Complex Manifolds - problem list no. 7 - zadania na 5.12

1 Decompose the representation $\Lambda^\bullet(V^* \otimes_{\mathbb{R}} \mathbb{C})$ into irreducible representations assuming that $\dim V = 1$, 2 or (maybe) 3.

2 [Fubini-Study metric on $\mathbb{P}^n$] Consider a local holomorphic section of the quotient bundle $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$

$$s : \mathbb{P}^n \supset U \to \mathbb{C}^{n+1} \setminus \{0\}$$

Show that the form

$$\omega = \frac{i}{2\pi} \partial\bar\partial \log(|s(z)|^2)$$

Does not depend on the choice of $\alpha(z)$. Hence can be glued to a global form on $\mathbb{P}^n$. In particular, in local coordinates on $U_0 = \{z_0 \neq 0\}$

$$\omega = \frac{i}{2\pi} \partial\bar\partial \log(1 + \sum_{k=1}^{n} |z_k|^2)\,.$$

Show that the resulting form is the Fubini-Study form.

Hint: Perform the calculation for $s(z) = \alpha(z)(1, z_1, \dots, z_n)\,.$

3 Prove the second Hodge identity $[L^*, \partial] = i\bar\partial^*$ for the flat metric: We decompose $\bar\partial = \sum \bar\partial_k$ and $L^* = \sum L^*_k$. Show that $\bar\partial^*_k = -2\iota_{\bar v_k} \frac{\partial}{\partial z_k}$, where $\bar v_k = \frac{\partial}{\partial \bar z_k}$. Note ∂_ℓ commutes with L^*_k for $k \neq \ell$. It remains to check $[L^*_k, \partial_k]$ for $\alpha = f dz_I \wedge d\bar z_J$, considering 4 cases $k \in$ or $\notin$ to I and J. For example: suppose $k \in I$, $k \in J$. That is $I = \{k\} \cup I'$, $J = \{k\} \cup J'$:

$$[L^*_k, \partial_k] f dz_k \wedge d\bar z_k \wedge dz_{I'} \wedge d\bar z_{J'} =$$

$$L^*_k \partial_k (f dz_k \wedge d\bar z_k \wedge dz_{I'} \wedge d\bar z_{J'}) - \partial_k L^*_k (f dz_k \wedge d\bar z_k \wedge dz_{I'} \wedge d\bar z_{J'}) =$$

$$2i\partial_k (f dz_{I'} \wedge d\bar z_{J'}) =$$

$$2i \tfrac{\partial f}{\partial z_k} dz_k \wedge dz_{I'} \wedge d\bar z_{J'} =$$

$$2i \tfrac{\partial f}{\partial z_k} \iota_{\bar v_k} (d\bar z_k \wedge dz_k \wedge dz_{I'} \wedge d\bar z_{J'}) =$$

$$i\bar\partial^* (f dz_k \wedge d\bar z_k \wedge dz_{I'} \wedge d\bar z_{J'}),$$

Compute another case, e.g. $k \notin I$, $k \in J$.

4 Referat dla BS: Holomorphic Poincaré Lemma (Huybrechts 1.3.7&8).