Complex Manifolds - problem list no. 5 - zadania na 21.11

1 Niech $X = \{(x,y) \in \mathbb{C}^2 \ : \ 2x^3 - 6x - 3y^2 = 0\}$ oraz $f : X \to \mathbb{R}$, $f(x,y) = \operatorname{Re} y$. Czy punkty krytyczne f są niezdegenerowane (tzn czy jeśli $D(f) = 0$ to $D^2(f)$ jest niezdegenerowana)? Jakie są indeksy?

2 Show that Hodge * satisfies
(i) $*^2 = (-1)^{k(d-k)}$ for k-forms,
(ii) $\langle \alpha, *\beta \rangle = (-1)^{k(d-k)} \langle *\alpha, \beta \rangle$.

3 Let $M = S^1$.
– Find the formula for d^* for 1-forms.
– Find the formula for Δ for 0-forms and 1-forms.
– Find all eigenvalues of Δ acting on $\mathcal{A}^k(S^1)$ for $k = 0, 1$.
– Solve the heat equation for $\alpha : \mathbb{R} \to \mathcal{A}^k(S^1)$:

$$\frac{d}{dt} \alpha_t = -\Delta \alpha_t$$

with the initial condition

$$\alpha_0 = \left(a + \sum_{k=1}^{\infty} b_k \sin(kt) + c_k \cos(kt) \right) dt\,.$$

4 Compute Laplasian for the torus $\mathbb{R}^n / \mathbb{Z}^n$ with the standard (flat) metric. Find the spaces of harmonic forms.

5 (Referat dla ML) Let α_t be a solution of the heat equation

$$\frac{d}{dt} \alpha_t = -\Delta \alpha_t$$

with the initial condition α_0. Show that if $d\alpha_0 = 0$, then $d\alpha_t = 0$ and the cohomology class $[\alpha_t] \in H^*(M)$ for $t \geqslant 0$ is constant. The limit is equal to the harmonic representative of $[\alpha_t]$.

[Donu Arapura, Algebraic Geometry over the Complex Numbers, Universitext 2012, Section 8.3]