Complex Manifolds - problem list no. 6 - zadania na 28.11

1 Show that Hodge * satisfies
(i) $*^2 = (-1)^{k(d-k)}$ for k-forms,
(ii) $\langle \alpha, *\beta \rangle = (-1)^{k(d-k)} \langle *\alpha, \beta \rangle$.

2 Let $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Solve the heat equation for $\alpha : \mathbb{R} \to \mathcal{A}^k(S^1)$:

$$\frac{d}{dt}\alpha_t = -\Delta \alpha_t$$

with the initial condition

$$\alpha_0 = \delta_0 dt\,,$$

where δ_0 is the Dirac delta. Show that for $t > 0$ the form α_t is smooth.

Uwaga: rozważamy tu formy o współczynnikach, które są dystrybucjami. (Takie formy nazywamy prądami, ang. „currents".) Dla n-wymiarowej rozmaitości takie k-formy działają na $(n-k)$-formach próbnych o zwartym nośniku w sposób oczywisty (?). Rozwinięcie dystrybucji w szereg Fouriera traktujemy czysto formalnie.

3 Compute Laplasian for the torus $(S^1)^n$ with the standard (flat) metric. Find the spaces of harmonic forms.

4 Let I be a standard complex structure in $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n \simeq \mathbb{R}^{2n}$. Consider the following subgroups of $GL_{2n}(\mathbb{R})$:

- $GL_n(\mathbb{C})$ consisting of A such that $AI = IA$,
- $Sp_n(\mathbb{R})$ consisting of A such that $A^T I A = I$,
- $O(2n)$ consisting of A such that $A^T A = id$.

Show that intersections of any two of three groups above is equal to $U_n \subset GL_n(\mathbb{C})$, the group of unitary matrices, i.e. the subgroup of $GL_n(\mathbb{C})$ preserving the hermitian product.

5 Let V be a complex vector space with a hermitian structure. Consider the space $\Lambda^\bullet = \Lambda^\bullet(V^* \otimes_{\mathbb{R}} \mathbb{C})$ with operations L, H, L^*:

- $L(\alpha) = \omega \wedge \alpha$,
- $H_{|\Lambda^k}$ =multiplication by $k - n$,
- $L^*(\alpha) = (-1)^k * L(*\alpha)$ for $\alpha \in \Lambda^k$ (show that indeed this is the adjoint operator to L).

Show that

- $[L, L^*] = H$,
- $[H, L] = 2L$,
- $[H, L^*] = -2L^*$.

That is, show that Λ^* is a representation of $\mathfrak{sl}_2(\mathbb{Z})$.
Hint: Check it for $\dim_{\mathbb{C}} V = 1$ and proceede by induction.

6 Decompose the representation $\Lambda^\bullet(V^* \otimes_{\mathbb{R}} \mathbb{C})$ into irreducible representations assuming that $\dim V = 1$, 2 or (maybe) 3.