Complex Manifolds - problem list no. 12 - zadania na 23.01.2023

1 Wykazać, że jeśli wiązka E, $\mathrm{rk}(E) = r$ ma nigdzie nie zerujący się przekrój, to $c_r(E) = 0$. (O bazie X zakładamy tylko, że jest przestrzenią topologiczną).

2 Let $E \to M$ be a bundle with connection, $U \subset M$ is an open subset. Suppose we have two sets of sections trivializing $E_{|U}$: $s_1, s_2, \dots, s_n$ and $s'_1, s'_2, \dots, s'_n$. Let $G \in GL_n(C^\infty(U))$ be the transition matrix. Suppose that

$$\nabla = d + A$$

with respect to the first trivialization. Find the formula for the connection form A' with respect to the second trivialization.

3 Construct the Euler exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}(1)^{\oplus n+1} \to T\mathbb{P}^n \to 0\,.$$

Hint: Let $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection. For any $v \in \mathbb{C}^{n+1} \setminus \{0\}$ we have a surjection

$$\mathbb{C}^{n+1} \simeq T_v\mathbb{C}^{n+1} \xrightarrow{T_v p} T_{[v]}\mathbb{P}^n\,.$$

Show that in fact we obtain a well defined surjection

$$\mathbb{C}^{n+1} \otimes \mathscr{O}(1) \to T\mathbb{P}^n\,.$$

4 Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d.
a) Compute $\chi(X; \mathscr{O}(k))$ for $n = 3$, $d = 4$, $k = 7$.
b) Compute the Hodge numbers $\dim H^q(X; \Omega^p)$ for $n = 3$, $d = 4$.
Hint a): Let $\mathscr{O}_X(d) := \mathscr{O}_{\mathbb{P}^n}(d)_{|X}$. Then

$$TX \oplus \mathscr{O}_X(d)_{|X} \simeq T\mathbb{P}^n_{|X}$$

Hint b): let

$$\lambda_y(X) = \sum_{p=0}^{\dim X} \Lambda^p T^*X\, y^p$$

be a formal sum (y is a formal variable). Show that

$$\lambda_y(X)(\mathscr{O}_X + y\mathscr{O}_X(d)) = \lambda_y(\mathbb{P}^n)\,.$$

5 Referat ML: klasyczne twierdzenie Riemanna-Rocha dla krzywych i powierzchni, [Huybrechts §5.1, przykład 5.1.2i-ii].