Complex Manifolds - problem list no. 9 - zadania na 19.12

1 Compute the Dolbeault cohomology $H^q(M;\Omega^p_M) = H^q(A^{p,\bullet}(M),\bar\partial)$ for $M = \mathbb{C}\setminus\{0\}$.
Wskazówka: Zostało do obliczenia $H^1(M,\mathscr{O}_M) \simeq H^1(M,\Omega^1_M)$. *Powinno wyjść 0, czyli trzeba wykazać, że każda forma* $f(z)d\bar z$ *jest różniczką. Proponuję najpierw sprawdzić czy jest to prawda dla* $M = \mathbb{C}$ *a potem wykorzystać* $\exp : \mathbb{C} \to \mathbb{C}\setminus\{0\}$.

2 Define the set $\mathcal{K}_X \subset H^{1,1}(X) \cap H^2(X;\mathbb{R})$ consisting of the classes $[\omega]$ such that ω is the minus imaginary part of a Kähler metric. Show that $\mathcal{K}_X$ is an open conical set, not containing any affine line.

3 Let M be the connected sum

$$\mathbb{P}^2\#\mathbb{P}^2 := (\mathbb{P}^2\setminus B_\epsilon)\sqcup(\mathbb{P}^2\setminus B_\epsilon)/\sim\,,$$

where we glue the points of the boundaries of the balls B_ϵ, so that the orientations of $\mathbb{P}^2$'s agree.
a) Compute the Betti numbers (i.e. the diminesions $\dim H^k(M)$) and the signature of M.
b) Using the formula relating Hodge numbers with the signature deduce that M does not admit any structure of Kähler manifold.

4 Compute Čech cohomology $\check{H}^1(\mathcal{U};\Omega^1)$, where $\mathcal{U} = \{U_0, U_1\}$ is the standard cover of $\mathbb{P}^1$, $U_i \simeq \mathbb{C}$ for $i = 0, 1$.

5 *(Trudniejsze)* Construct a lattice $\Lambda \subset \mathbb{C}^2$ such that $\mathbb{C}^2/\Lambda$ cannot be embedded into a projective space.

Wskazówka: Jeśli M *można zanurzyć w* $\mathbb{P}^n$*, to* ω_{FS} *pochodzi z* $H^2(M;\mathbb{Z})$*. Skonstruować taką kratę, by* $H^2(M;\mathbb{Q}) \cap H^{1,1}(M) = 0$ *(jako podprzestrzenie* $H^2(M;\mathbb{C})$*).*

6 Referat S.S.: Let $M = Bl_x\mathbb{P}^2$ be the blow-up of the projective plane, see [Huybrechts §2.5]. Compute Euler characteristic and the intersection form on $H^2(M)$.

(Alternativ description: M is homeomorphic to the connected sum

$$\mathbb{P}^2\#\overline{\mathbb{P}^2} := (\mathbb{P}^2\setminus B_\epsilon)\sqcup(\mathbb{P}^2\setminus B_\epsilon)/\sim\,,$$

where we glue the points of the boundaries of the balls B_ϵ, so that the orientations of $\mathbb{P}^2$'s do not agree.)