), $A = \{z_1, z_2, \dots, z_n\}$ a finite set. Let ω be a holomorphic form on $S \setminus A \to \mathbb{C}$, i.e. locally $\omega = f\,dz$ with f holomorphic. Then

$$\sum_{k=1}^{n} res_{z_k}(\omega) = 0\,.$$

• Proof from the Stokes theorem: Assume that the disks $D_{z_k} \ni z_k$ do not intersect:

$$\sum_{k=1}^{n}\int_{\partial D_{z_k}} f\,dz = -\int_{\partial(S\setminus\bigcup D_{z_k})} f\,dz = -\int_{S\setminus\bigcup D_{z_k}} d(f\,dz) = -\int_{S\setminus\bigcup D_{z_k}} \frac{\partial f}{\partial \bar z} d\bar z \wedge dz = 0.$$

1.9 A formula for the number of zeros in a disk (and its extension): If $f(z) \neq 0$ for $|z| = \epsilon$ then

$$\frac{1}{2\pi i}\int_{|z|=\epsilon} \frac{f'(\xi)}{f(\xi)} d\xi = \#\{z \in \mathbb{C}\ :\ |z| < \epsilon,\ f(z) = 0\}\,.$$

For $\ell > 0$ we have

$$\frac{1}{2\pi i}\int_{S_\epsilon} \frac{f'(\xi)}{f(\xi)} \xi^\ell d\xi = \sum_{|\alpha|<\epsilon,\ f(\alpha)=0} \alpha^\ell.$$

Complex manifolds - examples (mainly algebraic)

1.10 Projective spaces

1.11 Grassmannians (affine maps, Plücker embedding, $Gr(2,4)$ as a quadric in $\mathbb{P}^5$).

1.12 Elliptic curves in $\mathbb{P}^2$, hyperplane in $\mathbb{P}^n$ (in particular quadrics).

1.13 Non-algebraic example: Hopf surface. Fix $a \in \mathbb{C}$, $|a| > 1$. Let $A = \{a^k \in \mathbb{C}\ :\ k \in \mathbb{Z}\} \simeq \mathbb{Z}$:

$$H_a = (\mathbb{C}^2 \setminus \{0\})/A\,,$$

where the elements of A act as scalar multiplications on $\mathbb{C}^2$. We have

$$H_a \simeq S^3 \times S^1\,.$$

1.14 Complex manifolds as real manifolds are orientable since any linear complex map preserves the distinguished orientation of the underlying real vector space.

1.15 Basic information about topological coverings an induced complex structures: If $f : X \to Y$ is a topological covering, Y has a structure of a complex manifold, then X has a natural structure of a complex manifold and f is a holomorphic.

Curves

1.16 Riemann surfaces is an oriented surfaces with a Riemannian metric. Each Riemannian surface has a complex structure.

• The rotation by 90° in the tangent space allows to introduce a structure of complex vector space. This structure is „integrable" i.e. it comes from a structure of a complex manifold. It is a special (easy) case of Newlander-Nirenberg theorem.
(Vanishing of Nijenhuis [Nee-yen-house] tensor, see Theorem 4.16 in Dusa McDuff, Dietmar Salamon: *Introduction to Symplectic Topology*, Oxford University Press)

• The genus of compact Riemann surface determines the diffeomorphism type

$$g(S) = \dim\{\text{the space of holomorphic 1-forms}\}\,.$$

The relation of genus with the Euler characteristic:

$$\chi(S) = 2 - 2g(S)\,.$$

1.17 Riemann uniformization theorem: any complex curve is isomorphic to $\mathbb{P}^1$ or it is a quotient of $\mathbb{C}$ or $\mathbb{D} \simeq \mathbb{H} = \{z \in \mathbb{C} \,:\, im(z) > 0\}$.

• Another formulation: any simply connected complex curve is isomorphic to $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D}$. This is a generalization of the Riemann theorem for open subsets in $\mathbb{C}$.

1.18 The automorphism group of $\mathbb{P}^1$ is equal to $PGL_2(\mathbb{C})$. Any complex-analytic automorphism of $\mathbb{P}^1$ is given by a linear formula. (The same statement holds for $\mathbb{P}^n$.)

• Proof: Composing with a linear map we can assume that $f(0) = 0$, $f(\infty) = \infty$. Expanding at infinity we get an estimation $1/|f(z)| < c/|z|$. Hence the function $g(z) = z/f(z)$ is bounded. It has no poles, since at 0 the zero of the denominator cancels out and there are no more zeros of f. Hence by Liouville theorem $g(z)$ is constant.

• Hence each automorphism of $\mathbb{P}^1$ has a fixed point – the eigenvector of the linear map.
Topological proof: there are no nontrivial topological covering $\mathbb{P}^1 \simeq S^2 \to C$ except $C = \mathbb{RP}^2$. But the real projective plane is not orientable, so it cannot be a complex curve.

1.19 Automorphisms of $\mathbb{C}$ are given by affine maps $f(z) = az + b$. There are no fixed points only if $a = 1$.

• The map $f : \mathbb{C} \to \mathbb{C}$ extends to $\mathbb{P}^1$. It is continues at ∞. By Riemann extension theorem it is holomorphic ∞ and we apply (1.18).

1.20 The complex quotients of $\mathbb{C}$ are of the form $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$.

• The nontrivial diskrete subgroups of $\Lambda \subset (\mathbb{C}, +) \simeq \mathbb{R}^2$ are of the form $\Lambda = \langle a, b\rangle$ for $b/a \in \mathbb{H}$, (or $\Lambda = \langle a\rangle$). We can restrict our attention to subgroups of the form $\Lambda = \langle 1, \tau\rangle$, $\tau \in \mathbb{H}$.

• The group $PSL_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by homography: $\begin{pmatrix} s & t \\ u & v\end{pmatrix} \cdot z = \frac{sz+t}{uz+v}$. The subgroup $PSL_2(\mathbb{R})$ preserves upper half-plane $\mathbb{H}$.

• Suppose $\tau, \tau' \in \mathbb{H}$. Then $\mathbb{C}/\langle 1, \tau\rangle \simeq \mathbb{C}/\langle 1, \tau'\rangle$ if and only if τ and τ' belong to the same orbit of $PSL_2(\mathbb{Z})$.

• Proof: We define a function on the set of positively oriented $\mathbb{R}$-bases in $\mathbb{C}$: $\phi(a, b) = b/a = \tau \in \mathbb{H}$. Two positively oriented bases of Λ differ by the operations $(a, b) \mapsto (a, b + a)$ and $(a, b) \mapsto (b, -a)$. The corresponding matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ acting on $\tau = a/b$ generate $PSL_2(\mathbb{Z})$. This shows that

$$\{\text{latices in } \mathbb{C}\}/Aut(\mathbb{C}) \quad\simeq\quad \mathbb{H}/PSL_2(\mathbb{Z})\,.$$

The map

$$\begin{array}{ccc} \{\text{latices in } \mathbb{C}\}/Aut(\mathbb{C}) & \longrightarrow & \{\text{isomorphism classes of curves of genus 1}\} \\ \Lambda & \mapsto & \mathbb{C}/\Lambda \end{array}$$

is injective, since the if $\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda'$, then the pairs $(\mathbb{C}, \Lambda)$ and $(\mathbb{C}, \Lambda')$ are isomorphic via a linear map. To show that it is a surjection one has to argue, that every curve of genus 1 is a quotient of $\mathbb{C}$, i.e. it is not a quotient of $\mathbb{D}$. It is enough to check that $Aut(\mathbb{D})$ does not contain $\mathbb{Z}^2$.

- There is an embedding $\mathbb{C}/\Lambda \to \mathbb{P}^2$ — see Weierstrass function $\wp$ (exercises).

1.21 The disk automorphisms are the same as the automorphism of the upper hyperplane $\mathbb{H}$: $Aut(\mathbb{H}) = PSL_2(\mathbb{R})$.

- $Aut(\mathbb{D})$ consist of homographies by the Schwartz lemma (we can assume $f(0) = 0$ applying $z \mapsto \frac{z-a}{1-\bar{a}z}$).

1.22 diskrete subgroups of $PSL_2(\mathbb{R})$ are called Fuchsian groups (grupy Fuksa). The curves of higher genera $g > 1$ are quotients $\mathbb{H}/G$ where $G \subset PSL_2(\mathbb{R})$ is Fuchsian and acts without fixed points.

1.23 Read more: [Huybrechts, Complex Geometry, Chapter 2.1]

2 Weierstrass preparation theorem

Local theory: see [§1.1, Huybrechts], Springer link https://link.springer.com/book/10.1007/b137952, available from University computers.

2.1 Definition: a C^∞ function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if it is holomorphic with respect to every variable. Equivalently, if it can be developed in a power series at every point of the domain.

2.2 Cauchy integral formula Prop 1.1.2

2.3 Hartogs theorem Prop 1.1.4

2.4 Exercise: zero set of a holomorphic function ($Z(f) = \{z \in Domain : f(z) \neq 0\}$) has real codimension equal 2 or it is empty.

- Remark: any analytic set (eg zero set of a holomorphic functions) is triangulable by Łojasiewicz theorem, so there is no ambiguity about the notion of dimension.

2.5 Weierstrass preparation theorem (Th. 1.1.6).

2.6 Algebraic fact used in the proof: elementary symmetric functions σ_k can be expressed by power sums p_k.

2.7 Weierstrass preparation theorem – division version (Prop 1.1.17).

Local ring

2.8 The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is a unique factorization domain (Prop 1.1.15).

- Any Weierstrass polynomial is indecomposable in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z]$ iff it is indecomposable in $\mathcal{O}_{\mathbb{C}^n,0}$.

2.9 The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian (Prop 1.1.18).

- Proof: Any $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ is generated by $I \cap (\mathcal{O}_{\mathbb{C}^{n-1},0}[z])$ and any Weierstrass polynomial $g \in I$ (by division version of WPT).

2.10 Warning: If $\emptyset \neq U \subset \mathbb{C}^n$, $n > 0$ then $\mathcal{O}_{\mathbb{C}^n}(U)$ is not noetherian.

- Exercise: Give an example.

3 Germs of sets and intro to de Rham/Hodge theory

3.1 Germ of a set [Hu, 1,1,21] and the associated ideal $I(X)$ [Hu, 1.1.24] in the local ring.

- if $X_1 \subset X_2$ then $I(X_1) \supset I(X_2)$

3.2 The germ of the set $Z(J) = Z(f_1, f_2, \dots f_m)$ [Hu, Lem. 1.1.25] defined by an ideal $J = (f_1, f_2, \dots f_m) \subset \mathcal{O}_{\mathbb{C}^n,0}$.

- if $J_1 \subset J_2$ then $Z(J_1) \supset Z(J_2)$

3.3 $X \subset Z(I(X))$ for any set germ,

3.4 If X is analytic (i.e. of the form $X = Z(J)$), then $X = Z(I(X))$

- If $X = Z(f_1, f_2, \dots, f_k)$ then, then $f_1, f_2, \dots, f_k \in I(X)$, hence $X = Z(f_1, f_2, \dots, f_k) \supset Z(I(X))$.

3.5 In general $I(Z(J)) \supset J$.

- Hilbert nullstellensatz: $I(Z(J)) = \sqrt{J}$ (see sketch of a proof in Huybrechts p.20).

3.6 Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be indecomposable, then if $f_{|Z(g)} = 0$, then g divides f [Hu, Cor. 1.1.19]

- Proof from the division version of Weierstrass preparation theorem
- Key step: if g is indecomposable Weierstrass polynomial, then $g_w(z)$ generically (w.r. to w) has distinct roots.
- Corollary: Nullstellensatz for $J = (g)$.

Differential forms and de Rham cohomology – summary

3.7 Global differential forms on a C^∞-manifold M will be denoted by $A^\bullet(M) = \bigoplus_{k=0}^{\dim M} A^k(M)$. (The notation $\Omega^\bullet(M)$ is reserved for holomorphic forms.)

3.8 $A^\bullet(M)$ is a commutative algebra with gradation and a differential

- "commutativity with gradation" $xy = (-1)^{\deg(x)\deg(y)} xy$
- The differential satisfies the Leibniz rule $d(x \wedge y) = dx \wedge y + (-1)^{\deg(x)} x \wedge dy$

3.9 The linear space $A^k(M)$ is the space of the global sections of the sheaf of differential forms $\mathcal{A}^k_M$.

3.10 $\underline{\mathbb{R}}_M \hookrightarrow \mathcal{A}^0_M \to \mathcal{A}^1_M \to \mathcal{A}^2_M \to \dots$ is a resolution of the constant sheaf $\underline{\mathbb{R}}_M$. The resolution consists of soft (in particular acyclic) sheaves.
— Resolution – it means, that the complex of sheaves is locally exact. This is Poincaré lemma.
— The sheaf $\mathcal{S}$ is soft if for any closed $F \subset M$ the restriction map

$$\mathcal{S}(M) \to \mathcal{S}(F) = \lim_{V \supset F} \mathcal{S}(V) \qquad (V \text{ - open neighbourhood of } F)$$

is surjective. One can compute cohomology using soft resolutions for sheave over paracompact spaces.

- Therefore

$$H^k(A^\bullet(M), d) = H^k(M; \underline{\mathbb{R}}_M) \simeq \check{H}^k(M, \underline{\mathbb{R}}_M) \simeq H^k_{sing}(M; \mathbb{R}) .$$

That is: sheaf-theoretic, Čech and singular cohomology are the same as the de Rham cohomology, provided that we take coefficients in $\mathbb{R}$. We loose information about the torsion. The cohomology groups are denoted by $H^k(M)$, we skip $\mathbb{R}$ in the notation.

3.11 Convention: for a group with gradation we use * to denote the gradation, e.g.

$$H^*(M) = \bigoplus_{k=0}^{\dim M} H^k(M) .$$

If the group with gradation has a differential, then we rather write $\bullet$ to remember that it is a complex, e.g.

$$A^\bullet(M) = \bigoplus_{k=0}^{\dim M} A^k(M).$$

3.12 The complex $A^\bullet(M)$ with the product of forms is a commutative differential graded algebra over $\mathbb{R}$, in short $\mathbb{R}$-CDGA. "Commutative" means commutative for homogeneous elements with signs depending on the gradations.

• Wedge product induces multiplication in cohomology

$$H^k(M) \times H^\ell(M) \to H^{k+\ell}(M)\,,$$

so $H^*(M)$ is CGA.

3.13 If M is compact, $n = \dim M$ and M has a chosen orientation, then the integral of n-forms induces a map $\int_M : H^n(M) \to \mathbb{R}$. If M is connected, then $\int_M$ is an isomorphism.

3.14 (Poincaré Duality) if M is compact, oriented of dimension n, then the bilinear form

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R}$$

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$$

is a perfect pairing, i.e. it induces an isomorphism $H^k(M) \simeq H^{n-k}(M)^*$

3.15 if M is oriented (not necessarily compact), then we consider cohomology with compact supports

$$H^k_c(M) = H^k(A^\bullet_c(M)).$$

Then

$$\int_M - \wedge - : H^k_c(M) \times H^{n-k}(M) \to \mathbb{R}$$

is defined and it is a nondegenerate 2-linear form.

• It induces an isomorphism $H^k_c(M) \simeq H^{n-k}(M)^*$. The later term can be identified with homology $H_{n-k}(M)$.

3.16 Having a Riemannian metric on a compact manifold allows to define **harmonic** forms $\mathcal{H}^k(M)$ (see 3.23). The harmonic forms are closed and the resulting map $\mathcal{H}^k(M) \to H^k(M)$ is an isomorphism. However the product of harmonic forms does not have to be harmonic.

Hodge theory for $\mathbb{C}^\infty$ manifolds

See [Huybrechts, Appendix A].
Suppose an orientable manifold M is equipped with Riemannian metric, i.e. a scalar product at each tangent space T_xM. Let $n = \dim M$.

3.17 Volume form is denoted by $vol \in A^n(M)$. It depends on the choice of the Riemannian metric and orientation.

3.18 Hodge star:

$$* : A^k(M) \to A^{n-k}(M)$$

is defined pointwise. For $x \in M$

$$* : \Lambda^k T^*_x M \to \Lambda^{n-k} T^*_x M$$

is defined by the property

$$a \wedge *b = \langle a, b\rangle vol$$

for each $a, b \in \Lambda^k T^*_x M$.

3.19 We have (an exercise)
(i) $*^2 = (-1)^{k(n-k)}$ on k-forms.
(ii) $\langle \alpha, *\beta\rangle = (-1)^{k(n-k)}\langle *\alpha, \beta\rangle$,

3.20 Let's define $d^* = (-1)^{n(k+1)+1} * d* : A^k(M) \to A^{k-1}(M)$.

(I apologize for the sign, but this is how it is.)

3.21 For compact manifold M, $a \in A^{k-1}(M)$, $b \in A^k(M)$ we have

$$\langle da, b\rangle_M = \langle a, d^* b\rangle_M$$

We say that d^* is formally adjoint to d.

Proof. First note

$$0 = \int_M d(a \wedge b) = \int_M da \wedge b + (-1)^{k-1} \int_M a \wedge db\,,$$

hence

$$\int_M da \wedge b = \pm \int_M a \wedge db\,.$$

Now we check by the definitions of * and d^*

$$\begin{aligned}\langle a, d^* b\rangle_M &= \pm \int_M \langle a, * d * b\rangle vol \\ &= \pm \int_M a \wedge * * d * b \\ &= \pm \int_M a \wedge d * b \\ &= \pm \int_M da \wedge * b \\ &= \pm \int_M \langle da, b\rangle vol = \pm \langle da, b\rangle_M\end{aligned}$$

It is an exercise, to check that finally there is no sign.

3.22 Laplasian on forms is defined by

$$\Delta = dd^* + d^* d$$

It can be interpreted as the ,,super-commutator" $[d, d^*]_s$.

• In general the supercommutator of elements of a graded algebra $C^* = \bigoplus_{k\in\mathbb{Z}} C^k$ is defined by

$$[\phi, \psi]_s = \phi\psi - (-1)^{k\ell}\psi\phi \qquad \text{if} \qquad \phi \in C^k, \quad \psi \in C^\ell.$$

• The differentials d and d^* are treated as elements of the graded algebra $\mathrm{End}(A^*(M))$, they are of odd degree (i.e. lie in the odd gradations). With this point of view the Laplasian is a supercomutator.

3.23 Harmonic forms: $\mathcal{H} := ker\Delta$.

3.24 The operator $\Delta = dd^* + d^* d$ is formally self-adjoint

$$(\Delta a, b) = (a, \Delta b).$$

3.25 For a compact oriented C^∞-manifold M the following holds in $A^\bullet(M)$:
1) $\mathcal{H} = ker(d) \cap ker(d^*)$
2) $ker(d^*) = im(d)^\perp$, $\quad ker(d) = im(d^*)^\perp$, $\quad ker(\Delta) = im(\Delta)^\perp$, $\quad$ (hence $\mathcal{H} = ker(d) \cap im(d)^\perp$)
3) the spaces $\mathcal{H}$, $im(d)$ and $im(d^*)$ are perpendicular.

This is just a linear algebra, a statement about a vector space with a scalar product. We will write (a, b) instead of $\langle a, b\rangle_M$.

Proof:
1) suppose $a \in ker(\Delta)$:

$$0 = (\Delta a, a) = (dd^*a, a) + (d^*da, a) = (d^*a, d^*a) + (da, da) = ||d^*a||^2 + ||da||^2$$

2) Let $P = d$, d^* or Δ. If $a \in ker(P^*)$ then $0 = (P^*a, b) = (a, Pb)$, hence $a \in im(P)^\perp$.
Conversely, if $a \in im(P)^\perp$, then $0 = (a, PP^*a) = ||P^*a||^2$, so $P^*a = 0$.
3) It remains to show that the spaces $im(d)$ and $im(d^*)$ are perpendicular $(d^*a, db) = (a, d^2b) = 0$.
(Here we used that $d^2 = 0$, all the rest was an abstract properties of formally adjoint operators.)

3.26 Hodge decomposition (hard analysis)

$$A^\bullet(M) = \underbrace{im(d) \oplus \mathcal{H}}_{ker(d)} \oplus im(d^*).$$

This decomposition is orthogonal.

• The decomposition follows from a general property of elliptic differential operators, which we will not prove. We would have to extend the space of C^∞ forms and consider Sobolev spaces. See [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Cambridge Studies in Advanced Mathematics. Theorem 5.22, p.128-9]. For any *elliptic* operator $P : C^\infty(E) \to C^\infty(F)$

$$C^\infty(E) = ker(P) \oplus P^*(C^\infty(F))\,.$$

(Exercise: prove the corresponding statement for a linear map between finite dimensional spaces.)

• In our case $P = \Delta$, $P^* = \Delta$

$$A^*(M) = \mathcal{H} \oplus \Delta(A^*(M))\,.$$

Moreover we have

$$im(\Delta) \subset im(d) + im(d^*).$$

But from orthogonality $(im(d) \oplus im(d^*)) \cap \mathcal{H} = 0$, hence

$$im(\Delta) = im(d) \oplus im(d^*).$$

3.27 Corollary 1: $\mathcal{H} \to H^*(M)$ is an isomorphism.
Moreover: if $\Delta(a) = 0$ and $a' = a + db$, then $||a'|| \geq ||a||$.
Any harmonic form is the representative of its cohomology class, which has the smallest norm.

3.28 Corollary 2: Tricky proof of the Poincaré duality: Let $[\alpha] \neq 0 \in H^*(M)$, then there exists a class $[\beta]$ (in the complementary gradation) such that $\int_M \alpha \wedge \beta \neq 0$.

• Proof: let's assume that α is harmonic. Set $\beta = *\alpha$. Then β is harmonic as well $(d(*\alpha) = \pm * d^*(\alpha) = 0$ and $d^*(*\alpha) = \pm * d(\alpha) = 0)$. We have

$$\int_M \alpha \wedge *\alpha = \int_M ||\alpha||^2 vol = ||\alpha||^2_M\,.$$

4 Heat equation, differential forms on complex manifold

4.1 Heat equation $\alpha : \mathbb{R}_+ \to A^*(M)$ with the initial condition $\alpha(0) = \alpha$

$$\frac{d}{dt}\alpha(t) = -\Delta\alpha(t)\,,$$

4.2 Heuristic argument: Suppose V is a finite dimensional spaces with a scalar product, and suppose a linear map $\Delta : V \to V$ is self adjoint. Then

$$V = ker(\Delta) \overset{\perp}{\oplus} im(\Delta)$$

and the differential equation for a curve $\alpha : [0, +\infty) \to V$

$$\frac{d}{dt}\alpha(t) = -\Delta\alpha(t)$$

with arbitrary initial condition $\alpha(0) = \alpha_0$ has a solution.

- Assuming the eigenvalues of Δ are not negative, the $\lim_{t\to 0}\alpha(t)$ exists and lies in $\ker(\Delta)$. The map

$$\alpha_0 \to \lim_{t\to 0}\alpha(t)$$

is the orthogonal projection $V \to ker(\Delta)$.

- The same can be repeated for differential forms, but since $A^*(M)$ is of infinite dimension, an analytic argument is needed. See [D. Arapura, Algebraic Geometry over the Complex Numbers] §8.3.

$1\frac{1}{2}$–linear algebra — Huybrechts 1.2.1–1.2.6

4.3 Motivation: The tangent space to a complex manifold $V = T_xM$ has a structure of a complex vector space. We will consider V as a real vector space with and automorpism given by the multiplication by i, denoted by I. Further we tensor V with $\mathbb{C}$ over R, so we have

$$i : v \otimes 1 \mapsto v \otimes i,$$

$$I : v \otimes 1 \mapsto iv \otimes 1$$

and induced actions on $(\Lambda^k V^*)_{\mathbb{C}} = \Lambda^k(V_{\mathbb{C}})^*$.

4.4 Definition. A complex structure on a real vector space is an automorphism I satisfying $I^2 = -id$.

4.5 Any complex structure decomposes $V_{\mathbb{C}} := V \otimes \mathbb{C}$ into eigenspaces

$$V_{\mathbb{C}} = V_i + V_{-i}\,.$$

Notation

$$V^{1,0} = V_i\,, \qquad V^{0,1} = V_{-i}\,.$$

Necessarily $\dim V$ is even and one can find a real basis $e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_n$ of V, such that $I(e_k) = f_k$, $I(f_k) = -e_k$.

- The vectors $e_k - if_k$ form a basis of the complex space $V^{1,0}$: $\quad I(e_k - if_k) = f_k + ie_k = i(e_k - if_k)$
- The vectors $e_k + if_k$ form a basis of the complex space $V^{0,1}$: $\quad I(e_k + if_k) = f_k - ie_k = -i(e_k - if_k)$

4.6 We are more concerned about the dual space: Let V be a real vector space and $I \in \mathrm{End}(V)$ be a complex structure. Let

$$\mathbf{I} : \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})\,.$$

$$\mathbf{I}(f) = f \circ I\,.$$

which has the eigenspaces

$$\begin{aligned}(V^*)^{1,0} =& \{f \in \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \;:\; \forall v \in V \quad f(Iv) = if(v)\}, \qquad (\mathbb{C}\text{-linear forms})\\ (V^*)^{0,1} =& \{f \in \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \;:\; \forall v \in V \quad f(Iv) = -if(v)\}, \quad (\text{antilinear forms}).\end{aligned}$$

For $f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ let $f_{\mathbb{C}}$ denote the $\mathbb{C}$-linear extension to $V_{\mathbb{C}}$. Let $V^{1,0} \subset V_{\mathbb{C}}$ (resp. $V^{0,1}$) be the eigenspace of I with the eigenvalue i (resp. $-i$).

- If $f \in (V^*)^{0,1}$, then $f_{\mathbb{C}}(V^{1,0}) = 0$, if $f \in (V^*)^{1,0}$, then $f_{\mathbb{C}}(V^{0,1}) = 0$.
- The map

$$(V^*)^{1,0} \to (V^{1,0})^*,$$

$$f \mapsto (f_{\mathbb{C}})_{|V^{1,0}}$$

is an isomorphism.

- Similarly the map

$$(V^*)^{0,1} \to (V^{0,1})^*,$$

$$f \mapsto (f_{\mathbb{C}})_{|V^{0,1}}$$

is an isomorphism.

4.7 The dual basis is denoted by

$$dx_k := e_k^*\,, \quad dy_k := f_k^*\,.$$

We define

$$dz_k := dx_k + idy_k\,, \quad d\overline{z}_k := dx_k - idy_k\,.$$

The 1-forms dz_k are the basis of $(V^*)^{10}$, and $d\overline{z}_k$'s are the basis of $(V^*)^{01}$.

4.8 We have a $\mathbb{C}$-linear isomorphism

$$(V^*, I) \overset{\Phi}{\simeq} ((V^*)^{10}, i), \qquad \Phi(f)(v) = f(v) - if(Iv))\,,$$

$$\Phi(If)(v) = f(Iv) - if(I^2v) = f(Iv) + if(v) = i(f(v) - if(Iv)) = i\Phi(f)(v)\,.$$

And an anti-linear isomorphism:

$$(V^*, I) \overset{\psi}{\simeq} ((V^*)^{01}, i), \qquad \Psi(f)(v) = f(v) + if(Iv))\,,$$

$$\Psi(If)(v) = f(Iv) + if(I^2v) = f(Iv) - if(v) = -i(f(v) + if(Iv)) = -i\Psi(f)(v)\,.$$

Exterior forms — Huybrechts 1.2.7–1.2.10

4.9 General formula: if $W = W_1 \oplus W_2$, then

$$\Lambda^k W = \bigoplus_{p+q=k} \Lambda^p W_1 \otimes \Lambda^q W_2\,.$$

- We apply this general isomorphism to $W = V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}$.

4.10 The exterior forms of the type (p, q):

$$\Lambda^k V_{\mathbb{C}}^* = \bigoplus_{p+q=k} \Lambda^{pq} V^*\,, \qquad \Lambda^{pq} V^* := \Lambda^p (V^{1,0})^* \wedge \Lambda^q (V^{0,1})^* \simeq \Lambda^p (V^{1,0})^* \otimes \Lambda^q (V^{0,1})^*\,.$$

– Conjugation acts on $\Lambda^k(V^* \otimes \mathbb{C}) = (\Lambda^k V^*) \otimes \mathbb{C}$. We have

$$\overline{\Lambda^{pq}} = \Lambda^{qp}.$$

– The operator I (we neglect `\bf`) acts on $\Lambda^{p,q}V^*$ via multiplication by $i^{(p-q)}$.

4.11 We apply the above decomposition to $V = T_x M$, where M is a complex manifold $x \in M$. This gives a direct sum decomposition of the bundle of forms $\Lambda^k TM \otimes \mathbb{C}$, and results in the decomposition of the space of sections, i.e. differential forms

$$A^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} A^{pq}(M)$$

as well as a decomposition of sheaves

$$\mathcal{A}_M^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{A}_M^{pq}\,.$$

5 Dolbeault cohomology and deRham cohomology

5.1 An almost complex manifold (M, I) is a pair, where M is a real C^∞-manifold and $I \in End(TM)$ a (smooth) tensor satisfying $I^2 = -id$. That means that each T_pM has a structure of a vector space.

5.2 Example: If M is a complex manifold, then each tangent space T_pM in a natural way is a complex vector space.

- Not every almost complex structure arises in that way. The condition sufficient and necessary is vanishing of the Nijenhuis tensor $N_I \in T^*M \otimes T^*M \otimes \mathbb{C}$. That is the **Newlander-Nirenberg theorem**. *We skip the formulation and the proof. See* [Huybrechts 2.6.19].

5.3 From now on we assume that M is a complex manifold, then

$$d(A^{p,q}(M)) \subset A^{p+1,q}(M) \oplus A^{p,q+1}(M)$$

- Locally we have coordinates $z_1, z_2, \ldots, z_n$ and complex forms

$$dz_1, dz_2, \ldots, dz_n \in A^{1,0}(M), \qquad d\bar{z}_1, d\bar{z}_2, \ldots, d\bar{z}_n \in A^{0,1}(M)\,.$$

For a multiindex $A \subset \{1, 2, \ldots, n\}$ let dz_A and $d\bar{z}_A$ be the corresponding wedges od dz_i's and $d\bar{z}_i$'s. We define

$$\partial(f\, dz_A \wedge d\bar{z}_B) = \sum_{k=1}^{n} \tfrac{d}{dz_k} f dz_k \wedge dz_A \wedge d\bar{z}_B\,,$$

$$\overline{\partial}(f\, dz_A \wedge d\bar{z}_B) = \sum_{k=1}^{n} \tfrac{d}{d\bar{z}_k} f d\bar{z}_k \wedge dz_A \wedge d\bar{z}_B\,,$$

$$d = \partial + \overline{\partial}, \quad \partial^2 = 0 = \overline{\partial}^2, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0\,.$$

5.4 Dolbeault complex: for $0 \leq p \leq \dim_{\mathbb{C}} M$ we have a complex

$$0 \to A^{p,0}(M) \xrightarrow{\overline{\partial}} A^{p,1}(M) \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} A^{p,\dim M}(M) \to 0,,$$

$$ker(\overline{\partial} : A^{p,0}(M) \to A^{p,1}(M)) = \Omega^p(M)\,.$$

Here $\Omega^p(M)$ denotes the form of the type $(p, 0)$ with holomorphic coefficients.

5.5 We define Dolbeault cohomology [Huybrechts 2.6.20]:

$$H^q_{Dol}(M; \Omega^p) := H^q(A^{p,\bullet}(M), \overline{\partial})$$

5.6 Holomorphic Poincaré lemma [Hyubrechts 1.3.7]: the complex of sheaves

$$0 \to \Omega^p_M \to A^{p,0} \to A^{p,1} \to A^{p,2} \to \ldots$$

is exact.

- This means that if $\bar{\partial}\alpha = 0$, $\alpha \in A^{p,q}(U)$, then *locally* there exists β such that $\bar{\partial}\beta = \alpha$, i.e. for each point $p \in U$ there exists $V \subset U$, $p \in V$ and $\beta \in A^{p,q-1}(V)$ such that $\bar{\partial}\beta = \alpha_{|V}$.

Before giving a proof of the holomorphic Poncaré lemma we explain its importance.

5.7 Sheaf cohomology - a summary (MGiT recollection)
See [Huybrechts, Appendix B]

5.8 Cohomology with the coefficients in a sheaf $\mathcal{F}$ (as the derived functor of "global sections"):
1) we find a resolution of $\mathcal{F}$, i.e. an exact complex

$$0 \to \mathcal{F} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \dots$$

where each sheaf $\mathcal{A}^k$ is acyclic (in the manifold case *soft* is enough)
2) we apply the *functor of global sections*

$$\Gamma(\mathcal{A}^0) \to \Gamma(\mathcal{A}^1) \to \Gamma(\mathcal{A}^2) \to \dots$$

This complex is no longer exact.
3) W compute cohomology:

$$H^k(M;\mathcal{F}) = H^k(\Gamma(\mathcal{A}^\bullet))\,.$$

We have $H^0(M;\mathcal{F}) = \Gamma(\mathcal{F})$, because the functor Γ is left-exact.

5.9 Suppose M is a C^∞-manifold. Any sheaf which is a module over the ring of C^∞-functions is soft.
• The complex of C^∞-forms on C^∞-manifold $\mathcal{A}^0_M \to \mathcal{A}^1_M \to \mathcal{A}^2_M \to \dots$ is a resolution of the sheaf $ker(d : \mathcal{A}^0_M \to \mathcal{A}^1_M) = \mathbb{R}_M$, the sheaf of locally constant functions.

5.10 The sheaves $\mathcal{A}^{p,q}$ are $C^\infty(M)$-modules, hence they are soft as well. The holomorphic Poincaré lemma says that together with the differential $\overline{\partial}$ it is a resolution of

$$\Omega^p_M = ker(\overline{\partial} : \mathcal{A}^{p,0} \to \mathcal{A}^{p,1})\,.$$

We see that Ω^p_M is the sheaf of holomorphic forms. It consist of forms, which are locally presented as

$$\sum_{|A|=p} f_A(z) dz_A\,,$$

where f_A are holomorphic functions.

5.11 The complex of global sections is called the Dolbeault complex. Its cohomology is called Dolbeault cohomology.
•

$$H^k(M;\Omega^p) = H^k(A^{p,\bullet}(M))$$

i.e the Dolbeault cohomology is the sheaf cohomology in the sense of the homological algebra.

5.12 If M is a complex manifold, then $\mathcal{A}^\bullet_\mathbb{C} = \bigoplus_{p+q=\bullet} \mathcal{A}^{p,q}_M$ with the differential $d = \partial + \overline{\partial}$ is a resolution of the constant sheaf $\mathbb{C}_M$.
• The cohomology of the complex $A^\bullet(M)_\mathbb{C}$ is the de Rham cohomology, which coincides with sheaf cohomology. This is the de Rham theorem.

5.13 We have a bi-complex

$$\begin{array}{ccccccc}
 & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & \\
\stackrel{\partial}{\to} & A^{p-1,q+1}(M) & \stackrel{\partial}{\to} & A^{p,q+1}(M) & \stackrel{\partial}{\to} & A^{p+1,q+1}(M) & \stackrel{\partial}{\to} \\
 & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & \\
\stackrel{\partial}{\to} & A^{p-1,q}(M) & \stackrel{\partial}{\to} & A^{p,q}(M) & \stackrel{\partial}{\to} & A^{p+1,q}(M) & \stackrel{\partial}{\to} \\
 & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & \\
\stackrel{\partial}{\to} & A^{p-1,q-1}(M) & \stackrel{\partial}{\to} & A^{p,q-1}(M) & \stackrel{\partial}{\to} & A^{p+1,q-1}(M) & \stackrel{\partial}{\to} \\
 & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow &
\end{array}$$

• For $p \geq 0$ define the Hodge'a filtration on forms

$$F^pA^k(M) = \bigoplus_{p'+q=k,\ p'\geq p} A^{p',q}(M)\,.$$

Claim: $F^pA^\bullet(M)$ is a subcomplex of $A^\bullet(M)$.

• The resulting filtration in cohomology $H^k(M;\mathbb{C})$ is defined as the images if cohomologies of truncated complexes

$$F^pH^k(M;\mathbb{C}) = im(H^k(F^pA^\bullet(M)) \to H^k\big(A^\bullet_{\mathbb{C}}(M)\big))\,.$$

5.14 We have

$$F^pA^k(M)/F^{p+1}A^k(M) \simeq A^{p,k-p}(M)\,.$$

• The quotient map is a map of complexes (with a shift of the gradation)

$$(F^pA^\bullet(M), d) \to (A^{p,\bullet}(M), \overline{\partial})$$

• Passing to cohomology:

$$H^{p+q}((F^pA^\bullet/F^{p+1}A^\bullet)(M)) = H^q(A^{p\bullet}(M)) = H^q(M;\Omega^p_M)\,.$$

The relation between cohomologies of the quotients with cohomology of the entire sheaf is given by the spectral sequence

$$E_1^{p,q} = H^{p+q}(F^pA^\bullet(M)/F^{p+1}A^\bullet(M)) = H^q(M;\Omega^p_M) \ \Rightarrow H^{p+q}(M;\mathbb{C}).$$

5.15 Holomorphic Poincaré lemma [Hyubrechts 1.3.7], a clear proof in [Arapura 6.2.1].

5.16 1-dimensional case: Given a (0,1)-form $\alpha = f(z)d\bar{z}$ on a disk. It is $\overline{\partial}$-closed (of course). We have to show that $\alpha = \overline{\partial}\beta$ for some (0,0)-form, i.e. $\beta = g(z)$ is just a C^∞ function such that $\alpha = \overline{\partial}g$,

• This means $f = \frac{\partial}{\partial\bar{z}}g$.

• Since we work with germs, we can possibly shrink the radius of the disk.

5.17 Assume that f is defined on $\mathbb{C}$, since it is a C^∞-function (shrinking eventually the original disk we can extend it anyhow). Also assume that f has compact support. Let

$$g(z) = \tfrac{1}{2\pi i}\int_{\mathbb{C}} \tfrac{f(w)}{w-z}dw\wedge d\bar{w}\,.$$

• Exercise: This integral is convergent, because $\int_{D_\varepsilon} |\frac{1}{w}|dw\wedge d\bar{w} = 2\pi\varepsilon < \infty$.

• We change variables

$$g(z) = \tfrac{1}{2\pi i}\int_{\mathbb{C}} \tfrac{f(w)}{w-z}dw\wedge d\bar{w} = \tfrac{1}{2\pi i}\int_{\mathbb{C}} f(u+z)\tfrac{1}{u}du\wedge d\bar{u}$$

$$\tfrac{\partial}{\partial\bar{z}}g(z) = \tfrac{1}{2\pi i}\int_{\mathbb{C}} \tfrac{\partial f}{\partial\bar{u}}(u+z)\tfrac{1}{u}du\wedge d\bar{u}\,.$$

• Claim $\frac{\partial g}{\partial\bar{z}}(z) = f(z)$. We can assume for simplicity $z = 0$.

• Let $\omega = \frac{1}{2\pi i}\frac{f(u)}{u}du$. Note that

$$d\omega = -\tfrac{1}{2\pi i}\tfrac{\partial f}{\partial\bar{u}}(u)\tfrac{1}{u}du\wedge d\bar{u}\,.$$

Let D_ε be the disk with the center at 0 and radius ε. Then

$$\lim_{\epsilon\to 0}\int_{\partial D_\epsilon} \omega = f(0)\,.$$

On the other hand

$$\int_{\partial D_\epsilon} \omega = -\int_{\mathbb{C}\setminus D_\epsilon} d\omega = \tfrac{1}{2\pi i}\int_{\mathbb{C}\setminus D_\epsilon} \tfrac{\partial f}{\partial\bar{u}}(u)\tfrac{1}{u}du\wedge d\bar{u} \stackrel{\varepsilon\to 0}{\longrightarrow} \tfrac{\partial}{\partial\bar{z}}g(0)\,.$$

5.18 It is an analogy with the real case: for a real (compactly supported) $f : \mathbb{R} \to \mathbb{R}$ we define the primitive function

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{+\infty} K(t-x)f(t)dt,$$

where

$$K(t) = \begin{cases} 0 \text{ if } t < 0 \\ 1 \text{ if } t \geq 0 \end{cases}, \qquad \text{and} \qquad K'(t) = \delta_0 \,.$$

So the primitive function is expressed by the convolution with K, i.e $F(x) = (K * f)(x)$.

5.19 Poincaré lemma in many variables: See exercises 21.11.2025. (We apply convolution consecutively, for each coordinate.)

5.20 The most important cohomology class – Fubini-Study form. As it is known from the course in algebraic topology $H^*(\mathbb{P}^2)$ is generated by a class in gradation 2. This class will play further a crucial role, also after restriction to any complex submanifold in $\mathbb{P}^n$.

• Let $M = \mathbb{P}^n$, we define a form $\omega \in A^{1,1}(\mathbb{P}^n)$ by the following formula in the standard affine coordinates of j-th affine chart $u_k = z_k/z_j$

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}\log(1 + \sum_{k=1}^{n} |u_k|^2)$$

• Claim: the above formula does not depend on j. In the j'-coordinates $u'_k = \frac{u_k}{u_{j'}}$ (for convenience we set $u_j = 1$, $u'_{j'} = 1$)

$$\sum_{k=1}^{n} |u'_k|^2 = \frac{1}{|u_{j'}|^2}\left(\sum_{k=1}^{n} |u_k|^2\right) = \frac{1}{u_{j'}\bar{u}_{j'}}\left(1 + \sum_{k=1}^{n} |u_k|^2\right)$$

$$\log\left(1 + \sum_{k=1}^{n} |u_k|^2\right) = -\log(u_{j'}) - \log(\bar{u}_{j'}) + \log\left(1 + \sum_{k=1}^{n} |u_k|^2\right)$$

and

$$\partial\bar{\partial}u_{j'} = 0 \qquad \partial\bar{\partial}\bar{u}_{j'} = -\bar{\partial}\partial\bar{u}_{j'} = 0\,.$$

Obviously $d\omega = 0$.

5.21 For $n = 1$

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}log(1 + |w|^2) = \frac{i}{2\pi}\frac{1}{(1 + w\bar{w})^2}dw \wedge d\bar{w} = \frac{1}{\pi}\frac{1}{(1 + x^2 + y^2)^2}dx \wedge dy$$

• $\int_{\mathbb{P}^1} \omega = 1$, hence $[\omega] \in H^2(\mathbb{P}^1; \mathbb{R})$ is integral, i.e. it comes from $H^2(\mathbb{P}^1; \mathbb{Z})$.

Motivation leading to the notion of Čech cohomology :

[B. V. Shabath, Introduction to complex analysis II, Chapter IV].

5.22 Additive Cousin Problem: find a global meromorphic function with prescribed poles.
Let $M = \bigcup U_i$ be a covering. On each U_i there is given a meromorphic function f_i. We assume that the differences $g_{ij} = (f_i)_{|U_i \cap U_j} - (f_j)_{|U_i \cap U_j}$ are holomorphic. Does there exist a meromorphic function f on M such that each difference $f_{|U_i} - f_i$ is holomorphic?

5.23 Multiplicative Cousin Problem:
Let $\{U_i\}_{i \in I}$ be a covering of M. On each U_i there is given a meromorphic function f_i. We assume that the quotients $g_{ij} = \frac{(f_i)_{|U_i \cap U_j}}{(f_j)_{|U_i \cap U_j}}$ are holomorphic. Does there exist a meromorphic function f on M such that each quotient $\frac{f_{|U_i}}{f_i}$ is holomorphic?

5.24 The answer is in the language of Čech cohomology. For a covering $\mathscr{U} = \{U_i\}$ the Čecha complex is defined by:

$$\check{C}^k(\mathscr{U}) = \prod_{i_0<i_1<\cdots<i_k} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k})\,.$$

Notation: for a multiindex $I = \{i_0 < i_1 < \cdots < i_k\}$ let $U_I = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k}$. For $\{s_I\} \in \check{C}^{k-1}(\mathscr{U})$ define the differential

$$d(\{s_I\})_J = \sum_{a=1}^{k} (-1)^a (s_{J\setminus j_a})_{|U_J}$$

For example

$$d(\{s_i\})_{j_0,j_1} = (s_{j_1})_{|U_{j_0,j_1}} - (s_{j_0})_{|U_{j_0,j_1}}$$

$$d(\{s_{i_0,i_1}\})_{j_0,j_1,j_2} = s_{j_1,j_2} - s_{j_0,j_2} + s_{j_0,j_1} \qquad \text{restricted to } U_{j_0,j_1,j_2}$$

5.25 Čech cohomology is defined by $\check{H}^k(\mathscr{U};\mathcal{F}) = H^k(\check{C}^\bullet(\mathscr{U};\mathcal{F}), d)$.

5.26 Additive Cousin Problem : Let $\mathcal{F} = \mathcal{O}_M$, the collection of functions $\{g_{i,j}\} \in \check{C}^1(\mathscr{U};\mathcal{O}_M)$ satisfies the cocycle condition:

$$g_{ij} - g_{ik} + g_{jk} = 0.$$

It defines an element of Čech cohomology of the covering $H^1(\{U_i\};\mathcal{O}_M)$. The cohomology class is trivial if the cocycle is a coboundary, i.e. there exists a collection of elements $h_i \in \mathcal{O}_M(U_i)$ such that $g_{ij} = h_j - h_i$.

- the Cousin problem has a solution if and only if the cohomology class $[g_{ij}] = 0$.

Proof: If $g_{ij} = h_j - h_i$, then the meromorphic functions $\tilde{f}_i = f_i + h_i$ agree at the intersections:

$$\tilde{f}_i - \tilde{f}_j = f_i + h_i - f_j + h_j \qquad \text{on } U_i \cap U_j\,.$$

(The converse - exercise.)

5.27 Multiplicative Cousin problem has a positive solution if the cocycle g_i/g_j defines the trivial class in $H^1(\{U_i\};\mathcal{O}^*_M)$.

5.28 Passing to a finer cover defines a map of Čecha cohomology (it does not depend on inscribing function).

5.29 Theorem: If M is paracompact, then

$$\lim_{\substack{\longrightarrow \\ \mathscr{U}}} \check{H}^k(\mathscr{U};\mathcal{F}) \simeq H^k(M;\mathcal{F})$$

(The RHS is in the sense of homological algebra.)

5.30 If the covering is acyclic (i.e. $H^k(U_I;\mathcal{F}) = 0$ for any multiindex I and $k > 0$) then

$$H^k(\{U_i\};\mathcal{F}) \simeq H^k(M;\mathcal{F}).$$

5.31 Sufficient conditions for being acyclic:

- For locally constant sheaves on topological spaces: if all U_I are contractible,
- For coherent sheaves in algebraic geometry: if U_I are affine,
- For coherent sheaves in analytic geometry: if U_I are Stein spaces

Definition $U \subset M$ is Stein if:
– for any pair of points $p, q \in U$ there exists an analytic function $f \in \mathcal{O}_U$ such that $f(p) \neq f(q)$.
– (holomorphic convexity) for any compact set $K \subset U$ the set

$$\bar{K} := \{p \in U \mid \forall f \in \mathcal{O}_U \ |f(p)| \leq sup_{q\in K}|f(q)|\ \}$$

is compact.

5.32 In the cousin problems one can pass to a finer coverings. Since $H^1(\mathbb{P}^n;\mathcal{O}_M) = 0$, so on $\mathbb{P}^n$ the additive Cousin problem has always a positive solution. On curves of positive genus - not always: $genus = \dim H^1(C;\mathcal{O}_C)$.

6 GAGA

See http://www.mimuw.edu.pl/∼aweber/ComplexManifolds2025/GAGA.pdf

7 Hermitian linear algebra and Kähler manifolds

[Huybrechts §1.2] Suppose (V, I) is a real vector space with a complex structure.

7.1 Hermitian product

$$V \otimes V \to \mathbb{C}$$

$$\langle\!\langle v, w \rangle\!\rangle = \langle v, w\rangle - i\omega(v, w)$$

consists of:
– I-invariant scalar product $\langle v, w\rangle$,
– I-symplectic form $\omega(v, w)$
– the scalar product and the symplectic form determine each other $\omega(v, w) = \langle I(v), w\rangle = -\langle v, I(w)\rangle$.

7.2 The volume form is defined as the wedge of an orthonormal (positively oriented) basis vectors of V^*. The orientation is determined by the complex structure.

- suppose $\dim_{\mathbb{C}}(V) = n$

$$vol = (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n) = (\tfrac{i}{2})^n (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n)$$

-

$$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k = \tfrac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k\,.$$

$$\omega^n = n!vol\,.$$

7.3 The following calculus will be made in $\Lambda V^* := \Lambda_{\mathbb{R}} V^*$ or in $\Lambda V^*_{\mathbb{C}} := \Lambda_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C})^*$

- Remark: the form ω belongs to $\Lambda^2 V^* \cap \Lambda^{11} V^* \subset \Lambda^2 V^*_{\mathbb{C}}$.

7.4 Exercise $\Lambda^{10} \perp \Lambda^{01}$

7.5 The Lefschetz operator

$$L(\alpha) := \omega \wedge \alpha\,.$$

7.6 Let us define the adjoint operator L^*

$$\langle L\alpha, \beta\rangle = \langle \alpha, L^*\beta\rangle$$

L^* lowering the gradation by 2.

- Note that, since ω is a real form, the operator L and its adjoint L^* are real. Their complexifications are denoted by the same symbol.
- Exercise

$$L^* = *^{-1} L *$$

Note $*^{-1} = (-1)^{(2n-k)k} * = (-1)^k *$ on $\Lambda^k V^*$ since $\dim_{\mathbb{R}} V^* = 2n$.

7.7 Suppose $\dim_{\mathbb{C}} V = n$. Let us define $H \in End(\Lambda V^*)$ as the multiplication by $k - n$ on $\Lambda^k V$.

- **Theorem:**

$$[H, L] = 2L, \quad [H, L^*] = -2L^*, \quad [L, L^*] = H\,.$$

- Proof of the first identity: $a \in \Lambda^k V^*$: $HLa - LHa = (k + 2 - n)La - (k - n)La = 2La$.
- The second follows since $H^* = H$.

• Proof of the last identity: Let us check the identity for $\dim_{\mathbb{C}} V = 1$. The basis $lin\{1, dz, d\bar{z}, dz \wedge d\bar{z}\}$ of $\Lambda\mathbb{C}^*$ is orthonormal. The operator L has the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The operator L^* has the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Their commutator is equal to

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

• We show by induction on the dimension of V that $[L.L^*]$ restricted to $\Lambda^k V$ is the multiplication by $k-n$.

• For $V \simeq \mathbb{C}^*$ we have already checked that the claim holds.

• Assume that $V = V_1 \oplus V_2$ is a sum of Hermitian-orthogonal spaces. Let the corresponding operators be L_i, L_i^*, H_i, $i = 1, 2$. On the summand $\Lambda^{k_1} V_1^* \otimes \Lambda^{k_2} V_2^*$ the operator acts as follows

$$L(\alpha \otimes \beta) = L_1(\alpha) \otimes \beta + \alpha \otimes L_2(\beta)$$

i.e.

$$L = L_1 + L_2\,, \qquad \text{and also} \qquad L^* = L_1^* + L_2^*.$$

Observe that $[L_1, L_2^*] = [L_2, L_1^*] = 0$. Thus $[L, L^*] = [L_1, L_1^*] + [L_2, L_2^*] = (k_1 - n_1) + (k_2 - n_2) = k - n$ on $\Lambda^k V^* = \bigoplus_{k+1+k_2=k} \Lambda^{k_1} V_1^* \otimes \Lambda^{k_2} V_2^*$.

7.8 The vector space ΛV^* is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{Z})$:

$$\rho : \mathfrak{sl}_2(\mathbb{Z}) \to End(\Lambda V^*), \qquad \rho(h) = H, \qquad \rho(x) = L, \qquad \rho(y) = L^*\,,$$

where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

7.9 Recollection of representation theory:

• simple representations (not containing proper subrepresentations) are of the form $S_k = Sym^k(\mathbb{C}^2)$ (the same for the theory over $\mathbb{R}$).

• any $\mathfrak{sl}_2$ representation can be decomposed into simple representations

• all eigenvalues of h on a representation W are integers.

• x^k defines an isomorphism of the eigenspaces $W_{-k} \to W_k$ ($k \geq 0$).

• $x : W_k \to W_{k+2}$ is mono for $k < 0$, epi for $k + 2 > 0$

7.10 In our case when $W = \Lambda V^*$, the eigenvalues of H are integers by the definition. The eigenspaces are given by

$$(\Lambda V^*)_k = \Lambda^{k+n} V^* \quad \text{for} \quad k \in \mathbb{Z}\,.$$

7.11 FURTHER STRATEGY: we know that above analysis applies to any complex vector space with a Hermitian product. Hence it applies to tangent spaces of a complex manifold. We obtain a list operators, decompositions etc in the space of complex-valued differential forms. We will show that this structure survives in the cohomology of a complex projective variety. Instead of projectivity it is enough to assume that the manifold has Kähler structure – a notion defined in terms of differential forms.

Hodge theory for Hermitian manifolds

7.12 Hermitian structure on a complex manifold M is a choice of a Hermitian product in each tangent space.

• such structure is a smooth section of $T^*M \otimes \overline{T}^*M$ which is symmetric and positively definite.

• real part is a scalar product, the imaginary part - a differential 2-form (which does not have to be closed).

• Hermitian structures exist for paracompact manifolds: we can chose a Hermitian structure locally in maps and glue them using partition of unity.

7.13 We extend Hodge * $\mathbb{C}$-linearly

• If $\dim_{\mathbb{C}} M = 1$

$$*dz = *(dx+idy) = dy-idx = -i(dx+idy) = -idz\,, \qquad *d\bar{z} = *(dx-idy) = dy+idx = i(dx-idy) = id\bar{z}$$

$$*1 = \omega = dx \wedge dy = \tfrac{i}{2} dz \wedge d\bar{z}\,, \qquad *\omega = 1$$

• In higher dimensions

$$* : \Lambda^{p,q} \xrightarrow{\simeq} \Lambda^{n-q,n-p}$$

$$*(dz_I \wedge d\bar{z}_J) = const\, dz_{J^\vee} \wedge d\bar{z}_{I^\vee}$$

where $J^\vee = \{1,2,\ldots,n\} \setminus J$, $I^\vee = \{1,2,\ldots,n\} \setminus I$

• Exercise: compute *constants*.

7.14 We have real operators L, L^*, $H = [L, L^*] = (\deg - n)id$ acting on C^∞-forms $A^*(X)$. The adjoint operator is given by a local formula

$$L^* = *^{-1} L * = (-1)^{\deg} * L *\,.$$

(The sign should be $(-1)^{(\dim_{\mathbb{R}} M - \deg)\deg}$ but here $\dim_{\mathbb{R}} TM$ is even). Often in literature L^* is denoted by Λ, but it can be confused with the exterior power). The adjoint operator satisfies

$$\int_M \langle L\alpha, \beta \rangle dvol = \int_M \langle \alpha, L^* \beta \rangle dvol\,.$$

• The complexified operators L, L^*, H act on $A^*(X)$ and $A^*(X)_{\mathbb{C}}$. Hence $A^*(X)_{\mathbb{C}}$ becomes an infinite dimensional representation of $\mathfrak{sl}_2$.

• We take complexification, because we are also interested in the (p,q) bigradation, available only over $\mathbb{C}$.

7.15 We define operators

$$\partial^* = - * \bar{\partial} * : A^{p,q}(X) \to A^{p-1,q}(X)\,,$$

$$(p,q) \mapsto (n-q, n-p) \mapsto (n-q, n-p+1) \mapsto (p-1, q)$$

and

$$\bar{\partial}^* = - * \partial * : A^{p,q}(X) \to A^{p,q-1}(X)\,.$$

We have $d^* = \partial^* + \bar{\partial}^*$.

• explanation of signs: $d^* = (-1)^{\dim_{\mathbb{R}} M(\deg +1)+1} * d * = - * d *$

7.16 Kähler structure

It can be defined in three equivalent ways:

• Definition 1: in a neighbourhood of each point there exist local coordinates in which

$$\omega = \frac{i}{2} \sum_k dz_k \wedge d\bar{z}_k + \mathcal{O}(||x||^2)\,.$$

i.e. in some coordinates the Hermitian metric is the same as for flat the manifold $\mathbb{C}^n$ up to the terms of order 2.

- Definition 2: $d\omega = 0$
- Definition 3: locally $\omega = i\partial\bar{\partial} f$ for some real function $f : U \to \mathbb{R}$.
- Proofs 1) $\Rightarrow$ 2) and 3) $\Rightarrow$ 2) are obvious.

7.17 Proof 2) $\Rightarrow$ 1) [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Prop 3.14]

- How to construct good coordinates?

$$\omega = \tfrac{i}{2}\sum_k dz_k \wedge d\bar{z}_k + \sum_{k,l}(\epsilon^h_{k,l} + \epsilon^a_{k,l}) dz_k \wedge d\bar{z}_l + \mathcal{O}(|z|^2)$$

where $\epsilon^h_{k,l}$ is a holomorphic linear form, $\epsilon^a_{k,l}$ antiholomorphic liner form.

- $\overline{\epsilon^a_{k,l}} = \epsilon^h_{l,k}$ since ω is real.
- $\frac{\partial}{\partial z_j}\epsilon^h_{k,l} = \frac{\partial}{\partial z_k}\epsilon^h_{j,l}$ since ω is closed
- Let ϕ_l be a function such that $\phi_l(0) = 0$ and $\frac{\partial}{\partial z_j}\phi_l = \epsilon^h_{j,l}$. The new coordinates

$$z'_l = z_l + \phi_l$$

do the job.

7.18 The basic fact: $\mathbb{P}^n$ with the unique $U(n+1)$-invariant metric is a Kähler manifold.

- The the Fubini-Study form ω defined in (5.20).

7.19 Corollary: any complex submanifold of $M \subset \mathbb{P}^n$ has a Kähler structure

$$\omega_M = \omega_{|M}\,,$$

$$\underbrace{\omega_M \wedge \omega_M \wedge \cdots \wedge \omega_M}_{m} = \text{ volume form on M}$$

provided, that $\dim_{\mathbb{C}} M = m$.

7.20 Hodge identities: Suppose M is a compact Kähler maniforld. Then:

i) $[\bar{\partial}, L] = [\partial, L] = 0$ (since ω is closed)

i') equivalently $[L^*, \partial^*] = [L^*, \bar{\partial}^*] = 0$,

ii) $[\bar{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\bar{\partial}$,

ii') equivalently $[L^*, \bar{\partial}] = -i\partial^*$, $[L^*, \partial] = i\bar{\partial}^*$ (this is the most difficult, the rest follows),

iii) $[\partial, \bar{\partial}^*]_s = [\partial^*, \bar{\partial}]_s = 0$ (i.e $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$ etc, this is a formal consequence of ii)),

iv) $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$ and it commutes with ∂, $\bar{\partial}$, ∂^*, $\bar{\partial}^*$, L i L^* (formal algebraic proof).

- Proofs TBA.

7.21 Very important corollary: $\mathcal{H} \simeq H^*(M;\mathbb{C})$ is a representation of $\mathfrak{sl}_2(\mathbb{Z})$.

8 Cohomology of compact Kähler manifolds

i)	$[\bar\partial, L] = [\partial, L] = 0,$	i′)	$[L^*, \partial^*] = [L^*, \bar\partial^*] = 0,$
ii)	$[\bar\partial^*, L] = i\partial, \quad [\partial^*, L] = -i\bar\partial,$	ii′)	$[L^*, \bar\partial] = -i\partial^*, \quad [L^*, \partial] = i\bar\partial^*,$
iii)	$\partial\bar\partial^* + \bar\partial^*\partial = 0,$	iii′)	$\bar\partial\partial^* + \partial^*\bar\partial = 0,$
iv)	$\Delta_\partial = \frac{1}{2}\Delta$	iv′)	$\Delta_{\bar\partial} = \frac{1}{2}\Delta$
v)	$[\Delta, L]$	v′)	$[\Delta, L^*]$

8.1 Short proof of Hodge relations ii) from [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Prop 6.5].

- Assume according to Definition 1) that ω has a standard form up to the terms of order 2. Therefore in calculations involving only the **first derivatives** at a point we can assume that

$$\omega = \tfrac{i}{2}\sum dz_k \wedge d\bar z_k$$

- We show ii') i.e. $[L^*, \partial] = i\bar\partial^*$. It is enough to check

$$([L^*, \partial](\alpha))_{z=0} = i(\bar\partial^*\alpha)_{z=0}$$

- We decompose $\omega = \sum_k \omega_k$, $\omega_k = \frac{i}{2} dz_k \wedge d\bar z_k$.

The adjoint operator $L_k^* = (\omega_k\wedge)^*$ is expressed by the contraction of differential forms

$$L_k^* = -2i\iota_{\bar v_k}\iota_{v_k}\,,$$

where $v_k = \frac{\partial}{\partial z_k}$, $\bar v_k = \frac{\partial}{\partial \bar z_k}$.

- We decompose $\bar\partial = \sum \bar\partial_k$. The adjoint differentials

$$\partial_k^* = -2\tfrac{\partial}{\partial \bar z_k}\iota_{v_k}\,, \qquad \bar\partial_k^* = -2\tfrac{\partial}{\partial z_k}\iota_{\bar v_k},$$

A sample of check in dim=1

$$\partial^* f\,dz = -*\bar\partial * f\,dz = -*\bar\partial(-if dz) = i * \tfrac{\partial}{\partial \bar z} f d\bar z \wedge dz = -2\tfrac{\partial}{\partial \bar z} f \, * \tfrac{i}{2} dz \wedge d\bar z = -2\tfrac{\partial}{\partial \bar z} f$$

- The operators L_k^* and ∂_ℓ commute for $k \neq \ell$. It remains to check $[L_k^*, \partial_k]$ for $\alpha = f dz_I \wedge d\bar z_J$, considering 4 cases $k \in$ or $\notin I$ or J. For example: suppose $k \notin I$, $k \notin J$

$$[L_k^*, \partial_k] f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J =$$

$$L_k^*\partial_k(f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J) - \partial_k L_k^*(f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J) =$$

$$2i\partial_k(f dz_I \wedge d\bar z_J) =$$

$$2i\tfrac{\partial}{\partial z_k}(f dz_k \wedge dz_I \wedge d\bar z_J) =$$

$$-2i\tfrac{\partial}{\partial z_k}\iota_{\bar v_k}(f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J) =$$

$$i\bar\partial^*(f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J).$$

8.2 For a computational proof see Huybrechts.

- The Huybrechts' proof of ii'): an operator $d^c = I^{-1}dI$ is introduced and the adjoint operator $(d^c)^*$

$$d^c = -i(\partial - \bar\partial), \quad (d^c)^* = -*\,d^c\,*\,.$$

He shows ii") $[L^*, d] = -(d^c)^*$. The proof is computational, using Lefschetz decomposition into $L^k\alpha$, where α is primitive.

8.3 iii)

$$i[\partial, \bar\partial^*] \overset{ii)}{=} [\partial, [L^*, \partial]] = \partial L^*\partial - \partial^2 L^* + L^*\partial^2 - \partial L^*\partial = 0$$

8.4 To show and iv) it is convenient to introduce the language of supercommutators $[a,b] = ab - (-1)^{\deg(a)\deg(b)}ba$. In that notation

$$\Delta_\partial = [\partial, \partial^*] .$$

- Leibniz rule, equivalent to the graded Jacobi identity

$$[a,[b,c]] = [[a,b],c] + (-1)^{\deg(a)\deg(b)}[b,[a,c]] ,$$

$$[[a,b],c] = [a,[b,c]] + (-1)^{\deg(b)\deg(c)}[[a,c],b] .$$

-

$$\Delta_\partial = [\partial^*, \partial] \overset{ii')}{=} i[[L^*, \bar\partial], \partial] \overset{Leibniz}{=} i([L^*, \underbrace{[\bar\partial, \partial]}_{0}] - [[L^*, \partial], \bar\partial]) \overset{ii)}{=} [\bar\partial^*, \bar\partial] = \Delta_{\bar\partial}$$

and from iii) $\Delta = \Delta_\partial + \Delta_{\bar\partial}$.

$$[L, \Delta_\partial] = [L, [\partial, \partial^*]] \overset{Leibniz}{=} [\underbrace{[L, \partial]}_{0}, \partial^*] + [\partial, [L, \partial^*]] \overset{ii)}{=} i[\partial, -i\bar\partial] = 0$$

8.5 Corollary from $\mathfrak{sl}_2$ action – Lefschetz decomposition

- For $k \geq 0$ let us define the primitive subspace

$$P_k = \{w \in W_{-k} \,|\, x^{k+1}w = 0\}.$$

For any $\mathfrak{sl}_2$ representation W we have the Lefschetz decomposition of weight spaces

$$W_{-k} = P_k \oplus xP_{k+2} \oplus x^2 P_{k+4} \oplus \dots .$$

- The maps

$$W_{-k} \xrightarrow{L} W_{-k+2} \xrightarrow{L} \dots \xrightarrow{L} W_{k-2} \xrightarrow{L} W_k$$

are injective (the first $\lceil k/2 \rceil$ steps) and further the are surjective. The composition L^k is an isomorphism.

8.6 The primitive forms (attention at the gradation shift): for $0 \leq k \leq n$ let us define

$$P^{n-k} = \{\alpha \in \Lambda^{n-k}V^* \mid L^{k+1}\alpha = 0\} \subset (\Lambda V^*)_{-k}$$

$$P^{p,q} = \Lambda^{p,q} \cap P^{p+q}_{\mathbb{C}} .$$

We have

$$P^{n-k}_{\mathbb{C}} = \bigoplus_{p+q=n-k} P^{p,q} .$$

8.7 Lefschetz decomposition of forms:

$$\Lambda^{n-k}V^* = P^{n-k} \oplus L(P^{n-k-2}) \oplus \dots \oplus L^j(P^{n-k-2j}) \oplus \dots .$$

- Practical consequences:

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism (**Hard Lefschetz Theorem**).

- It follows

$$\dim H^k(M) \leq \dim H^{k+2}(M) \qquad \text{if } k+1 \leq n ,$$

$$\dim H^k(M) \geq \dim H^{k+2}(M) \qquad \text{if } k+1 \geq n .$$

8.8 Hodge decomposition for the operator $\bar\partial$ (again, it follows from PDE, $\Delta_{\bar\partial}$ is an elliptic operator):

$$\mathcal{A}^{p,q}(M) = \underbrace{im(\bar\partial) \oplus \mathcal{H}^{p,q}_{\bar\partial}}_{ker(\bar\partial)} \oplus im(\bar\partial^*).$$

$$\bar\partial : A^{p,q-1} \to A^{p,q}, \qquad \bar\partial^* : A^{p,q+1} \to A^{p,q}.$$

- $$H^q(M;\Omega^p) \simeq \mathcal{H}^{p,q}_{\bar\partial},$$
- Since $\Delta_{\bar\partial} = \frac{1}{2}\Delta$, we have
$$\mathcal{H}^{p,q}_{\bar\partial} = \mathcal{H}^{p,q},$$
$$\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}, \qquad *\mathcal{H}^{p,q} = \mathcal{H}^{n-q,n-p}.$$
- $$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}.$$

8.9 Hodge (p,q) decomposition in cohomology:

- Taking harmonic representatives we conclude that
$$H^k(M;\mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(M;\Omega^p).$$

It is possible to construct a zig-zag map $H^k(M;\mathbb{C}) \to H^q(M;\Omega^p)$ not involving the harmonic representatives.

- Let
$$F^pA^\bullet(M) = A^{\geq p,\bullet}(M).$$

This is a subcomplex. On the level on complexes we have the maps

$$(A^{\bullet,p}(M),\bar\partial) \overset{\text{surjective}}{\longleftarrow} (F^pA^\bullet(M),d) \overset{\text{injective}}{\longrightarrow} (A^\bullet(M),d)$$

The above maps are independent from the metric.

- Let us define the Hodge filtration in $H^*(M)$:
$$F^pH^k(M) = image(H^k(F^pA^\bullet(M),d) \to H^k(M)$$
$$H^{p,q}(M) := F^pH^{p+q}(M)) \cap \overline{F^qH^{p+q}(M)}.$$
- **Theorem**
$$H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M), \qquad H^{p,q}(M) \simeq H^q(M;\Omega^p)$$

8.10 Let $h^{p,q} = \dim H^{p,q}(M)$.

- Hard Lefschetz implies inequalities
$$h^{p,q} \leq \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \leq n,$$
$$h^{p,q} \geq \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \geq n,$$
- The symmetries $h^{p,q} = h^{n-p,n-q} = h^{q,p}$ are organized in the ,,Hodge diamond"
- For example for $n = 3$

$$\begin{array}{ccccccc} & & & h^{33} & & & \\ & & h^{32} & & h^{23} & & \\ & h^{31} & & h^{22} & & h^{13} & \\ h^{30} & & h^{21} & & h^{12} & & h^{03} \\ & h^{20} & & h^{11} & & h^{02} & \\ & & h^{10} & & h^{01} & & \\ & & & h^{00} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & \spadesuit & & \spadesuit & & \\ & \diamondsuit & & \clubsuit & & \diamondsuit & \\ \square & & \heartsuit & & \heartsuit & & \square \\ & \diamondsuit & & \clubsuit & & \diamondsuit & \\ & & \spadesuit & & \spadesuit & & \\ & & & 1 & & & \end{array}$$

- Hard Lefschetz implies inequalities

$$h^{p,q} \le \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \le n\,,$$

$$h^{p,q} \ge \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \ge n\,,$$

8.11 Moreover
– If $k = n-(p+q) \ge 0$ then $L^k : H^{p,q}(M) \to H^{p+k,q+k}(M)$ is an isomorphism
– If $p+q \le n$ then

$$H^{p,q}(M) = P^{p,q}(M) \oplus L(P^{p-1,q-1}(M) \oplus L^2(P^{p-2,q-2}(M) \oplus \dots$$

8.12 Generalities about spectral sequence: if $C^\bullet$ is a complex with decreasing filtration

$$C^\bullet = F^0C^\bullet \supset F^1C^\bullet \supset F^2C^\bullet \supset \dots,$$

then one wishes to relate cohomologies $H^*(F^pC^\bullet/F^{p+1}C^\bullet)$ with $H^*(C^\bullet)$.

- There exists a spectral sequence (under some boundness of degree assumptions)

$$E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}, \qquad E_1^{p,q} = H^{p+q}(F^pC^\bullet/F^{p+1}C^\bullet)\,, \qquad \dots.$$

- There exists a sequence of tables $E_r^{p,q}$ with differentials of degree $(1-r,r)$, such that

1) $H^*(E_r^{\bullet,\bullet}) = E_{r+1}^{\bullet,\bullet}$
2) $E_\infty^{p,q} = F^pH^{p+q}(\mathbb{C}^\bullet)/F^{p+1}H^{p+q}(\mathbb{C}^\bullet)$

8.13 For the total complex of the bicomplex $A^{p,q}(M)$ with the Hodge filtration $F^pA^\bullet(M) = A^{\ge p,\bullet}(M)$ the resulting spectral sequence is called the **Frölicher spectral sequence**.

- Corollary: If M Kähler and compact, then the Frölicher spectral sequence degenerates on E_1, i.e.

$$E_1^{p,q} = H^q(M;\Omega^p) = E_\infty^{p,q}\,.$$

(the higher differentials vanish).

8.14 Corollary: Suppose M Kähler and compact: if $\alpha \in \Omega^p(M)$ then $\partial\alpha = 0$.

- Holomorphic implies closed.
- This is a generalization of: global holomorphic function is constant.

8.15 Integral structure

- The cohomology with complex coefficients can be expressed as

$$H^*(M;\mathbb{C}) = H^*(M;\mathbb{Z}) \otimes \mathbb{C}\,.$$

We have an inclusions

$$H^*(M)_\mathbb{Z} := H^*(M;\mathbb{Z})/(torsion) \hookrightarrow H^*(M;\mathbb{R}) \hookrightarrow H^*(M;\mathbb{C})\,,$$

it is a lattice, $\mathrm{rk}(H^*(M;\mathbb{Z})) = \dim(H^*(M;\mathbb{R}))$.

- Note that if $M \subset \mathbb{P}^N$ then

$$[\omega_{|M}] \in H^{1,1}(M) \cap H^*(M)_\mathbb{Z}$$

8.16 Example/exercise: Let V be a complex vector space, $A \subset V$ a lattice. Then canonically

$$H^*(V/A;\mathbb{C}) \simeq \Lambda V^*_\mathbb{C}$$

$$H^*(V/A)_\mathbb{Z} = \Lambda A^\vee\,,$$

$$A^\vee = \{f \in \mathrm{Hom}(V,\mathbb{R}) \mid \forall a \in A \quad f(a) \in \mathbb{Z}\}$$

- For a generic lattice $H^{1,1}(M) \cap H^*(M)_\mathbb{Z} = \{0\}$.