Complex Manifolds 2022/23

Lecture summary. **This is not a replacement for a textbook.**

`aweber at mimuw.edu.pl`

http://www.mimuw.edu.pl/∼aweber/ComplexManifolds

Date January 27, 2023

The main reference: Daniel Huybrechts, Complex Geometry, An introduction. (Springer 2005)

Also:
Donu Arapura, Algebraic Geometry over the Complex Numbers (Universitext)
Claire Voisin, Hodge Theory and Complex Algebraic (Cambridge Studies in Advanced Mathematics)

1 Introduction

1.1 Definition of complex manifolds

1.2 Projective spaces

1.3 Grassmannians $Gr_k(\mathbb{C}^n)$. Affine maps: for $I = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$

$$U_I = \{V \in Gr_k(\mathbb{C}^n) \mid \text{projection } V \to I\text{-coordinates is an isomorphism }\}$$

$$U_I \simeq \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k})\,.$$

1.4 Plücker embedding, $Gr_2(\mathbb{C}^4)$ as a quadric in $\mathbb{P}^5 = \mathbb{P}(\Lambda^2\mathbb{C}^4)$.

1.5 Hyperplane in $\mathbb{P}^n$ e.g. elliptic curve in $\mathbb{P}^2$

$$y^3 + pxz^2 + qz^3 - x^2z = 0$$

with p, q fixed.

1.6 Complex manifolds as real manifolds are orientable since any linear complex map preserves the distinguished orientation of the underlying real vector space.

1.7 Basic information about topological coverings an induced complex structures: If $f : X \to Y$ is a topological covering, Y has a structure of a complex manifold, then X has a natural structure of a complex manifold and f is a holomorphic.

Curves

1.8 Riemann surfaces (= oriented surfaces with a Riemannian metric) and complex surfaces: each Riemannian surface has a complex structure. Genus of Riemann surface.

• The rotation by 90° in the tangent space allows to introduce a structure of complex vector space. This structure is ,,integrable" i.e. it comes from a structure of a complex manifold (a proof will be later, it follows trivially from Newlander-Nirenberg theorem).

1.9 Riemann uniformization theorem: any complex curve is isomorphic to $\mathbb{P}^1$ or it is a quotient of $\mathbb{C}$ or $\mathbb{D} \simeq \mathbb{H}$.

• Another formulation: any simply connected complex curve is isomorphic to $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D}$. This is a generalization of the Riemann theorem for open subsets in $\mathbb{C}$.

1.10 The automorphism group of $\mathbb{P}^1$ is equal to $PGL_2(\mathbb{C})$. Any complex-analytic automorphism of $\mathbb{P}^1$ is given by a linear formula. (The same statement holds for $\mathbb{P}^n$.)

• Proof: Composing with a linear map we can assume that $f(0) = 0$, $f(\infty) = \infty$. Expanding at infinity we get an estimation $1/|f(z)| < c/|z|$. Hence the function $g(z) = z/f(z)$ is bounded. It has no poles, since at 0 the zero of the denominator cancels out and there are no more zeros of f. Hence by Liouville theorem $g(z)$ is constant.

• Hence each automorphism of $\mathbb{P}^1$ has a fixed point – the eigenvector of the linear map.
Topological proof: there are no nontrivial topological covering $\mathbb{P}^1 \simeq S^2 \to C$ except $C = \mathbb{RP}^2$. But the real projective plane is not orientable, so it cannot be a complex curve.

1.11 Automorphisms of $\mathbb{C}$ are given by affine maps $f(z) = az + b$. There are no fixed points only if $a = 1$.

• The map $f : \mathbb{C} \to \mathbb{C}$ extends to $\mathbb{P}^1$. It is continues at ∞. By Riemann extension theorem it is holomorphic ∞ and we apply (1.10).

1.12 The complex quotients of $\mathbb{C}$ are of the form $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$.

• The nontrivial discrete subgroups of $\Lambda \subset (\mathbb{C}, +) \simeq \mathbb{R}^2$ are of the form $\Lambda = \langle a, b\rangle$ for $b/a \in \mathbb{H}$, (or $\Lambda = \langle a\rangle$). We can restrict our attention to subgroups of the form $\Lambda = \langle 1, \tau\rangle$, $\tau \in \mathbb{H}$.

• The group $PSL_2(\mathbb{C}) := SL_2(\mathbb{C})/\{\pm I\}$ acts on $\mathbb{P}^1$ by homography: $\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \cdot z = (sz + t)/(uz + v)$. The subgroup $PSL_2(\mathbb{R})$ preserves the upper hyperplane $\mathbb{H}$.

• Suppose $\tau, \tau' \in \mathbb{H}$. Then $\mathbb{C}/\langle 1, \tau\rangle \simeq \mathbb{C}/\langle 1, \tau'\rangle$ if and only if τ and τ' belong to the same orbit of $PSL_2(\mathbb{Z})$. (Exercise)

1.13 The group of disk automorphisms is isomorphic to the group of the upper hyperplane $\mathbb{H}$ automorphisms: $Aut(\mathbb{H}) = PSL_2(\mathbb{R})$.

• $Aut(\mathbb{D})$ consist of homographies (apply the Schwartz lemma, assuming $f(0) = 0$).

1.14 Discrete subgroups of $PSL_2(\mathbb{R})$ are called Fuchsian groups (grupy Fuksa). The curves of higher genera $g > 1$ are quotients $\mathbb{H}/G$ where $G \subset PSL_2(\mathbb{R})$ is Fuchsian and acts without fixed points.

1.15 Read more: [Huybrechts, Complex Geometry, Chapter 2.1]

2 Weierstrass preparation

Local theory: see [§1, Huybrechts].

2.1 Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ and complex differential $\frac{1}{2}\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$.

2.2 Differentials $dz = dx + idy$ and $d\bar{z} = dx - idy$.

• For any C^∞ function on $f : \mathbb{C} \to \mathbb{C}$ the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}\,.$$

(Hint if $A = \left(\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right)$ then $(A^T)^{-1} = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix}\right)$.)

2.3 Recollection of theorems for complex analytic functions in one variable

• series expansion

• Cauchy integration formula

• maximum principle

• identity principle

• Liouville theorem

2.4 Residue $res_{z_0}(f) = \frac{1}{2\pi i}\int_{\partial D_{z_0}} f\,dz$, where D_{z_0} is a small disk around z.

2.5 Residue theorem: for a meromorphic function f (enough to assume: holomorphic away from a discrete set $\{z_1, z_2, \ldots, z_n\}$) on a compact Riemann surface S

$$\sum_k res_{z_k}(f) = 0\,.$$

• Proof from the Stokes theorem: Assume that the discs D_{z_k} for $z \in Sing(f)$ do not intersect:

$$\sum_{z_k}\int_{\partial D_{z_k}} f\,dz = -\int_{\partial(S\setminus\bigcup D_{z_k})} f\,dz = -\int_{S\setminus\bigcup D_{z_k}} d(f\,dz) = -\int_{S\setminus\bigcup D_{z_k}} \frac{\partial f}{\partial \bar z} d\bar z\wedge dz = 0.$$

2.6 A formula for the number of zeros in a disk has a generalization which will be used later. If $f(z)\neq 0$ for $|z|=\varepsilon$ then for $\ell \geq 0$ we have

$$\frac{1}{2\pi i}\int_{S_\varepsilon}\frac{f'(\xi)}{f(\xi)}\xi^\ell d\xi = \sum_{|\alpha|<\varepsilon,\ f(\alpha)=0}\alpha^\ell.$$

Many variables - references to [Huybrechts §1.1]

2.7 Definition: a C^∞ function $f:\mathbb{C}^n\to\mathbb{C}$ is holomorphic if $\partial_{\bar z_k f} = 0$ for $k=1,2,\ldots,n$.

2.8 Cauchy integral formula Prop 1.1.2

2.9 Hartogs theorem Prop 1.1.4

2.10 Corollary: zero set of a holomorphic function ($f\not\equiv 0$) has real codimension equal 2 or it is empty.

• Remark: any analytic set (eg zero set of a holomorphic function) is triangulable by Łojasiewicz theorem, so there is no ambiguity with the notion of dimension.

2.11 Weierstrass preparation theorem (Th. 1.1.6).

2.12 Algebraic fact used in the proof: elementary symmetric functions σ_k can be expressed by power sums p_k.

Local ring

2.13 The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is a unique factorization domain (Prop 1.1.15).

• Key argument: Weierstrass polynomial is indecomposable in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z]$ iff it is indecomposable in $\mathcal{O}_{\mathbb{C}^n,0}$.

3 Weierstrass II

3.1 Weierstrass preparation theorem – division version (Prop 1.1.17).

3.2 The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian (Prop 1.1.18).

3.3 Remark: If $\emptyset\neq U\subset\mathbb{C}^n$, $n>0$ then $\mathcal{O}_{\mathbb{C}^n}(U)$ is not noetherian.

• Proof: Any $I\subset\mathcal{O}_{\mathbb{C}^n,0}$ is generated by $I\cap(\mathcal{O}_{\mathbb{C}^{n-1},0}[z])$ and any Weierstrass polynomial $g\in I$ (by division version of WPT).

3.4 Germ of sets and ideals in the local ring:

• The germ of the set $Z(J)$ defined by an ideal $J\subset\mathcal{O}_{\mathbb{C}^n,0}$.
— if $J_1\subset J_2$ then $Z(J_1)\supset Z(J_2)$

• The ideal of function germs vanishing on the germ of a set $I(X)$. We have:
— if $X_1\subset X_2$ then $I(X_1)\supset I(X_2)$

3.5 Compositions of Z and I

- $X \subset Z(I(X))$ for any set germ,
- $J \subset I(Z(J))$ for any ideal,
- $X = Z(I(X))$ for analytic set germs (i.e. of the form $X = Z(J)$)

— since $J \subset I(Z(J))$ then $X = Z(J) \supset Z(I(Z(J))) = Z(I(X))$.

- Hilbert nullstellensatz: $I(Z(J) = \sqrt{J}$ (see sketch of a proof in Huybrechts p.20).

3.6 Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be indecomposable, then if $f_{|Z(g)} = 0$, then g divides f (Cor. 1.1.9)

- Proof from the division version of Weierstrass preparation theorem.
- Key step: if g is indecomposable Weierstrass polynomial, then $g_w(z)$ generically (w/r to w) has distinct roots.

— let K be the quotient field of $\mathcal{O}_{\mathbb{C}^{n-1},0}$. The polynomials $g_w(z)$ and $g'_w(z)$ are coprime (by Gauss lemma), so there exist $\alpha(z), \beta(z) \in K(z)$ such that $\alpha(z)g_w(z) + \beta(z)g'_w(z) = 1$. Passing to $\mathcal{O}_{\mathbb{C}^{n-1},0}$, removing the denominators

$$\tilde{\alpha}(z)g_w(z) + \tilde{\beta}(z)g'_w(z) = \gamma$$

with $0 \neq \gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$. At the points where $\gamma(w) \neq 0$ the polynomial g_w does not have multiple roots.

3.7 The germ of a set is indecomposable (also called irreducible) if and only if $I(X)$ is a prime ideal (Lemma 1.1.28)

Rough notes on GAGA (dla absolwentów teorii snopów)

J-P.Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier 6: 1-42, (1956)

See also: Amnon Neeman, Algebraic and analytic geometry. Cambridge University Press (2007)

3.8 For an algebraic manifold X (a scheme in general) we define ,,analytification" X^{an}.

- As a set $X = X^{an}$.
- While X has Zariski topology, X^{an} has classical topology (glued from the open subsets $U \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$). The identity map $\iota : X^{an} \to X$ is continuous (every Zariski open set is open in the classical topology). [Serre §5 Lemma 1]
- Both spaces are ringed. We have distinguished sheaves of rings $\mathcal{O}_X$ (algebraic functions) and $\mathscr{H}_X$ (holomorphic functions), the stalks are local rings. We have a map

$$\theta_X : \iota^{-1}\mathcal{O}_X \to \mathscr{H}_X,$$

i.e. ι extends to a map of ringed spaces. Here ι^{-1} denotes the pull-back of a sheaf. The map θ_X is injective, flat, an isomorphism after completion in $\mathfrak{m}$. [Serre §6, prop 4]

3.9 For an algebraic sheaf $\mathcal{F}$ over an algebraic manifold we define ,,analytification"

$$\mathcal{F}^{an} = \mathscr{H}_X \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1}\mathcal{F}\,.$$

Of course $\mathcal{O}_X^{an} = \mathscr{H}_X$. [Serre §9, Prop 10]

3.10 Definition: Let $(Y, \mathscr{R}_Y)$ be a ringed space. The sheaf $\mathscr{R}_Y$–modules $\mathcal{F}$ is coherent iff

1) locally there is a surjective map $(\mathscr{R}_Y^N)_{|U} \to \mathcal{F}_{|U}$ for some N (i.e. $\mathcal{F}$ is locally finitely generated),

2) for any map $(\mathscr{R}_Y^M)_{|U} \to \mathcal{F}_{|U}$ the kernel is finitely generated.

By $Coh(Y)$ we denote the category of coherent sheaves.

- Mind the difference comparing with the definition for algebraic varieties.

3.11 Oka Theorem: $\mathcal{F} = \mathscr{H}_X$ is coherent. (This is not a tautology!) References in [Serre §3 Prop.1]

3.12 Analytification of sheaves is a functor preserving coherent sheaves [Serre §9]

$$(-)^{an} : Sh(X) \to Sh(X^{an})$$

3.13 (Serre) If X is projective, $\mathcal{F}$ coherent then the natural map $H^*(X;\mathcal{F}) \to H^*(X^{an};\mathcal{F}^{an})$ is an isomorphism. [Serre §12 Th. 1]

• Relative version: Let $f : X \to Y$ be a projective morphism of algebraic varieties. Then f induces a functor

$$f_*^{an} : Coh(X^{an}) \to Coh(Y^{an})$$

and

$$(f_*\mathcal{F})^{an} = f_*^{an}\mathcal{F}^{an}$$
$$(R^k f_*\mathcal{F})^{an} = R^k f_*^{an}\mathcal{F}^{an}$$

If $Y = pt$ then we recover the previous formulation.

3.14 (Serre cont.) If X is a projective variety, then $(-)^{an}$ restricted to $Coh(X)$ is an equivalence of categories.
The above means:
(i) $Hom_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to Hom_{\mathscr{H}_X}(\mathcal{F}^{an},\mathcal{G}^{an})$ is an isomorphism. [Serre §12 Th. 2]
(ii) For any analytic coherent sheaf G there exists an algebraic sheaf $\mathcal{F}$ such that $G \simeq \mathcal{F}^{an}$. [Serre §12 Th. 3]

3.15 The proofs can be reduced to $X = \mathbb{P}^n$. To check the equality $H^*(X;\mathcal{F}) \simeq H^*(X^{an};\mathcal{F}^{an})$ we can assume (by various cohomology exact sequences) that $\mathcal{F} \simeq \mathcal{O}(m)$.

3.16 For a proof of (i) use the equality of sheaf-Homs

$$(\underline{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))^{an} = \underline{Hom}_{\mathscr{H}_X}(\mathcal{F}^{an},\mathcal{G}^{an})$$

which holds for algebraic coherent sheaves. Then apply the general principle

$$Hom_Y(F,G) = H^0(Y;\underline{Hom}_Y(F,G))\,,$$

and apply 3.13.

3.17 For a proof of (ii) have to show that any analytic sheaf F on $X = \mathbb{P}^n$ after tensoring with $\mathscr{H}_X(m)$ for some big m is globally generated, i.e. there exists k and a surjection

$$\mathscr{H}_X^k \to F(m) := F \otimes_{\mathscr{H}_X} \mathscr{H}_X(m)\,,$$

which is equivalent to: for each point $x \in X$

$$\text{Global sections of } F(m) \to F(m)_x$$

is a surjection, [Serre §16 Lemma 8]. Then $F(m) = coker(\mathscr{H}_X^\ell \to \mathscr{H}_X^k)$, thus by (i) it is algebraic, [Serre §17].

3.18 Corollary (Chow Theorem): Any analytic subvariety $\mathbb{P}^n$ is described by a set of polynomial equations..

4 Morse theory for C^∞-manifolds and weak Lefschetz

[Milnor – Morse theory, 1963]

4.1 Def: Morse function $f : M \to \mathbb{R}$ is a proper smooth function such that if $Df(p) = 0$ for $p \in M$ then $D^2f(p)$ is nondegenerate. Additionally we assume that for each critical value there exist only one critical point of f (critical values of distinct points do not collide).

4.2 $ind(p)$= the index of a critical point = the number of minuses after diagonalization of $D^2f(p)$.

4.3 for $t \in \mathbb{R}$ let

$$M_{\leq t} = \{p \in M \mid f(p) \leq t\}\,.$$

4.4 Theorem:
1) If there is no critical value in the interval $[a,b]$, then the inclusion $M_{\leq a} \subset M_{\leq b}$ is a homotopy equivalence
2) If $f(p) = c \in [a,b]$ is the only one critical value in the interval $[a,b]$ then $M_{\leq b}$ is homeomorphic to $M_{\leq a}$ with attached $I^{ind(p)} \times I^{n-ind(p)}$ along $\partial I^{ind(p)} \times I^{n-ind(p)}$, (up to homotopy we attach a cell of the dimension $k = ind(p)$).

4.5 The effect of attaching k-dimensional cell:

$$M_{\leq b} = M_{\leq a} \cup_\phi D^k\,, \qquad \phi : S^{k-1} \to M_{\leq a}\,.$$

There is an exact sequence

$$0 \;\to\; H^{k-1}(M_{\leq b}) \;\to\; H^{k-1}(M_{\leq a}) \;\stackrel{\phi^*}{\to}\; \underset{\displaystyle\mathbb{Z}}{\overset{}{\underset{\|}{H^{k-1}(S^{k-1})}}} \;\to\; H^k(M_{\leq b}) \;\to\; H^k(M_{\leq a}) \;\to\; 0$$

The for the remaining gradations $H^i(M_{\leq b}) \simeq H^i(M_{\leq a})$ (the case $i = 0$ needs a separate discussion). For real (or rational) coefficients: replace $\mathbb{Z}$ by $\mathbb{R}$ (or $\mathbb{Q}$). Then there are two cases: $\phi = 0$ or not.

- If $\phi = 0$, then $H^k(M_{\leq b}) \simeq H^k(M_{\leq a}) \oplus \mathbb{R}$, and the remaining gradations are not changed.
- If $\phi \neq 0$, then $H^{k-1}(M_{\leq b}) \simeq ker(\phi)$, and the remaining gradations are not changed.

4.6 Corollary: If all the cells are of even dimension, then

$$H^{odd}(M) = 0\,, \qquad H^{2k}(M) \simeq \mathbb{Z}^{\#\text{ of } 2k \text{ cells}}\,.$$

4.7 Suppose $M \subset \mathbb{R}^N$ is a compact submanifold, let $f_q(x) = dist(q,x)^2$ for a fixed $q \in \mathbb{R}^N \setminus M$.

4.8 For almost all $q \in \mathbb{R}^N$ the function f_q is Morse.

4.9 Assume that $q = 0$, $p = (a, 0, \dots, 0)$ with $a \in \mathbb{R}_+$, $T_pM = \{x_{n+1} = x_{n+2} = \cdots = 0\}$; then M locally is the graph of a function $g = (a + g_1, g_2, \dots g_{N-n} : \mathbb{R}^n \to \mathbb{R}^{N-n}$, $g_1(0) = 0$, $g_k(0) = 0$ for $k > 1$, $Dg(0) = 0$;
Parametrization of M:

$$\underline{x} = (x_1, x_2, \dots, x_n) \mapsto (a + g_1(\underline{x}), g_2(\underline{x}), \dots g_{N-n}(\underline{x}), x_1, x_2, \dots, x_n)\,.$$

- then

$$f_q(x) = (a + g_1(\underline{x}))^2 + \sum_{j=2}^{N-n} g_j(\underline{x})^2 + \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 + 2aQ(\underline{x}) + \mathcal{O}(||\underline{x}||^4)\,,$$

where Q is a quadratic form of g_1, hence

$$D^2 f_q(p) = 2(I + 2aQ)\,.$$

Therefore

$$ind(p) = \#\{\lambda \in spec\, Q \mid \lambda < -\tfrac{1}{2a}\}$$

Weak Lefschetz

4.10 Lemma: If $M \subset \mathbb{C}^N$ is a complex submanifold, $q \notin M$ and p is a critical point of f_q, then

$$index(p) \leq \dim_{\mathbb{C}}(M)$$

- Proof: we assume as before that $q = 0$, $p = (a, 0, \ldots, 0)$, $a \in \mathbb{R}_+ \subset \mathbb{C}$.
- Very easy algebraic lemma: Suppose Q is a nondegenerate quadratic form on $\mathbb{C}^n$. If v is an eigenvector of the real part $\mathrm{Re}(Q)$ with the eigenvalue λ, then iv is an eigenvector with the eigenvalue $-\lambda$. Hence the eigenvalues are symmetrically distributed with respect to 0.
- Corollary the index of $2(I + a\,\mathrm{Re}(D^2g(0)) = 2(I + 2a\,\mathrm{Re}(Q))$ is at most $\frac{1}{2}\dim_{\mathbb{R}}(M)$.

4.11 If $M \subset \mathbb{C}^N$ is a complex submanifold of the complex dimension n, then M has the homotopy type of n-dimensional CW-complex. Hence $H^k(M; R) = 0$ for $k > n$ (with coefficient in any ring R).

4.12 ,,Weak Lefschetz" aka ,,Lefschetz hyperplane theorem" [Milnor, Morse Theory §7]: If $X \subset \mathbb{P}^N$ is a complex submanifold of dimension n, $i : Y = X \cap \mathbb{P}^{N-1} \to X$, then X is a sum of Y with cells of dimension $k \geq n$. Thus

- $i^* : H^k(X) \to H^k(Y)$ is an isomorphism for $k < n - 1$ and mono for $k = n - 1$,
- $i_* : H_k(Y) \to H_k(X)$ is an isomorphism for $k < n - 1$ and epi for $k = n - 1$.

- Moreover $i_* : \pi_1(Y) \to \pi_1(X)$ is an isomorphism if $2 < n$, epimorphism if $2 = n$.

4.13 If $X \subset \mathbb{P}^N$, and M is a smooth hypersurface of degree d, then $M \cap X \simeq \iota(X) \cap H$, where $\iota : \mathbb{P}^N \to \mathbb{P}(Sym^d(\mathbb{C}^{N+1}))$ is the Veronese embedding and H is a linear hypersurface in $\mathbb{P}(Sym^d(\mathbb{C}^{N+1}))$.
- Hence for complete intersection $X \subset \mathbb{P}^N$ we have information about all Betti numbers, except the middle one:

$$X = X_{N-n} \subset X_{N-n-1} \subset \cdots \subset X_{N-1} \subset X_N = \mathbb{P}^N$$

$\dim(X_i) = N - i$, since $k < n < \dim(X_i)$ for $i < N - n$, we have isomorphisms $H^k(X_i) \simeq H^k(x_{i+1})$.

$$H^k(X) = \begin{cases} \mathbb{Z} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$$

for $k < n$, and from Poincaré duality $H^k(X) \simeq H_{2n-k}(X)$ we get the same result for $n > k$.

4.14 Exercise: compute $\dim(H^n(Q_n))$ for a nonsingular quadric $Q_n \subset \mathbb{P}^{n+1}$.

5 Hodge theory

Differential forms and de Rham cohomology – summary

5.1 Global differential forms on a C^∞-manifold M will be denoted by $A^\bullet(M) = \bigoplus_{k=0}^{\dim M} A^k(M)$. (The notation $\Omega^\bullet(M)$ is reserved for holomorphic forms.)

5.2 $A^\bullet(M)$ is a commutative algebra with gradation $ab = (-1)^{\deg(a)\deg(b)}ba$)

5.3 differential satisfies the Leibniz rule $d(ab) = ad(b) + (-1)^{\deg(a)}b$

5.4 the linear space $A^k(M)$ is the space of the global sections of a sheaf A^k_M.

5.5 $\underline{\mathbb{R}}_M \hookrightarrow A^0_M \to A^1_M \to A^2_M \to \ldots$ is a soft (in particular acyclic) resolution of the constant sheaf $\underline{\mathbb{R}}_M$, therefore

$$H^k(A^\bullet(M), d) = H^k(M; \underline{\mathbb{R}}_M) \simeq H^k_{sing}(M; \mathbb{R}) .$$

The cohomology groups are denoted by $H^k(M)$, we skip $\mathbb{R}$ in the notation.

5.6 exterior product of forms induces multiplication in cohomology $H^k(M) \times H^\ell(M) \to H^{k+\ell}(M)$

5.7 if M is compact, $n = \dim M$ and M has a chosen orientation, then the integral of n-forms induces a map $\int_M : H^n(M) \to \mathbb{R}$. If M is connected, then $\int_M$ is an isomorphism.

5.8 (Poincaré Duality) if M is compact, oriented of dimension n, then the bilinear form

$$\int_M - \wedge - : H^k(M) \times H^{n-k}(M) \to \mathbb{R}$$

is nondegenerate.

5.9 if M is oriented (not necessarily compact), then we consider cohomology with compact supports

$$H^k_c(M) = H^k(A^\bullet_c(M)).$$

Then

$$\int_M - \wedge - : H^k_c(M) \times H^{n-k}(M) \to \mathbb{R}$$

is defined and it is a nondegenerate 2-linear form.

5.10 Having a Riemannian metric on a compact manifold allows to define **harmonic** forms $\mathcal{H}^k(M)$ (see 5.17). The harmonic forms are closed and the resulting map $\mathcal{H}^k(M) \to H^k(M)$ is an isomorphism. However the product of harmonic forms does not have to be harmonic.

Hodge theory for C^∞ manifolds

Suppose M is equipped with Riemannian metric, i.e. a scalar product at each tangent space T_xM. Let $n = \dim M$.

5.11 Volume form is denoted by $vol \in A^n(M)$.

5.12 Hodge star: for $x \in M$

$$* : \Lambda^k T^*_x M \to \Lambda^{n-k} T^*_x M$$

It is defined by the property

$$a \wedge *b = \langle a, b \rangle vol$$

for each $a, b \in \Lambda^k T^*_x M$. The Hodge star extends to

$$* : A^k(M) \to A^{n-k}(M)$$

pointwise.

5.13 We have
(i) $*^2 = (-1)^{k(n-k)}$ on k-forms.
(ii) $\langle \alpha, *\beta \rangle = (-1)^{k(n-k)} \langle *\alpha, \beta \rangle$,

5.14 Let's define $d^* = (-1)^{n(k+1)+1} * d * : A^k(M) \to A^{k-1}(M)$.

5.15 For compact manifold M, $a \in A^{k-1}(M)$, $b \in A^k(M)$ we have

$$\langle da, b \rangle_M = \langle a, d^* b \rangle_M$$

We say that d^* is formally adjoint to d.
Proof

$$0 = \int_M d(a \wedge *b) = \int_M da \wedge *b + (-1)^{k-1} \int_M a \wedge d(*b)\,.$$

Hence

$$\int_M da \wedge *b = (-1)^k \int_M a \wedge d(*b)\,.$$

$$\begin{aligned}
\langle a, d^*b\rangle_M &= \int_M \langle a, (-1)^{d(k+1)+1} * d * b\rangle vol \\
&= (-1)^{d(k+1)+1}\int_M a\wedge * * d * b && \deg(*d*b)=k-1 \\
&= (-1)^{d(k+1)+1+(k-1)(d-k+1)}\int_M a\wedge d*b && d(k+1)+1+(k-1)(d-k+1)\equiv_2 k \\
&= (-1)^k\int_M a\wedge d(*b) = \int_M da\wedge *b = \int_M \langle da, b\rangle vol
\end{aligned}$$

5.16 Laplacian on forms is defined by

$$\Delta = dd^* + d^*d$$

It can be interpreted as the ,,super-commutator" $[d, d^*]_s$.

- In general the supercommutator of elements of a graded algebra $A = \bigoplus_{k\in\mathbb{Z}} A^k$ is defined by

$$[\phi,\psi]_s = \phi\psi - (-1)^{k\ell}\psi\phi \qquad \text{if} \qquad \phi\in A^k, \quad \psi\in A^\ell.$$

5.17 Harmonic forms: $\mathcal{H} := ker\Delta$.

5.18 The operator $\Delta = dd^* + d^*d$ is formally self-adjoint

$$(\Delta a, b) = (a, \Delta b).$$

5.19 For a compact oriented C^∞-manifold M the following holds in $A^\bullet(M)$
1) $\mathcal{H} = ker(d)\cap ker(d^*)$
2) $ker(d^*) = im(d)^\perp$, $ker(d) = im(d^*)^\perp$, $ker(\Delta) = im(\Delta)^\perp$,
(hence $\mathcal{H} = ker(d)\cap im(d)^\perp$)
3) the spaces $\mathcal{H}$, $im(d)$ and $im(d^*)$ are perpendicular.

Proof:
1) suppose $a\in ker(\Delta)$:

$$0 = (\Delta a, a) = (dd^*a, a) + (d^*da, a) = (d^*a, d^*a) + (da, da) = ||d^*a||^2 + ||da||^2$$

2) Let $P = d,\ d^*$ of Δ. If $a\in ker(P^*)$ then $0 = (P^*a, b) = (a, Pb)$, hence $a\in im(P)^\perp$.
Conversely, if $a\in im(P)^\perp$, then $0 = (a, PP^*a) = ||P^*a||^2$, so $P^*a = 0$.
3) It remains to show that the spaces $im(d)$ and $im(d^*)$ are perpendicular $(d^*a, db) = (a, d^2b) = 0$.
(Here we used that $d^2 = 0$, all the rest was an abstract properties of formally adjoint operators.)

5.20 Hodge decomposition

$$\mathcal{A}^\bullet(M) = \underbrace{im(d)\oplus\mathcal{H}}_{ker(d)}\oplus im(d^*).$$

This decomposition is orthogonal.

- The decomposition follows from a general property of elliptic differential operators, which we will not prove. We would have to extend the space of C^∞ forms and consider Sobolev spaces. See [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Cambridge Studies in Advanced Mathematics. Theorem 5.22, p.128-9]. For any *elliptic* operator $P: C^\infty(E)\to C^\infty(F)$

$$C^\infty(E) = ker(P)\oplus P^*(C^\infty(F)).$$

(Exercise: prove the corresponding statement for a linear map between finite dimensional spaces.)

- In our case $P = \Delta$, $P^* = \Delta$

$$A^\bullet(M) = \mathcal{H}\oplus\Delta(A^\bullet(M)).$$

Moreover we have

$$im(\Delta)\subset im(d) + im(d^*).$$

But from orthogonality $(im(d)\oplus im(d^*))\cap\mathcal{H} = 0$, hence

$$im(\Delta) = im(d)\oplus im(d^*).$$

5.21 Corollary 1: $\mathcal{H} \to H^*(M)$ is an isomorphism.
Moreover: if $\Delta(a) = 0$ and $a' = a + db$, then $||a'|| \geq ||a||$.
Any harmonic form is the representative of its cohomology class, which has the smallest norm.

5.22 Corollary 2: Tricky proof of the Poincaré duality: Let $[\alpha] \neq 0 \in H^*(M)$, then there exists a class $[\beta]$ (in the complementary gradation) such that $\int_M \alpha \wedge \beta \neq 0$.

- Proof: let's assume that α is harmonic. Set $\beta = *\alpha$. Then β is harmonic as well ($d(*\alpha) = \pm * d^*(\alpha) = 0$ and $d^*(*\alpha) = \pm * d(\alpha) = 0$). We have

$$\int_M \alpha \wedge *\alpha = \int_M ||\alpha||^2 vol = ||\alpha||^2_M .$$

5.23 Heat equation $\alpha : \mathbb{R}_+ \to A^*(M)$ with the initial condition $\alpha(0) = \alpha$

$$\frac{d}{dt}\alpha(t) = -\Delta\alpha(t) ,$$

see [D. Arapura, Algebraic Geometry over the Complex Numbers] §8

- the solution exists for $t \geq 0$
- $\alpha_H := \lim_{t\to\infty} \alpha(t)$ exists and is a harmonic form.

(Laplacian has nonnegative eigenvalues: if $\Delta(\alpha) = \lambda\alpha$ then

$$\lambda||\alpha|| = (\Delta\alpha, \alpha) = ||d\alpha||^2 + ||d^*\alpha||^2 \geq 0 .$$

hence the limit exists.)

- $\alpha = \alpha_H + \Delta G(\alpha)$, where $G(\alpha) = \int_0^\infty (\alpha(t) - \alpha_H)dt$ is the Green operator $G : \mathcal{H}^\perp \to A^\bullet(M)$.
- Let's check for α being an eigenvector $\Delta\alpha = \lambda\alpha$, $\lambda \neq 0$: The solution is of the form $\alpha(t) = e^{-\lambda t}\alpha$. Then

$$\Delta\left(\int_0^\infty e^{-\lambda t}\alpha dt\right) = \int_0^\infty e^{-\lambda t}\lambda\alpha\, dt = \left(\int_0^\infty e^{-\lambda t}\lambda\, dt\right)\alpha = \alpha .$$

- If $\beta(t)$ is a solution with the initial condition β, then $d\beta(t)$ is a solution with the initial condition $d\beta$ (because $d\Delta = ddd^* + dd^*d = dd^*d = dd^*d + d^*dd = \Delta d$).
- If $\alpha = \alpha_H + d\beta$ then $\alpha_t = \alpha_h + d\beta_t$.
- If $d\alpha = 0$, then $d\alpha_t = 0$ and $[\alpha_t] = [\alpha]$

Proof $\alpha = \alpha_h + d\beta$, $(\alpha_t - \alpha_h)' = -\Delta(d\beta_t) = -d\Delta(\beta_t)$

Hermitian linear algebra

Suppose (V, I) is a real vector space with a complex structure.

5.24 Hermitian product

$$V \otimes V \to \mathbb{C}$$
$$\langle\langle v, w\rangle\rangle = \langle v, w\rangle - i\omega(v, w)$$

consists of:
– I-invariant scalar product $\langle v, w\rangle$,
– I-symplectic form $\omega(v, w)$
– the scalar product and the symplectic form determine each other $\omega(v, w) = \langle I(v), w\rangle = -\langle v, I(w)\rangle$.

5.25 The volume form is defined as the wedge of an orthonormal (positively oriented) basis vectors of V^*:

- suppose $\dim_{\mathbb{C}}(V) = n$

$$vol = (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n) = (\tfrac{i}{2})^n (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n)$$

-

$$\omega = \sum_{k=1}^n dx_k \wedge dy_k = \tfrac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k .$$
$$\omega^n = n! vol .$$

5.26 ω as a differential form on $\mathbb{C}^n$ is closed and U_n invariant.

6

6.1 General picture:

- 1) Manifolds with Riemannian metric $\rightsquigarrow$ harmonic forms represent cohomology classes
- 2) Complex manifolds $\rightsquigarrow$ complex coordinates, forms dz and $d\bar{z}$, decomposition of differential forms into types (p, q)
- 1) & 2) hermitian manifolds $\rightsquigarrow$ the differential form ω of type (1,1)
- 3) Kähler manifolds (the condition $d\omega = 0$) $\rightsquigarrow$ decomposition of cohomology into types and $\mathfrak{sl}_2$ action.

Main example - the projective space

6.2 The projective space $\mathbb{P}^n$ can be obtained as the quotient S^{2n+1}/S^1. The tangents space $T_{[z]}\mathbb{P}^n = T_z S^{2n+1}/T_z(S^1 z)$.

6.3 the form ω is well defined on the quotient space: tangent vector space $T_z(S^1 z)$ is spanned by the vector $(\frac{d}{dt}e^{it}z)_{t=0} = Iz$. Therefore for $w \in T_z S^{2n+1} = z^\perp$

$$\omega(w, Iz) = \langle Iw, Iz\rangle = \langle w, z\rangle = 0\,.$$

Since ω is S^1 invariant the choice of $z \in [z]$ leads to the same form.

6.4 We define a 2-form $\omega_{FS}(w_1, w_2) = \omega(\tilde{w}_1, \tilde{w}_2)$, where $\tilde{w}_1, \tilde{w}_2$ are any lifts of $w_1, w_2 \in T_{[z]}\mathbb{P}^n$ to $T_z S^{2n+1}$. Let $p : S^{2n+1} \to \mathbb{P}^n$ be the projection.

- The form ω_{FS} satisfies $p^*(\omega_{FS}) = \omega$.
- The form ω_{FS} is closed because p^* is injective on forms and $d\omega = 0$.

6.5 For $n = 1$, on $U_0 = \{z_0 \neq 0\} \simeq \mathbb{C}$ there is a section

$$(s_1, s_2) : U_0 \to S^3 \subset \mathbb{C}^2$$

$$s_1(z) = \frac{1}{\sqrt{1 + |z|^2}}\,, \quad s_2(z) = \frac{z}{\sqrt{1 + |z|^2}}\,.$$

Then

$$\omega_{FS}(z) = \frac{i}{2}(s_1^*(dz_1 \wedge d\bar{z}_1) + s_2^*(dz_2 \wedge d\bar{z}_2))$$

Since the image of s_1 is contained in $\mathbb{R}$, thus $s_1^*(dz_1 \wedge d\bar{z}_1) = 0$. The second summand is equal to Jacobian times $dx \wedge dy$,

$$(x, y) \mapsto \left(\frac{x}{\sqrt{1 + x^2 + y^2}}, \frac{y}{\sqrt{1 + x^2 + y^2}}\right)$$

$$J(x, y) = \det\left(\frac{1}{(1 + x^2 + y^2)^{3/2}}\begin{bmatrix} 1 + y^2 & -xy \\ -xy & 1 + x^2 \end{bmatrix}\right)$$

Hence

$$\omega_{FS}(z) = \frac{1}{(1 + x^2 + y^2)^2} dx \wedge dy\,.$$

The volume of $\mathbb{P}^1$:

$$\int_{\mathbb{R}^2} \frac{1}{(1 + x^2 + y^2)^2} dx \wedge dy = 2\pi \int_{\mathbb{R}_+} \frac{r}{(1 + r^2)^2} dr \overset{r^2=u}{=} \pi \int_{\mathbb{R}_+} \frac{1}{(1 + u)^2} du = \pi\,.$$

6.6 Often we normalize

$$\omega_{FS} := \frac{1}{\pi}\omega_{FS}\,.$$

With this normalization $\int_{\mathbb{P}^n}\omega_{FS}^n = 1$.

6.7 The class $[\omega_{FS}] \in H^2(\mathbb{P}^n)$ is a generator of $H^*(\mathbb{P}^n;\mathbb{R}) \simeq \mathbb{R}[h]/(h^{n+1})$

6.8 The normalized class $[\omega_{FS}]$ is represented by the (Poincaré dual) of $[\mathbb{P}^{n-1}]$ with $\mathbb{P}^{n-1}$ embedded as a linear hypersurface.

$1\frac{1}{2}$–linear algebra

6.9 Complex structure on a real vector space is an automorphism I satisfying $I^2 = -id$. It decomposes $V_{\mathbb{C}} := V \otimes \mathbb{C}$ into eigenspaces

$$V_{\mathbb{C}} = V_i + V_{-i}\,.$$

Necessarily $\dim V$ is even and one can find a real basis $e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_n$ of V, such that $I(e_k) = f_k$, $I(f_k) = -e_k$.

- The vectors $e_k - if_k$ form a basis of V_i: $\quad I(e_k - if_k) = f_k + ie_k = i(e_k - if_k)$
- The vectors $e_k + if_k$ form a basis of V_{-i}: $\quad I(e_k + if_k) = f_k - ie_k = -i(e_k - if_k)$

6.10 We are more concerned about the dual space: $\mathbb{C}$-linear form are said to have the type (1,0)

$$\Lambda^{10}V^* := \{\phi \in Hom_{\mathbb{R}}(V,\mathbb{C}) \mid \phi(Iv) = i\phi(v)\ \}\,,$$

the antilinear forms are said to have the type (0,1)

$$\Lambda^{01}V^* := \{\phi \in Hom_{\mathbb{R}}(V,\mathbb{C}) \mid \phi(Iv) = -i\phi(v)\ \}\,,$$

We have

$$V^* \otimes \mathbb{C} = \Lambda^{10}V^* \oplus \Lambda^{01}V^*\,.$$

6.11 The dual basis is denoted by

$$dx_k := e_k^*\,, \quad dy_k := f_k^*\,.$$

We define

$$dz_k := dx_k + idy_k\,, \quad d\overline{z}_k := dx_k - idy_k\,.$$

The 1-forms dz_k are the basis of $\Lambda^{10}V^*$, and $d\overline{z}_k$'s are the basis of $\Lambda^{01}V^*$.

6.12 We have a $\mathbb{C}$-linear isomorphism $(V^*, I) \overset{\Phi}{\simeq} (\Lambda^{10}V^*, i)$, $\Phi(f)(v) = f(v) - if(Iv))$
$\Phi(If)(v) = f(Iv) - if(I^2v) = f(Iv) + if(v) = i(f(v) - if(Iv)) = i\Phi(f)(v)$

And an anti-linear isomorphism: $(V^*, I) \overset{\psi}{\simeq} (\Lambda^{01}V, i)$, $\Psi(f)(v) = f(v) + if(Iv))$
$\Psi(If)(v) = f(Iv) + if(I^2v) = f(Iv) - if(v) = -i(f(v) + if(Iv)) = -i\Psi(f)(v)$

6.13 The exterior forms of the type (p,q):

$$\Lambda^k V_{\mathbb{C}}^* = \bigoplus_{p+q=k} \Lambda^{pq}\,, \qquad \Lambda^{pq} := \Lambda^p(\Lambda^{10}V^*) \wedge \Lambda^q(\Lambda^{01}V^*)\,.$$

– Conjugation acts on $\Lambda^k(V^* \otimes \mathbb{C}) = (\Lambda^k V^*) \otimes \mathbb{C}$. We have

$$\overline{\Lambda^{pq}} = \Lambda^{qp}.$$

– The operator I acts on $\Lambda^{p,q}V^*$ via multiplication by $i^{(p-q)}$

6.14 Remark: the form ω belongs to $\Lambda^2 V^* \cap \Lambda^{11}V^* \subset \Lambda^2 V_{\mathbb{C}}^*$.

6.15 Exercise $\Lambda^{10} \perp \Lambda^{01}$

Linear algebra on the tangent space

6.16 Assume that M is a complex manifold, then tangent space T_pM at each point $p \in M$ is a complex vector space. We treat it as a real vector space wit an automorphism I given by the multiplication by i. Globally $I \in \mathrm{End}(TM)$, i.e. is an endomorphism of the tangent bundle.

6.17 Our method: Linear algebra $\rightsquigarrow$ differential/complex manifolds structure

6.18 An almost complex manifold (M, I) is a pair, where M is a real C^∞-manifold and $I \in \mathrm{End}(TM)$ a tensor satisfying $I^2 = -id$ (i.e. a complex structure in each T_pM smoothly depending on the point $p \in M$.)
[W tym roku nie będziemy rozważać rozmaitości niemal zespolonych w ogólności, ale od razu zakładamy, że mamy rozmaitość zespoloną. Patrz [Huybrechts §1.2], w szczególniści [Huybrechts 2.6.19]]

6.19 The eigenspace of I acting on $T^*M \otimes \mathbb{C}$ decomposes this bundle into a direct sum of complex subbundles:

$$(T^*M \otimes \mathbb{C})_i \oplus (T^*M \otimes \mathbb{C})_{-i}\,.$$

- The global sections of the above bundles will be denoted by $A^{10}(M)$ and $A^{01}(M)$.
- Locally a form in $A^{10}(M)$ can be written as $\sum_k a_k(z)dz_k$. If we change the coordinate chart it can be written in the same form. This is because for a holomorphic map $\phi : U' \to U$ the composition $z_k \circ \phi : U' \to \mathbb{C}$ is holomorphic, so $d(z_k \circ \phi) = \sum_k a'_k(z')dz'_k$ for some functions $a'_k(z')$.

6.20 The complexified space of forms decomposes as a direct sum $A^k(M)_\mathbb{C} = \bigoplus_{p+q=k} A^{p,q}(M)$.
- (p, q)-form locally can be written as

$$\sum_{|A|=p} \sum_{|B|=q} a_{A,B}(z)dz_A \wedge d\bar{z}_B$$

Hermitian structure

Assume that M has a hermitian structure, this is equivalent of having Riemannian metric, which is I-invariant.

6.21 The form ω is of the type (1,1), in addition it has real coefficients.

6.22 The Lefschetz operator

$$L(\alpha) := \omega \wedge \alpha\,.$$

6.23 Suppose $\dim V = n$. Let us define $H \in \mathrm{End}(\Lambda V^*)$ as the multiplication by $k - n$ on $\Lambda^k V$.
- We have $[H, L] = 2L$.

6.24 Let us define the adjoint operator L^*

$$\langle L\alpha, \beta\rangle = \langle \alpha, L^*\beta\rangle$$

lowering the gradation by 2. We have:
- $[H, L] = 2L, \quad [H, L^*] = -2L^*$
- $[L, L^*] = H\,.$

6.25 The vector space of forms at a point $p \in M$, i.e. $\bigoplus_k \Lambda^k T^*_p(M)$ is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{Z})$. We obtain a representation on the global forms.

$$\rho : \mathfrak{sl}_2(\mathbb{Z}) \to \mathrm{End}(A^\bullet(M)), \qquad \rho(h) = H, \qquad \rho(\ell) = L, \qquad \rho(\ell^*) = L^*\,,$$

where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \ell = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \ell^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Recollection from representation theory

See e.g. [Fulton-Harris, Representation Theory: A First Course, §11]

- Representation of a Lie algebra $\mathfrak{g}$ on V is (by definition) a morphism of Lie algebras $\mathfrak{g} \to \mathrm{End}(V)$.
- Having a representation of $\mathfrak{sl}_2(\mathbb{Z})$ is equivalent to having three linear maps L, H, L^* such that

$$[H, L] = 2L, \quad [H, L^*] = -2L^* \quad [L, L^*] = H\,.$$

It costs nothing to extend linearly such representation to $\mathfrak{sl}_2(K)$ if V is a vector space over the field $K = \mathbb{R}$ or $\mathbb{C}$.

- Any finite dimensional representation of $\mathfrak{sl}_2(\mathbb{Z})$ is a direct sum of simple subrepresentations. („Simple" means that it has no nontrivial subrepresentations.)
- Simple representations are of the form $S_k = Sym^k(\mathbb{C}^2)$ (the same for the theory over $\mathbb{R}$). Other description:

$$S_k = \{\text{homogeneous polynomials of degree } k \text{ in variables } x, y\}$$

$$\ell(f) = y\tfrac{df}{dx}\,, \quad \ell^*(f) = x\tfrac{df}{dy}\,.$$

6.26 With the assumption that $d\omega = 0$ we will show that $\mathfrak{sl}_2$ action on forms induces an action on cohomology and deduce very important consequences.

7 Differential on complex manifolds

7.1 If M is a complex manifold, then

$$d(A^{p,q}(M)) \subset A^{p+1,q}(M) \oplus A^{p,q+1}(M)$$

$$d = \partial + \overline{\partial}, \quad \partial^2 = 0 = \overline{\partial}^2, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0\,.$$

7.2 Let $\Omega^p(M)$ denote the form of the type $(p, 0)$ with holomorphic coefficients.

- Lemma:

$$\Omega^p(M) = ker(\overline{\partial} : A^{p,0}(M) \to A^{p,1}(M))\,.$$

7.3 Dolbeault complex: for $0 \le p \le \dim_{\mathbb{C}} M$ we have a complex

$$0 \to A^{p,0}(M) \overset{\overline{\partial}}{\to} A^{p,1}(M) \overset{\overline{\partial}}{\to} \dots \overset{\overline{\partial}}{\to} A^{p,\dim M}(M) \to 0, ,$$

7.4 We define Dolbeault cohomology [Huybrechts 2.6.20]:

$$H^q_{Dol}(M; \Omega^p) := H^q(A^{p,\bullet}(M), \overline{\partial})$$

7.5 Holomorphic Poincaré lemma [Huybrechts 1.3.7]: the complex of sheaves on M

$$0 \to \Omega^p \to A^{p,0} \to A^{p,1} \to A^{p,2} \to \dots$$

is exact.

- This means that if $\bar{\partial}\alpha = 0$, $\alpha \in A^{p,q}(U)$, then *locally* there exists β such that $\bar{\partial}\beta = \alpha$, i.e. for each point $p \in U$ there exists $V \subset U$, $p \in V$ and $\beta \in A^{p,q-1}(V)$ such that $\bar{\partial}\beta = \alpha_{|V}$.

7.6 It is enough to solve the following problem:

- Holomorphic Poincaré lemma in 1 variable: Let $\overline{\mathbb{D}}_\varepsilon \subset U \subset \mathbb{C}$, where U is open, and let $f \in C^\infty(U; \mathbb{C})$ be a smooth function. Suppose $\frac{\partial}{\partial \bar{z}} f = 0$, then there exists $g \in C^\infty(\mathbb{D}_\varepsilon; \mathbb{C})$ such that $\frac{\partial}{\partial \bar{z}} g = f$.
- The solution to the previous problem, with $f = f_w$ depending smoothly on a parameter can be found in a way that g_w depends smoothly on the parameter.

$$g(z) = \mathcal{I}(f)(z) = \frac{1}{2\pi i}\int_{\mathbb{D}_\varepsilon} \tfrac{f(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}.$$

- Analogy with the real case:

— for a real (compactly supported) $f : \mathbb{R} \to \mathbb{R}$ we define the primitive function

$$\mathcal{I}(f)(x) = \int_{-\infty}^{x} f(\xi)d\xi = \int_{-\infty}^{+\infty} K(\xi - x)f(\xi)d\xi,$$

where

$$K(\xi) = \begin{cases} 0 \text{ if } \xi < 0 \\ 1 \text{ if } \xi \geq 0 \end{cases} \qquad \text{and} \qquad K'(\xi) = \delta_0\,.$$

So the primitive function is expressed by the convolution with K, i.e $\mathcal{I}(f)(x) = (K * f)(x)$. (In general $(f_1 * f_2)' = f_1' * f_2$.)

— similarly for complex, compactly supported function $f : \mathbb{C} \to \mathbb{C}$

$$g(z) = (K * f)(z),$$

where $K(z) = \frac{1}{2\pi i}\frac{1}{z}$, which has the property $\frac{\partial}{\partial \bar{z}} K = \delta_0$

Sheaf cohomology - a summary, see eg [Huybrechts, Appendix B]

7.7 Cohomology with the coefficients in a sheaf $\mathcal{F}$: there are two important construction

- Čech cohomology
- Sheaf cohomology as the derived functor of Γ - taking the global sections.

1) we find a resolution of $\mathcal{F}$, i.e. an exact complex

$$0 \to \mathcal{F} \to I^0 \to I^1 \to I^2 \to \dots$$

with the sheave A^k sufficiently good (acyclic, e.g. injective)

2) we apply the functor of global sections (and cut off the first term)

$$\Gamma(I^0) \to \Gamma(I^1) \to \Gamma(I^2) \to \dots$$

This complex is no longer exact.

3) W compute cohomology:

$$H^k(M; \mathcal{F}) = H^k(\Gamma(I^\bullet))\,.$$

We have $H^0(M; \mathcal{F}) = \Gamma(\mathcal{F})$, because the functor Γ is left-exact.

7.8 In our case, when the base is paracompact any *soft* resolution is acyclic. („Soft" means, that sections defined on a closed set can be extended to global sections.)

- Suppose M is a C^∞-manifold. Any sheaf which is a module over the ring of C^∞-functions is soft.
- The complex of C^∞-forms on C^∞-manifold $A^0 \to A^1 \to A^2 \to \dots$ is a resolution of the sheaf $ker(d : A^0 \to A^1) = \underline{\mathbb{R}}_M$, the sheaf of locally constant functions.

7.9 The sheaves $A^{p,q}$ are A^0-modules, hence they are soft.

- The Dolbeault complex is a resolution of $\Omega^p_M = ker(\bar{\partial} : A^{p,0} \to A^{p,1})$
-

$$H^k(M; \Omega^p) = H^k(A^{p,\bullet}(M))$$

i.e the Dolbeault cohomology is the sheaf cohomology in the sense of the homological algebra.

7.10 If M is a complex manifold, then $A^\bullet_{\mathbb{C}} = \bigoplus_{p+q=\bullet} A^{p,q}$ is a resolution of the sheaf $\mathbb{C}_M$.

• For $p \geq 0$ define the Hodge'a filtration (on the sheaf level)

$$F^p A^k = \bigoplus_{p'+q=k,\, p'\geq p} A^{p',q}\,.$$

Claim: $F^p A^\bullet$ is a subcomplex of $A^\bullet$.

• The resulting filtration in cohomology $H^k(M;\mathbb{C}) = H^k\big(A^\bullet(M)_{\mathbb{C}}\big)$

$$F^p H^k(M;\mathbb{C}) = im(H^k(F^p A^\bullet(M)) \to H^k\big(A^\bullet(M)_{\mathbb{C}}\big))\,.$$

7.11 We have

$$F^p A^k / F^{p+1} A^k \simeq A^{p,k-p}\,.$$

• The quotient map is a map of complexes (with a shift of the gradation)

$$(F^p A^{p+\bullet}, d) \to (A^{p,\bullet}, \bar\partial)$$

• We have maps of complexes (I denote the shift of gradations by $[i]$. i.e. $(F[i]^k = F^{k+i})$

$$A^\bullet \leftarrow F^p A^\bullet \to A^{p,\bullet}[-p]$$

• Passing to cohomology:

$$H^k(M;\mathbb{C}) \leftarrow H^k(M; F^p A^\bullet) \to H^{k-p}(X;\Omega^p)\,.$$

7.12 The relation between cohomologies of the quotients with cohomology of the entire sheaf is given by the spectral sequence

$$E_1^{p,q} = H^{p+q}(F^p A^\bullet(M)/F^{p+1}A^\bullet(M)) = H^q(M;\Omega^p_M) \;\Rightarrow H^{p+q}(M;\mathbb{C}).$$

Generalities about spectral sequence

If $C^\bullet$ is a complex with decreasing filtration

$$C^\bullet = F^0 C^\bullet \supset F^1 C^\bullet \supset F^2 C^\bullet \supset \dots,$$

then one wishes to relate cohomologies $H^*(F^p C^\bullet / F^{p+1} C^\bullet)$ with $H^*(C^\bullet)$.

• There exists a spectral sequence (under some boundness of degree assumptions)

$$E_0^{p,q} = F^p C^{p+q}/F^{p+1}C^{p+q}, \qquad E_1^{p,q} = H^{p+q}(F^p C^\bullet / F^{p+1} C^\bullet)\,, \qquad \dots .$$

• There exists a sequence of tables $E_r^{p,q}$ with differentials of degree $(1-r, r)$, such that
1) $H^*(E_r^{\bullet,\bullet}) = E_{r+1}^{\bullet,\bullet}$
2) $E_\infty^{p,q} = F^p H^{p+q}(\mathbb{C}^\bullet)/F^{p+1}H^{p+q}(\mathbb{C}^\bullet)$

7.13 For the total complex of the bicomplex $A^{p,q}(M)$ with the Hodge filtration $F^p A^\bullet(M) = A^{\geq p,\bullet}(M)$ the resulting spectral sequence is called the **Frölicher spectral sequence**.

Hodge theory for Hermitian manifolds

7.14 Hermitian structure on a complex manifold M is a choice of a Hermitian product in each tangent space.

• such structure is a section of $T^*M \otimes \overline{T}^* M$ which is symmetric and positively definite. We assume that it is a C^∞

• real part is a scalar product, the imaginary part - a differential 2-form (which does not have to be closed).

• Hermitian structures exist for paracompact manifolds: we can chose a Hermitian structure locally in maps and glue them using partition of unity.

7.15 We extend Hodge * $\mathbb{C}$-linearly

- If $\dim M = 1$

$$*dz = *(dx+idy) = dy-idx = -i(dx+idy) = -idz\,, \qquad *d\bar{z} = *(dx-idy) = dy+idx = i(dx-idy) = id\bar{z}$$

$$*1 = \omega = dx \wedge dy = \tfrac{i}{2} dz \wedge d\bar{z}\,, \qquad *\omega = 1$$

- In higher dimensions

$$* : \Lambda^{p,q} \xrightarrow{\simeq} \Lambda^{n-q,n-p}$$

$$*dz_I \wedge d\bar{z}_J = c\, dz_{[n]\setminus J} \wedge d\bar{z}_{[n]\setminus I}$$

Exercise: compute c.

- Occasionally will appear antilinear star

$$\bar{*} : \Lambda^{p,q} \xrightarrow{\simeq} \Lambda^{n-p,n-q}, \qquad \bar{*}(\alpha) = *\bar{\alpha} = \overline{*\alpha}\,.$$

7.16 We have operators real L, L^*, $H = [L, L^*] = (\deg - n)id$ acting on C^∞-forms $A^*(X)$. The adjoint operator

$$L^* = *^{-1} L * = (-1)^{\deg} * L * \ .$$

(The sign should be $(-1)^{(\dim_{\mathbb{R}} M - \deg)\deg}$ but here $\dim_{\mathbb{R}} TM$ is even). Often in literature L^* is denoted by Λ, but it can be confused with the exterior power). The adjoint operator satisfies $(L\alpha, \beta\rangle = \langle \alpha, L^*\beta)$.

- The complexified operators L, L^*, $H = [L, L^*] = (\deg - n)id$ act on $A^*(X)_{\mathbb{C}}$. Hence $A^*(X)_{\mathbb{C}}$ becomes a (infinite dimensional) representation of $\mathfrak{sl}(2)$.
- We take complexification, because we are also interested in the bigradation, available only over $\mathbb{C}$.

7.17 We define operators

$$\partial^* = - * \bar{\partial} * : A^{p,q}(X) \to A^{p-1,q}(X)\,,$$

$$(p,q) \mapsto (n-q, n-p) \mapsto (n-q, n-p+1) \mapsto (p-1, q)$$

and

$$\bar{\partial}^* = - * \partial * : A^{p,q}(X) \to A^{p,q-1}(X)\,.$$

We have $d^* = \partial^* + \bar{\partial}^*$.

- explanation of signs: $d^* = (-1)^{\dim_{\mathbb{R}} M(\deg + 1) + 1} * d * = - * d *$

7.18 Kähler structure

It can be defined in equivalent ways:

- Definition 1: locally there exists local coordinates in which $\omega = \frac{i}{2}\sum_k dz_k \wedge d\bar{z}_k + \mathcal{O}(||x||^2)$.

i.e. in some coordinates the Hermitian metric is the same as for flat the manifold $\mathbb{C}^n$ up to the terms of order 2.

- Definition 2: $d\omega = 0$
- Proof 1) $\Rightarrow$ 2) obvious.

7.19 Proof 2) $\Rightarrow$ 1) [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Prop 3.14]

- How to construct good coordinates?

$$\omega = \tfrac{i}{2} \sum_k dz_k \wedge d\bar{z}_k + \sum_{k,l} (\varepsilon^h_{k,l} + \varepsilon^a_{k,l}) dz_k \wedge d\bar{z}_l + \mathcal{O}(|z|^2)$$

where $\varepsilon^h_{k,l}$ is a holomorphic linear form, $\varepsilon^a_{k,l}$ antiholomorphic liner form.

- $\overline{\varepsilon^a_{k,l}} = \varepsilon^h_{l,k}$ since ω is real.
- $\frac{\partial}{\partial z_j}\varepsilon^h_{k,l} = \frac{\partial}{\partial z_k}\varepsilon^h_{j,l}$ since ω is closed

7.20 Hodge identities:
i) $[\bar\partial, L] = [\partial, L] = 0$ (since ω is closed)
i') equivalently $[L^*, \partial^*] = [L^*, \bar\partial^*] = 0$
ii) $[\bar\partial^*, L] = i\partial$, $[\partial^*, L] = -i\bar\partial$
ii') equivalently $[L^*, \bar\partial] = -i\partial^*$, $[L^*, \partial] = i\bar\partial^*$ (this is the most difficult, the rest follows)
iii) $[\partial, \bar\partial^*]_s = [\partial^*, \bar\partial]_s = 0$ (i.e $\partial\bar\partial^* + \bar\partial^*\partial = 0$ etc, this is a formal consequence of ii))
iv) $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$ and it commutes with ∂, $\bar\partial$, ∂^*, $\bar\partial^*$, L i L^* (formal algebraic proof)

7.21 Short proof from [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Prop 6.5].

- Assume according to Definition 1) that ω has a standard form up to the terms of order 2. Therefore in calculations involving only the **first derivatives** at a point we can assume that

$$\omega = \frac{i}{2}\sum dz_k \wedge d\bar z_k$$

- We show ii') i.e. $[L^*, \partial] = i\bar\partial^*$. It is enough to check

$$([L^*, \partial](\alpha))_{z=0} = i(\bar\partial^*\alpha)_{z=0}$$

- We decompose $\omega = \sum_k \omega_k$, $\omega_k = \frac{i}{2}dz_k \wedge d\bar z_k$.

The adjoint operator $L_k^* = (\omega_k\wedge)^*$ is expressed by the contraction of differential forms

$$L_k^* = -2i\iota_{\bar v_k}\iota_{v_k}\,,$$

where $v_k = \frac{\partial}{\partial z_k}$, $\bar v_k = \frac{\partial}{\partial \bar z_k}$.

- We decompose $\bar\partial = \sum \bar\partial_k$. The adjoint differentials

$$\partial_k^* = -2\frac{\partial}{\partial \bar z_k}\iota_{v_k}\,, \qquad \bar\partial_k^* = -2\frac{\partial}{\partial z_k}\iota_{\bar v_k},$$

A sample of check in dim=1

$$\partial^* f\,dz = -*\bar\partial * f\,dz = -*\bar\partial(-if dz) = i*\frac{\partial}{\partial\bar z}f d\bar z\wedge dz = -2\frac{\partial}{\partial \bar z}f\, *\frac{i}{2}dz\wedge d\bar z = -2\frac{\partial}{\partial\bar z}f$$

7.22 Second Hodge identity $[L^*, \partial] = i\bar\partial^*$ for the flat metric: We decompose $\bar\partial = \sum\bar\partial_k$ and $L^* = \sum L_k^*$. Show that $\bar\partial_k^* = -2\iota_{\bar v_k}\frac{\partial}{\partial z_k}$, where $\bar v_k = \frac{\partial}{\partial \bar z_k}$. Note ∂_ℓ commutes with L_k^* for $k \neq \ell$. It remains to check $[L_k^*, \partial_k]$ for $\alpha = f dz_I \wedge d\bar z_J$, considering 4 cases $k \in$ or $\notin$ to I and J. For example: suppose $k \in I$, $k \in J$. That is $I = \{k\} \cup I'$, $J = \{k\}\cup J'$:

$$[L_k^*, \partial_k] f dz_k \wedge d\bar z_k \wedge dz_{I'} \wedge d\bar z_{J'} =$$
$$L_k^*\partial_k(f dz_k \wedge d\bar z_k \wedge dz_{I'} \wedge d\bar z_{J'}) - \partial_k L_k^*(f dz_k \wedge d\bar z_k \wedge dz_{I'} \wedge d\bar z_{J'}) =$$
$$2i\partial_k(f dz_{I'} \wedge d\bar z_{J'}) =$$
$$2i\frac{\partial f}{\partial z_k} dz_k \wedge dz_{I'} \wedge d\bar z_{J'} =$$
$$2i\frac{\partial f}{\partial z_k}\iota_{\bar v_k}(d\bar z_k \wedge dz_k \wedge dz_{I'} \wedge d\bar z_{J'}) =$$
$$i\bar\partial^*(f dz_k \wedge d\bar z_k \wedge dz_{I'} \wedge d\bar z_{J'}),$$

- It remains to check 3 other cases.

7.23 For a computational proof see Huybrechts.

- The Huybrechts' proof of ii'): an operator $d^c = I^{-1}dI$ is introduced and the adjoint operator $(d^c)^*$

$$d^c = -i(\partial - \bar\partial), \quad (d^c)^* = -*\,d^c\,*\,.$$

He shows ii") $[L^*, d] = -(d^c)^*$. The proof is computational, using Lefschetz decomposition into $L^k\alpha$, where α is primitive.

8 Kähler identities cont.

8.1 Proof of iii) and iv) from i)&ii)

- iii)

$$i[\partial, \bar{\partial}^*] \overset{ii)}{=} [\partial, [L^*, \partial]] = \partial L^* \partial - \partial^2 L^* + L^* \partial^2 - \partial L^* \partial = 0$$

- To show and iv) it is convenient to introduce the language of supercommutators $[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba$. In that notation

$$\Delta_\partial = [\partial, \partial^*] .$$

- Leibniz rule, equivalent to the graded Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)} [b, [a, c]] .$$

$$[[a, b], c] = [a, [b, c]] + (-1)^{\deg(b)\deg(c)} [[a, c], b] .$$

-

$$\Delta_\partial = [\partial^*, \partial] \overset{ii)}{=} i[[L^*, \bar{\partial}], \partial] \overset{Leibniz}{=} i([L^*, \underbrace{[\bar{\partial}, \partial]}_{0}] - [[L^*, \partial], \bar{\partial}]) \overset{ii)}{=} [\bar{\partial}^*, \bar{\partial}] = \Delta_{\bar{\partial}}$$

and from iii) $\Delta = \Delta_\partial + \Delta_{\bar{\partial}}$.

$$[L, \Delta_\partial] = [L, [\partial, \partial^*]] \overset{Leibniz}{=} [\underbrace{[L, \partial]}_{0}, \partial^*] + [\partial, [L, \partial^*]] \overset{ii)}{=} i[\partial, -i\bar{\partial}] = 0$$

Cohomology of Kähler manifold

- Corollary $H^*(M) \simeq \mathcal{H}$ is a representation of $\mathfrak{sl}_2(\mathbb{Z})$.

8.2 STRATEGY: We obtain a list operators, decompositions etc. We have shown that this structure, initially defined on forms, survives in cohomology of a complex Kähler variety.

Lefschetz decomposition

8.3 Let W be a representation of $\mathfrak{sl}_2$,

- the eigenspaces of h are equal $W_{k-n} = \Lambda^k T_x^* M \otimes \mathbb{C}$,
- L^k defines an isomorphism $W_{-k} \to W_k$ $(k \geq 0)$,
- $L : W_k \to W_{k+2}$ is mono for $k < 0$, epi for $k + 2 > 0$.
- Lefschetz decomposition: For $k \geq 0$ let us define the primitive subspace

$$P_k = \{w \in W_{-k} \,|\, L^{k+1} w = 0\}.$$

We have

$$W_{-k} = P_k \oplus L P_{k+2} \oplus L^2 P_{k+4} \oplus \dots .$$

8.4 The primitive cohomology classes (attention at the gradation shift): for $0 \leq k \leq n$ let us define

$$P^{n-k} = \{\alpha \in H^{n-k}(M) \mid L^{k+1}\alpha = 0\}$$

$$P^{p,q} = \mathcal{H}^{p,q} \cap P^{p+q}_{\mathbb{C}} .$$

We have

$$P^{n-k}_{\mathbb{C}} = \bigoplus_{p+q=n-k} P^{p,q} .$$

Practical consequences:

8.5 Hard Lefschetz Theorem Let M be a Kähler manifold od dimension n and let$0 \leq k \leq n$. Then

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism.

- It follows

$$\dim H^k(M) \leq \dim H^{k+2}(M) \qquad \text{if } k+1 \leq n\,,$$
$$\dim H^k(M) \geq \dim H^{k+2}(M) \qquad \text{if } k+1 \geq n\,.$$

8.6 Hodge decomposition for the operator $\bar\partial$

$$\mathcal{A}^{p,q}(M) = \underbrace{im(\bar\partial) \oplus \mathcal{H}^{p,q}_{\bar\partial}}_{ker(\bar\partial)} \oplus im(\bar\partial^*).$$

$$\bar\partial : A^{p,q-1} \to A^{p,q}, \qquad \bar\partial^* : A^{p,q+1} \to A^{p,q}\,.$$

-

$$H^q(M;\Omega^p) \simeq \mathcal{H}^{p,q}_{\bar\partial}\,,$$

- Since $\Delta_{\bar\partial} = \frac{1}{2}\Delta$, we have

$$\mathcal{H}^{p,q}_{\bar\partial} = \mathcal{H}^{p,q}\,,$$
$$\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p}\,, \qquad *\mathcal{H}^{p,q} = \mathcal{H}^{n-q,n-p}\,.$$

-

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}\,.$$

8.7 Hodge decomposition in cohomology

- Recall the Hodge filtration

$$F^p A^k(M) = \bigoplus_{p'\geq p,\, p+q=k} A^{p,q}(M)$$

and the induced filtration in cohomology

$$F^p H^k(M) = im\big(H^k(F^p A^\bullet(M) \to H^*(M)\big)\,.$$

The definition is independent from the metric and

$$F^p H^k(M) = \text{image of} \bigoplus_{p'\geq p,\, p+q=k} \mathcal{H}^{p,q}\,.$$

- Conjugating we obtain

$$\overline{F^p H^k(M)} = \text{image of} \bigoplus_{p'\geq p,\, p+q=k} \mathcal{H}^{q,p}\,.$$

- Define

$$H^{p,q}(M) = F^p H^{p+q}(M)) \cap \overline{F^q H^{p+q}(M)}\,.$$

This definition does not depend on the Kähler metric.

$$H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M)\,,$$

8.8 Let $h^{p,q} = \dim H^{p,q}(M)$.

- Hard Lefschetz implies inequalities

$$h^{p,q} \leq \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \leq n\,,$$

$$h^{p,q} \geq \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \geq n\,,$$

- The symmetries $h^{p,q} = h^{n-p,n-q} = h^{q,p}$ are organized in the ,,Hodge diamond"
- For example for $n = 3$

$$\begin{array}{ccccccc} & & & h^{33} & & & \\ & & h^{32} & & h^{23} & & \\ & h^{31} & & h^{22} & & h^{13} & \\ h^{30} & & h^{21} & & h^{12} & & h^{03} \\ & h^{20} & & h^{11} & & h^{02} & \\ & & h^{10} & & h^{01} & & \\ & & & h^{00} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & \spadesuit & & \spadesuit & & \\ & \diamondsuit & & \clubsuit & & \diamondsuit & \\ \square & & \heartsuit & & \heartsuit & & \square \\ & \diamondsuit & & \clubsuit & & \diamondsuit & \\ & & \spadesuit & & \spadesuit & & \\ & & & 1 & & & \end{array}$$

- Hard Lefschetz implies inequalities

$$h^{p,q} \leq \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \leq n\,,$$

$$h^{p,q} \geq \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \geq n\,,$$

8.9 Moreover

– If $k = n - (p+q) \geq 0$ then $L^k : H^{p,q}(M) \to H^{p+k,q+k}(M)$ is an isomorphism

– If $p + q \leq n$ then

$$H^{p,q}(M) = P^{p,q}(M) \oplus L(P^{p-1,q-1}(M) \oplus L^2(P^{p-2,q-2}(M) \oplus \dots$$

- Corollary: If M Kähler and compact, then the (Frölicher) spectral sequence

$$H^q(M;\Omega^p) \Rightarrow H^{p+q}(M;\mathbb{C})$$

degenerates on E_1, i.e.

$$E_1^{p,q} = H^q(M;\Omega^p) = E_\infty^{p,q}\,.$$

(the higher differentials vanish).

8.10 Corollary: Suppose M Kähler and compact: if $\alpha \in \Omega^p(M)$ then $\partial\alpha = 0$.

- Holomorphic implies closed.
- This is a generalization of: global holomorphic function is constant.

8.11 We say that M is Calabi-Yau if $\Omega^n \simeq \mathcal{O}_M$
(according to more restrictive definitions it is assumed additionally $H^0(M,\Omega^p) = 0$ for $0 < p < n$)

- Thus $h^{n,q} = h^{0,q}$.
- For $n = 3$ the Hodge diamond looks like this

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \clubsuit & & 0 & \\ 1 & & \heartsuit & & \heartsuit & & 1 \\ & 0 & & \clubsuit & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

- We say that M^* is a cohomological mirror of M if $h^{pq}(M^*) = h^{n-p,q}(M)$.
- For 3-manifolds this means $h^{12}(M^*) = h^{11}(M)$ i $h^{11}(M^*) = h^{12}(M)$.
- Problem: how to find M^*?

8.12 Serre duality: the exterior product

$$\wedge : \Omega^p \times \Omega^q \to \Omega^{p+q}$$

defines a bilinear map

$$H^k(M;\Omega^p) \times H^\ell(M;\Omega^q) \to H^{k+l}(M;\Omega^{p+q}) .$$

If $k+\ell = p+q = n$ we obtain compose it with the integral $\int : H^n(M;\Omega^n) \simeq H^{2n}(M;\mathbb{C}) \to \mathbb{C}$.

- By Poincaré duality this form is nondegenerate

$$H^k(M;\Omega^p) \simeq H^{n-k}(M;\Omega^{n-p})^* .$$

- More generally: we have a nondegenerate form

$$H^k(M;E) \times H^{n-k}(M;E^* \otimes \Omega^n) \to H^n(M;\Omega^n) \to \mathbb{C}$$

for a locally free sheaf E. In particular for $\Omega^p = E$:

$$\Omega^{n-p} \simeq \underline{Hom}(\Omega^p,\Omega^n) = (\Omega^p)^* \otimes \Omega^n$$

and we recover the previous formula.

9 Signature, Cousin problems

Signature

9.1 If V is a real vector space with a symmetric nondegenerate form ϕ, then the signature

$$\sigma(V,\phi) := \dim\{\text{maximal positive definite subspace}\} - \dim\{\text{maximal negative definite subspace}\} ,$$

i.e. $\#\{+\} - \#\{-\}$ after diagonalization.

- If there exists $Z \subset V$ such that $Z^{\perp_\phi} = N$, then $\sigma(\phi) = 0$.

9.2 For oriented compact C^∞-manifold M of dimension $4m$ the intersection pairing in $H^{2m}(M;\mathbb{R})$

$$[\alpha]\cdot[\beta] = \int_M \alpha \wedge \beta$$

is symmetric and nondegenerate. Its signature is called the signature of M, denoted $sgn(M)$ or $\sigma(M)$.

$$\sigma(M) := \sigma(H^{2m}(M), \text{intersection form}) .$$

- Instead of $H^{2m}(M)$ we can take $H^{even}(M)$ declaring $\alpha\cdot\beta = 0$ if $\deg(\alpha)+\deg(\beta) \neq \dim(M)$.
- Exercise: the signature is multiplicative: $\sigma(M\times N) = \sigma(M)\sigma(N)$.
- If M is a boundary of an oriented $4m+1$-manifold W, then $\sigma(M) = 0$.

Proof: let $\iota : M = \partial W \to W$. Define

$$Z = \iota^*(H^{2m}(W)) \subset \iota^*(H^{2m}(M)) = V .$$

For $[\alpha],[\beta] \in H^{2m}(W)$ by Stokes

$$\int_M \iota^*\alpha \wedge \iota^*\beta = \int_W d(\alpha\wedge\beta) = 0 .$$

It remains to show, that if

$$(*)\ [\alpha]\cdot[\iota^*\beta] = 0 \text{ for all } [\beta] \in H^{2m}(W),$$

then $[\alpha] = \iota^*[\tilde\alpha]$.
The condition (*) is equivalent to

$$[\alpha] \in ker\big(H^{2m}(M) \stackrel{d}{\to} H^{2m+1}(W,M) \simeq (H^{2m}(W))^*\big) .$$

From the exact sequence

$$H^{2m}(W) \stackrel{\iota^*}{\to} H^{2m}(M) \stackrel{d}{\to} H^{2m+1}(W,M)$$

we get the conclusion.

9.3 Instead the real intersection form we consider $H^*(M;\mathbb{C})$ with the hermitian form. The resulting signature is the same.

9.4 Hodge'a-Riemann relations [Huybrechts 3.3.15]: Define the hermitian form $B(\alpha,\beta)$ on $H^k(M)$ as:

$$B(\alpha,\beta)=\int_M \alpha\wedge\bar\beta\wedge\omega^{n-k}\,.$$

This form is symmetric or antisymmetric depending on the parity of k

$$B(\alpha,\beta)=(-1)^k\overline{B(\alpha,\beta)}\,.$$

- It is nondegenerate: for $\alpha\in H^k(M)$ there exists $\beta\in H^k(M)$ such that $B(\alpha,\beta)\neq 0$.

— Let $\gamma\in H^{2n-k}(M)$ such that $\int_M \alpha\wedge\bar\gamma\neq 0$ (e.g. $\gamma=\bar{*}\alpha$)
— By Hard Lefschetz $\gamma=L^{n-k}\beta$ for some $\beta\in H^k(M)$

$$B(\alpha,\beta)=\int_M \alpha\wedge\bar\gamma\neq 0\,.$$

- The pairing B restricted to $H^{p,q}(M)$ is non degenerate. The form $\gamma=\bar{*}\alpha$ is of the type $(n-q,n-p)$, hence $L^{k-n}\gamma$ is of the type $(n-q-n+k,n-p-n+k)=(p,q)$.

9.5 Antisymmetric forms over $\mathbb{C}$ can be turned into symmetric:

- if ϕ jest antisymmetric, i.e.

$$\phi(a,b)=-\overline{\phi(b,a)}\,,$$

then $\psi(a,b):=i\phi(a,b)$ is symmetric.

- If hermitian form ψ is symmetric then $\psi(a,a)=\overline{\psi(a,a)}$, hence $\psi(a,a)\in\mathbb{R}$
- We say that such form is positive definite if

$$\psi(a,a)>0\qquad\text{for } a\neq 0\,.$$

9.6 Theorem [Hodge-Riemann relations]: Let $k=p+q$. The form

$$i^{p-q}\cdot(-1)^{k(k-1)/2}B(\alpha,\beta)$$

restricted to the primitive space

$$P^{p,q}(M)=P^k(M)\cap H^{p,q}(M)$$

is symmetric and positive definite.

9.7 Proof reduces to calculations for $\Lambda^\bullet\mathbb{C}^n$: one has to check the sign of the form B_0 restricted to $P^{p,q}\subset\Lambda^{p,q}\subset\Lambda(\mathbb{C}^n)^*\otimes\mathbb{C}$. Here B_0 is defined by the formula

$$\alpha\wedge\bar\beta\wedge\omega^{n-k}=B_0(\alpha,\beta)dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n\,.$$

- We check the following identity for $\alpha\in P^k$:

$$(***)\qquad L^{n-k}\alpha=(-1)^{\frac{k(k-1)}{2}}(n-k)!*I(\alpha)\,,$$

or equivalently as in [Huybrechts]

$$*L^{n-k}\alpha=(-1)^{\frac{k(k+1)}{2}}(n-k)!\,I(\alpha)\,,$$

where I is the complex structure acting on $\Lambda(\mathbb{C}^n)^*\otimes_{\mathbb{R}}\mathbb{C}$. On the (p,q) forms it acts by the multiplication by i^{p-q}.

- We show inductively

$$L^j\alpha = (-1)^{\frac{k(k-1)}{2}} \frac{j!}{(n-k-j)!} * L^{n-k-j} I(\alpha).$$

- Having $(***)$:

$$\alpha \wedge \overline{\alpha} \wedge \omega^{n-k} = \alpha \wedge L^{n-k}(\overline{\alpha}) = \alpha \wedge (-1)^{\frac{k(k-1)}{2}} (n-k)! * I(\overline{\alpha}) =$$

$$= (-1)^{\frac{k(k-1)}{2}} \alpha \wedge *(n-k)! I(\overline{\alpha}) = i^{q-p} (-1)^{\frac{k(k-1)}{2}} (n-k)! < \alpha, \alpha > vol$$

9.8 Corollary [Huybrechts 3.3.18]: Let $n = 2m$. Then M is a real manifold of dimension $4m$. The intersection form in the middle dimension $2m$ is symmetric. It coincides with $B(\alpha, \beta)$.

- The signature of M is defined as the signature of the intersection form $H^{2m}(M)$ is equal to

$$\sum_{p+q \leq m,\ 2|p+q} (-1)^{\frac{k(k-1)+p-q}{2}} \dim(P^{p,q}(M))$$

- We have equality $\dim P^{p,q}(M) = h^{p,q} - h^{p-1,q-1}$. Using symmetries of Hodge diamond $h^{p,q} = h^{q,p} = h^{n-p,n-q}$ we obtain a formula for the signature

$$sgn(M) = \sum_{p,q=0,\ 2|p+q}^{\dim(M)} (-1)^p h^{p,q}\,.$$

- Example: let $n = 4$: we sum up the terms for which $p - q$ is even:

$$sgn(M) = \begin{array}{ccccc} +p^{4,0} & -p^{3,1} & +p^{2,2} & -p^{1,3} & +p^{0,4} \\ & +p^{2,0} & -p^{1,1} & +p^{0,2} & \\ & & +p^{0,0} & & \end{array}$$

$$= \begin{array}{ccccc} +h^{4,0} & -h^{3,1} + h^{2,0} & +h^{2,2} - h^{1,1} & -h^{1,3} + h^{0,2} & +h^{0,4} \\ & +h^{2,0} & -h^{1,1} + h^{0,0} & +h^{0,2} & \\ & & +h^{0,0} & & \end{array}$$

$$= \begin{array}{ccccc} & & +h^{4,4} & & \\ & +h^{4,2} & -h^{3,3} & +h^{2,4} & \\ +h^{4,0} & -h^{3,1} & +h^{2,2} & -h^{1,3} & +h^{0,4} \\ & +h^{2,0} & -h^{1,1} & +h^{0,2} & \\ & & +h^{0,0} & & \end{array}$$

- We can neglect the remaining summands with $p+q$ odd, since $(-1)^q h^{p,q}$ cancels with $(-1)^p h^{q,p}$

$$sgn(M) = \sum_{p,q=0}^{\dim(M)} (-1)^p h^{p,q}$$

Further we can transform the formula:

$$sgn(M) = \sum_{p,q=0}^{\dim(M)} (-1)^q h^{p,q} = \sum_{p=0}^{\dim M} \chi(M; \Omega^p)\,.$$

- Example: For the connected surfaces the intersection form is of the type $(2h^{2,0} + 1, h^{1,1} - 1)$.

Motivation leading to the notion of Čech cohomology :

[B. V. Shabath, Introduction to complex analysis II, Chapter IV].

9.9 Additive Cousin Problem: find a global meromorphic function with prescribed poles.
Let $M = \bigcup U_i$ be a covering. On each U_i there is given a meromorphic function f_i. We assume that the differences $g_{ij} = (f_i)_{|U_i \cap U_j} - (f_j)_{|U_i \cap U_j}$ are holomorphic. Does there exist a meromorphic function f on M such that each difference $f_{|U_i} - f_i$ is holomorphic?

9.10 Multiplicative Cousin Problem:
Let $\{U_i\}_{i \in I}$ be a covering of M. On each U_i there is given a meromorphic function f_i. We assume that the quotients $g_{ij} = \frac{(f_i)_{|U_i \cap U_j}}{(f_j)_{|U_i \cap U_j}}$ are holomorphic. Does there exist a meromorphic function f on M such that each quotient $\frac{f_{|U_i}}{f_i}$ is holomorphic?

9.11 The answer is in the language of Čech cohomology. For a covering $\mathscr{U} = \{U_i\}$ the Čech complex is defined by:

$$\check{C}^k(\mathscr{U}) = \prod_{i_0 < i_1 < \cdots < i_k} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k})\,.$$

Notation: for a multiindex $I = \{i_0 < i_1 < \cdots < i_k\}$ let $U_I = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k}$. For $\{s_I\} \in \check{C}^{k-1}(\mathscr{U})$ define the differential

$$d(\{s_I\})_J = \sum_{a=1}^{k} (-1)^a (s_{J \setminus j_a})_{|U_J}$$

For example

$$d(\{s_i\})_{j_0,j_1} = (s_{j_1})_{|U_{j_0,j_1}} - (s_{j_0})_{|U_{j_0,j_1}}$$

$$d(\{s_{i_0,i_1}\})_{j_0,j_1,j_2} = s_{j_1,j_2} - s_{j_0,j_2} + s_{j_0,j_1} \qquad \text{restricted to } U_{j_0,j_1,j_2}$$

9.12 Čech cohomology is defined by $\check{H}^k(\mathscr{U}; \mathcal{F}) = H^k(\check{C}^\bullet(\mathscr{U}; \mathcal{F}), d)$.

9.13 Additive Cousin Problem : Let $\mathcal{F} = \mathcal{O}_M$, the collection of functions $\{g_{i,j}\} \in \check{C}^1(\mathscr{U}; \mathcal{O}_M)$ satisfies the cocycle condition:

$$g_{ij} - g_{ik} + g_{jk} = 0.$$

It defines an element of Čech cohomology of the covering $H^1(\{U_i\}; \mathcal{O}_M)$. The cohomology class is trivial if the cocycle is a coboundary, i.e. there exists a collection of elements $h_i \in \mathcal{O}_M(U_i)$ such that $g_{ij} = h_j - h_i$.

- the Cousin problem has a solution if and only if the cohomology class $[g_{ij}] = 0$.

Proof: If $g_{ij} = h_j - h_i$, then the meromorphic functions $\tilde{f}_i = f_i + h_i$ agree at the intersections:

$$\tilde{f}_i - \tilde{f}_j = f_i + h_i - f_j + h_j \qquad \text{on } U_i \cap U_j\,.$$

(The converse - exercise.)

9.14 Multiplicative Cousin problem has a positive solution if the cocycle g_i/g_j defines the trivial class in $H^1(\{U_i\}; \mathcal{O}^*_M)$.

9.15 Passing to a finer cover defines a map of Čech cohomology (it does not depend on inscribing function).

9.16 Theorem: If M is paracompact, then

$$\lim_{\substack{\longrightarrow \\ \mathscr{U}}} \check{H}^k(\mathscr{U}; \mathcal{F}) \simeq H^k(M; \mathcal{F})$$

(The RHS is in the sense of homological algebra.)

9.17 If the covering is acyclic (i.e. $H^k(U_I;\mathcal{F}) = 0$ for any multiindex I and $k > 0$) then

$$H^k(\{U_i\};\mathcal{F}) \simeq H^k(M;\mathcal{F}).$$

9.18 Sufficient conditions for being acyclic:

- For locally constant sheaves on topological spaces: if all U_I are contractible,
- For coherent sheaves in algebraic geometry: if U_I are affine,
- For coherent sheaves in analytic geometry: if U_I are Stein spaces

Definition $U \subset M$ is Stein if:
– for any pair of points $p, q \in U$ there exists an analytic function $f \in \mathcal{O}_U$ such that $f(p) \neq f(q)$.
– (holomorphic convexity) for any compact set $K \subset U$ the set

$$\bar{K} := \{p \in U \mid \forall f \in \mathcal{O}_U \ |f(p)| \leq sup_{q\in K}|f(q)|\ \}$$

is compact.

9.19 In the cousin problems one can pass to a finer coverings. Since $H^1(\mathbb{P}^n;\mathcal{O}_M) = 0$, so on $\mathbb{P}^n$ the additive Cousin problem has always a positive solution. On curves of positive genus - not always: $genus = \dim H^1(C;\mathcal{O}_C)$.

10 Vector bundles and connection

10.1 Let $Vect^1(X)$ denotes the set of isomorphism classes of (topological) complex linear bundles X. Looking at the definition of Čech cohomology we discover a bijection

$$Vect^1(X) = H^1(X; C(-,\mathbb{C}^*)\,,$$

where $C(-,\mathbb{C}^*)$ denotes the sheaf of continuous functions with values in $\mathbb{C}^*$.

- Similarly the isomorphism classes of holomorphic vector bundles over complex manifolds are identified with $H^1(X;\mathcal{O}^*_X)$.

10.2 The exponential exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to C(-,\mathbb{C}) \stackrel{exp}{\to} C(-,\mathbb{C}^*) \to 0$$

induces the map

$$c_1 : Vect^1(X) = H^1(X; C(-,\mathbb{C}^*)) \to H^2(X,\mathbb{Z})\,.$$

This is the first Chern class, we will give a differential definition later.

Divisors and line bundles, [Huybrechts §2.3]

We identify holomorphic bundles with sheaves of holomorphic sections. Locally free sheaves of $\mathcal{O}_X$-modules are identified with holomorphic vector bundles.

10.3 Divisor $D = \sum a_i D_i$ is a formal combination of codimension 1 indecomposable subvarieties (we assume that X is an analytic manifold).

- We define a restriction of divisors to open sets: $D_{|U} = \sum_{D_i\cap U\neq\emptyset} a_i(D_i \cap U)$
- D is an effective divisor iff all $a_i \geq 0$, we write $D \geq 0$.

10.4 A meromorphic function defines a principal divisor $div(f) = zeros(f) - poles(f)$.

10.5 Any divisor D defines a line bundle $\mathcal{O}_X(D)$, viewed as a subsheaf of the sheaf $Mero_X$ of meromorphic functions: for each open $U \subset X$

$$\mathcal{O}_X(D)(U) = \{f \in Mero_X(U) \ : \ div(f) + D_{|U} \text{ is effective in } U\}$$

- If $D_1 = D_2 + div(g)$, where g is a global meromorphic function, then $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$. The multiplication by g defines an isomorphism.
- We have an injection

$$\{Divisors\}/\{Principal\ divisors\} \hookrightarrow \{Holomorphic\ Line\ Bundles\}\,.$$

The image consists of line bundles admitting a meromorphic section.

- Suppose $s : X \dashrightarrow L$ is a meromorphic section. Define $D = zeros(s) - poles(s)$. Then $L \simeq \mathcal{O}_X(D)$
- If $L \to X$ is an algebraic bundle, then it admits a meromorphic section.

10.6 Example: the tautological bundle over $\mathbb{P}^1$.
On $U_0 = \{z_0 \neq 0\}$ we have a section $s_0([1 : z]) = (1, z)$, on $U_1 = \{z_1 \neq 0\}$ we have a section $s_1([w : 1]) = (w, 1)$. These sections do not vanish, so they define local trivializations. The transition function $g_{1,0}s_0 = s_1$ satisfies

$$g_{1,0}(z)(1, z) = (1/z, 1)\,.$$

Hence

$$g_{1,0} : U_0 \cap U_1 = \mathbb{C}^* \to GL_1(\mathbb{C}) = \mathbb{C}^*\,,$$
$$g_{0,1}(z) = z^{-1}\,.$$

- The section s_0 has pole at ∞, hence the tautological bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(D)$, where $D = -\{[0 : 1]\}$. Equally well we could have $D = -\{[1 : 0]\}$ or any other point.

10.7 Taking the transition function $g_{1,0}(z) = z^k$ we obtain $\mathcal{O}_{\mathbb{P}^1}(k)$

10.8 Example: $\mathcal{O}_{\mathbb{P}^n}(kH)$, where $H \simeq \mathbb{P}^{n-1}$ is the divisor at infinity. If $k \geq 0$ the global sections $H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(kH))$ are naturally identified with $\mathbb{C}[z_1, z_2, \ldots, z_n]_{\deg \leq k} \simeq \mathbb{C}[z_0, z_1, \ldots, z_n]_{\deg = k}$ and

$$\mathcal{O}_{\mathbb{P}^n}(kH) \simeq (tautological^*)^{\otimes k} =: tautological^{\otimes -k}\,.$$

The only section for $k < 0$ is 0 and

$$\mathcal{O}_{\mathbb{P}^n}(kH) \simeq tautological^{\otimes -k}\,.$$

10.9 The bundle $\mathcal{O}_{\mathbb{P}^n}(kH) \simeq tautological^{\otimes -k}$ is denoted $\mathcal{O}_{\mathbb{P}^n}(k)$.

- If Y is a hypersurface in $\mathbb{P}^n$ of degree d, then $\mathcal{O}_{\mathbb{P}^n}(Y) \simeq \mathcal{O}_{\mathbb{P}^n}(d)$.

Connection for a vector bundle over C^∞-manifold

10.10 Connection is a linear map $\nabla : C^\infty(X; E) \to C^\infty(X; T^*_X \otimes E) =: A^1_X(E)$ satisfying the Leibniz rule

$$\nabla(fs) = df \otimes (s) + fs\,.$$

10.11 Let ∇ and ∇' be two connections. The difference $\nabla - \nabla'$ is A^1_X linear. [Huybrechts 4.2.3]

- Locally, every connection is of the form $\nabla = d + a$, where $a \in A^1(X, \mathrm{End}(E))$.
- If ∇ is a connection and $a \in A^1(X, \mathrm{End}(E)) = C^\infty(X, T^*X \otimes \mathrm{End}(E))$, then $\nabla + a$ is a connection.
- Affine combination of connections $t\nabla_1 + (1 - t)\nabla_2$ is a connection.
- Applying a partition of unity associated to the trivializing atlas of E we glue together local connections and obtain a global one.
- The space of connections is isomorphic to $A^1(X, \mathrm{End}(E))$. (But no connection is distinguished.)

Connections concordant with structures

10.12 Suppose E is a hermitian bundle. A connection is Hermitian if

$$d < s_1, s_2 >=< \nabla s_1, s_2 > + < s_1, \nabla s_2 > .$$

(again the Leibniz formula) [Huybrechts 4.2.9]

10.13 Let V be a Hermitian vector space. By $\mathrm{End}(V,h)$ denote the endomorphism a satisfying

$$< a(v), w > + < v, a(w) >= 0 .$$

If $V = \mathbb{C}^n$ with the standard hermitian product, then $\mathrm{End}(V,h) = \mathfrak{u}_n = \{A \in M_{n\times n}(\mathbb{C}) \mid A + \bar{A}^T = 0\}$.

- For a Hermitian vector bundle $\mathrm{End}(V,h)$ is a real vector bundle of the dimension $= \dim(\mathfrak{u}_{\mathrm{rk}\, E})$.
- As before we prove that the space of Hermitian connection is a real vector space isomorphic to $A^1(X, \mathrm{End}(E,h))$. (But no connection is distinguished.)

10.14 If $\mathrm{rk}E = 1$**.** Then $\mathrm{End}(E,h) \simeq \mathbb{R}$.

10.15 Suppose X is a complex manifold, E is a holomorphic bundle (the transition functions are holomorphic). Let $A^k(X,E) = \Gamma(A^k_X \otimes E)$, $A^k(X,E) = \bigoplus_{p+q=k} A^{p,q}(X,E)$. The operator $\bar\partial$ is well defined

$$\bar\partial_E : A^{p,q}(X,E) \to A^{p,q+1}(X,E) .$$

Warning: the operator ∂ does not commute with the transition functions. Thus ∂_E is not defined, unless the transition functions are locally constant.

10.16 The connection decomposes into components $\nabla^{10} + \nabla^{01}$. We say that ∇ is compatible with the complex structure if $\nabla^{0,1} = \bar\partial_E$.

10.17 The space of connections compatible with complex structure is isomorphic to $A^{1,0}(X, \mathrm{End}(E))$.

10.18 Theorem [Huybrechts 4.2.14]: For a Hermitian holomorphic bundle there exists exactly one connection compatible with the complex structure.

- In local coordinates: let H be the matrix of the Hermitian product, $\nabla = d + A$, (we identify A locally with a matrix, we call it connection matrix)

$$A \in M_{n\times n}(A^{1,0}(X)) , \qquad H \in M_{n\times n}(C^\infty(X)) , \qquad \bar H = H^T , \qquad n = \mathrm{rk}(E).$$

- the Hermitian condition reads

$$dH = A^T H + H\overline{A} .$$

Hence

$$\partial H = A^T H ,$$

so

$$A = \bar H^{-1} \partial(\bar H) .$$

- If $n = 1$, $H = [h]$. then $a = \partial \log(h)$.

10.19 Example $L = \mathcal{O}(-1)$ on $\mathbb{P}^n$, i.e. the tautological bundle, $L \subset \mathbb{C}^{n+1} \times \mathbb{P}^n$ has the induced Hermitian structure from the trivial bundle $\mathbb{C}^{n+1}$. The connection form

$$A = \partial \log(||s||^2) ,$$

where s is any section (trivialization) of L.

- For example on the chart $\{z_0 \neq 0\} \simeq \mathbb{C}^n$ there is a section

$$s([1 : z_1 : \cdots : z_n]) = (1 : z_1 : \cdots : z_n) ,$$

the differential

$$F_\nabla = d\partial \log(1 + ||z||^2) = -\partial\bar\partial \log(1 + ||z||^2) ,$$

is called the curvature.

- Note:

$$\frac{i}{2\pi} F_\nabla = -\omega_{FS} .$$

11 Chern classes

11.1 We extend the connection using Leibniz formula to obtain the operator $\nabla_E : A^k(X,E) \to A^{k+1}(X,E)$.

11.2 Theorem: The curvature $F_\nabla = \nabla^2 : A^0(E) \to A^2(E)$ is $A^0(X)$-linear, hence it defines a section o the bundle $\Lambda^2 T^*X \otimes \mathrm{End}(E))$.

11.3 Locally in the matrix notation

$$F_\nabla = dA + A \wedge A \in M_{n\times n}(A^2(X)).$$

11.4 For a line bundle $E = L = \mathbb{C} \times X$: we have $\mathrm{End}(L) = \mathbb{C}$ and $A \wedge A = 0$ (since A is a 1×1 matrix). Then $H = [h]$, $h : X \to \mathbb{R}$

$$F_\nabla = dA = d\partial \log(h) = \bar\partial\partial \log(h)\,.$$

11.5 Example $L = \mathcal{O}(-1)$ on $\mathbb{P}^n$, i.e. the tautological bundle, $L \subset \mathbb{C}^{n+1} \times \mathbb{P}^n$ has the induced Hermitian structure from the trivial bundle $\mathbb{C}^{n+1}$:

$$F_\nabla = d\partial \log(||s||^2) = -\partial\bar\partial \log(||s||^2),$$

where v is any section of L,

$$\frac{i}{2\pi} F_\nabla = -\omega_{FS}\,.$$

- For example on the chart $\{z_0 \neq 0\} \simeq \mathbb{C}^n$ there is a section

$$s([1 : z_1 : \cdots : z_n]) = (1 : z_1 : \cdots : z_n)\,,$$

hence

$$F_\nabla = -\partial\bar\partial \log(1 + ||z||^2)\,.$$

- Note: $c_1(\mathcal{O}(-1)) = -[\omega_{FS}]$.

11.6 Connection on E induces a connection on $\mathrm{End}(E)$

$$(\nabla f)(s) := \nabla(f(s)) - f\nabla s = [\nabla, f]s\,.$$

In particular we can apply ∇ to F_∇.

11.7 Bianchi identity:

$$\boxed{\nabla(F_\nabla) = 0 \in A^3(X, \mathrm{End}(E)),}$$

because $[\nabla, \nabla \circ \nabla] = 0$.

- Locally for $\nabla = d + A$ we have

$$dF_\nabla = d(dA + A \wedge A) = dA \wedge A - A \wedge dA = [dA, A] = [F_\nabla, A]\,.$$

Hence

$$0 = \nabla(F_\nabla) = dF_\nabla + [A, F_\nabla]\,.$$

We obtain a formula for the differential

$$\boxed{dF_\nabla = [F_\nabla, A]\,.}$$

Differential definition of Chern classes

Huybrechts §4.4

11.8 Theorem: For any polynomial map $P : \mathrm{End}(\mathbb{C}^n) \to \mathbb{C}$ which is invariant with respect to conjugation the form $P(\nabla_E^2) \in A^{2\deg(P)}(X)$ is closed.

• Lemma (see Milnor-Stasheff, Appendix C, p.297) For $X = (x_{ij})_{i,j}$ define the matrix $P'(X) = (\frac{\partial P}{\partial x_{ji}})_{i,j}$ (note, that the indices i, j are exchanged). We have:
(1) $dP(X) = tr(P'(X) \cdot dX)$.
(2) if P is Ad-invariant, then the matrices $P'(X)$ and X commute.

Proof:
ad (1) easy
ad (2) $P((I + tE_{ij})X) = P(X(I + tE_{ij}))$, hence
$\sum_k x_{i,k} \frac{\partial P}{\partial x_{jk}} = \sum_k \frac{\partial P}{\partial x_{ki}} x_{k,j}$

• Proof of theorem:

$$dP(F_\nabla) \overset{(1)}{=} tr(P'(F_\nabla)dF_\nabla) = tr(P'(F_\nabla)[F_\nabla, A]) = tr(P'(F_\nabla) \wedge F_\nabla \wedge A - P'(F_\nabla) \wedge A \wedge F_\nabla) =$$

$$\overset{(2)}{=} tr(F_\nabla \wedge (P'(F_\nabla) \wedge A) - (P'(F_\nabla) \wedge A) \wedge F_\nabla) = tr([F_\nabla, P'(F_\nabla) \wedge A]) = 0\,.$$

11.9 Remark: the map

$$\mathbb{C}[M_{n\times n}(\mathbb{C})]^{GL_n} \to \mathbb{C}[\text{diagonal matrices}]^{\Sigma_n} = \mathbb{C}[\sigma_1, \sigma_2, \dots, \sigma_n]$$

is an isomorphism. If P is Ad-invariant, then it can be expressed by the coefficients of the characteristic polynomial. Equivalently, $P(A)$ is a symmetric function in eigenvalues of A.

11.10 The 2-form $P(F_\nabla)$ defines a cohomology class, which does not depend on the connection (dowód TBA).

• For $P = \left(\frac{i}{2\pi}\right)^k \sigma_k$, ($(-1)^k \sigma_k$ is ($\mathrm{rk}E - k$)-th coefficient of the characteristic polynomial) the resulting forms represent the Chern classes.

The first Chern class $c_1(L)$ of a line bundle - various constructions

• Axiomatic definition, see Milnor-Stasheff:

a)	$c_1 : Vect^1(-) \to H^2(-, \mathbb{Z})$ is a natural transformation of functors $Top \to GrAb$
b)	$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$
c)	$c_1(\mathcal{O}_{\mathbb{P}^1}(1)) =$ the distinguished generator $[pt] \in H^2(\mathbb{P}^1)$

• The identification $Vect^1(X) = [X, \mathbb{P}^\infty] = [X, K(\mathbb{Z}, 2)] = H^2(X; \mathbb{Z})$,
where $[-, -]$ denotes the set of homotopy classes of maps.

• Via the differential in the long exact sequence of cohomologies associated to the short exact sequence of sheaves

$$0 \to 2\pi i\mathbb{Z} \to C(X, \mathbb{C}) \overset{\exp}{\to} C(X, \mathbb{C}^*) \to 0$$

$$0 = H^1(X, C(X, \mathbb{C})) \to H^1(X, C(X, \mathbb{C}^*)) \overset{\simeq}{\to} H^2(X; 2\pi i\mathbb{Z}) \to H^2(X, C(X, \mathbb{C})) = 0$$

Note, that we have an identification $Vect^1(X) = \check{H}^1(X, H^1(X, C(X, \mathbb{C}^*))$.

• via the obstruction theory: the obstruction to the existence of a nonzero section belongs to

$$H^2(X; \pi_1(\mathbb{C}^*)) \simeq H^2(X; \mathbb{Z})$$

• $c_1(L) = [\mathit{zero\ section}] \in H^2(L) \simeq H^2(X)$

• a definition via connection (when X is a manifold) t.j.w. $\frac{i}{2\pi}[F_\nabla] = \frac{i}{2\pi}[\partial\bar\partial \log h] \in H^2(X; \mathbb{C})$, [Huybrechts §4].

Generalities about characteristic classes for higher rank bundles

- Let

$$Vect^n(X) = \{\text{isomorphism classes of } n\text{-dimensional complex vector bundles over } X\}$$

- Def: a characteristic class on n-dimensional bundle is a transformation of functors $hTop \to Sets$

$$Vect^n(-) \quad\longrightarrow\quad H^*(-)\,.$$

Since $Vect^n(-)$ is representable,

$$Vect^n(X) = \{\text{homotopy classes } f : X \to Grass_n(\mathbb{C}^\infty)\}$$

for finite CW-complexes, by Yoneda lemma

$$\{\text{Characteristic classes of } n\text{-bundles}\} = H^*(Grass_n(\mathbb{C}^\infty)) = \mathbb{Z}[c_1, c_2, \ldots, c_n]\,.$$

- More generally, for a compact Lie group (or a reductive algebraic group) G, for cohomology with coefficients in $\mathbb{C}$:

Let $Bun^G(X)$ be the set of isomorphism classes of G-bundles over X. This functor $hTop \to Set$ is representable by BG, thus

$$\{\text{Characteristic classes of } G\text{-bundles}\}_{\mathbb{C}} = H^*(BG;\mathbb{C}) = \mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W\,.$$

by Borel theorem. Here $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{t}$ the Lie algebra of the maximal torus, and $W = NT/T$ is the Weyl group.

- For $G = GL_n$ we have

$$\mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[t_1, t_2, \ldots t_n]^{\Sigma_n}\,,$$

the ring of symmetric functions in n variables.

11.11 Axioms of Chern classes

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)\,, \qquad r = rk(E)\,, \qquad c_k(E) \in H^{2k}(X;\mathbb{Z})\,.$$

1) Functoriality: $c(-)$ is a transformation of functors

$$Vect^r(-) \to H^*(-;\mathbb{Z})\,,$$

2) Whitney formula:

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2),$$

where $\cup$ denotes product in cohomology, sometimes written simply as $\cdot$.

3) Normalization:

$$c(\mathcal{O}_{\mathbb{P}^1}(1)) = 1 + [pt]\,,$$

where $[pt]$ is the generator of $H^2(\mathbb{P}^1;\mathbb{Z})$, which agrees with the complex orientation. (In de Rham cohomology $[pt] = [\omega_{FS}]$).

- By splitting principle we can assume that a vector bundle is a sums of line bundles. The cost is that we replace the base X by $Fl(E)$, the bundle of flags over X, which is harmless, since it is mono on cohomology. Topologically every line bundle can be pulled back from $\mathbb{P}^\infty$ (which has the same cohomology H^2 as $\mathbb{P}^1$). Hence the axioms determine $c(E)$.

12 Chern classes and others

The total $c(E)$ is associated to the Ad-invariant (nonhomogeneous) polynomial $P : X \mapsto \det(X+I)$

12.1 Let $f : Y \to X$ be a C^∞-map, $E \to X$ a complex bundle with a connection ∇. The **pull back**: if locally $\nabla = d_X + A$, then $f^*\nabla = d_y + f^*A$ - (Huybrechts 4.2.6.v)

$$\begin{array}{ccccccc}
\mathrm{End}(f^*E)\otimes T^*Y_{|f^{-1}U} & \longleftarrow & \mathrm{End}(f^*E)\otimes f^*T^*X_{|f^{-1}U} & \longrightarrow & \mathrm{End}(E)\otimes T^*X_{|U} & & \\
{\scriptstyle f^*A}\ \nwarrow & & \Big\downarrow & & {\scriptstyle A}\uparrow\Big\downarrow & & \\
 & & f^{-1}(U) & \longrightarrow & U & \subset & X
\end{array}$$

12.2 Let P be an Ad-invariant polynomial. The class of $P(F_\nabla)$ in $H^*(X)$ does not depend on the connection. (Assumption: X is a real manifold.)
Proof: for two connections ∇_0, ∇_1 define a connection on $X\times\mathbb{R}$ by the formula $\tilde{\nabla} = t\,p^*\nabla_1 + (1-t)p^*\nabla_0$, where $p : X\times\mathbb{R} \to X$ is the projection. Inclusions $i_0, i_1 : X \to X\times\mathbb{R}$ on submanifolds $t=0$ and $t=1$ are homotopic, so $[P(F_{\nabla_1})] = i_1^*P([F_{\tilde{\nabla}})] = i_0^*[P(F_{\tilde{\nabla}})] = [P(F_{\nabla_0})]$.

12.3 Verification of axioms:
1) Functoriality (the connection can be pulled back)
2) Whitney formula

$$c(E_1\oplus E_2) = c(E_1)c(E_2)\,,$$

Let connection on $E_1\oplus E_2$ be of the product form $\nabla(s_1,s_2) = (\nabla_1 s_1, \nabla_2 s_2)$. Then

$$F_\nabla = F_{\nabla_1}\oplus F_{\nabla_2} \in (\mathrm{End}(E_1)\oplus\mathrm{End}(E_2))\otimes A^2(X) \subset \mathrm{End}(E_1\oplus E_2)\otimes A^2(X)\,.$$

3) Normalization $c_1(\mathcal{O}(-1)) = -[\omega_{FS}]$, by the definition of the Fubini-Study form.

12.4 Remark: The differential forms obtained by the above constructions are integral (i.e. come from $H^*(X;\mathbb{Z})$.

12.5 Suppose X is complex manifold, L a holomorphic line bundle with a Hermitian structure. Then $F_\nabla = dA = \bar{\partial}\partial\log h$ is a (1,1)-form.

- If X is Kähler manifold, then $c_1(L) \in H^{1,1}(X)\cap image(H^2(X;\mathbb{Z}))$.

12.6 Theorem: Let X be a Kähler manifold, E a holomorphic bundle, then $c_k(E) \in H^{2k}(X;\mathbb{C})$ is represented by a (k,k)-form $(\frac{i}{2\pi})^k\sigma_k(F_\nabla)$.

- Splitting principle: Let $p : Fl(E) \to X$ be a bundle of flag spaces over X.

$$Fl(E) = \{(x, V_1, V_2, \dots V_{\mathrm{rk}E}) \mid x\in X,\ V_i\subset E_x,\ \dim(V_i) = i,\ V_i\subset V_{i+1}\ \}\,.$$

Let $L_i = V_i/V_{i-1}$. The Hermitian structure defines an isomorphism $V_i = L_i\oplus V_{i-1}$. (Note: This isomorphism is not holomorphic.)

Topologically $p^*E = \bigoplus_{i=1}^{\mathrm{rk}E} L_i$. The Chern classes are topological invariants, hence

$$c_\bullet(p^*E) = \prod_{i=1}^{\mathrm{rk}E}(1 + c_1(L_i))\,.$$

Each $c_1(L_i)$ is of the type (1,1) thus $c_k(p^*E)$ is of the type (k,k).

Fact: $p^* : H^*(X) \hookrightarrow H^*(Fl(E))$ is a monomorphism. Moreover it preserves types. Conclusion: $c_k(E)$ is of the type (k,k).

12.7 Huybrechts 4.2.18: in general one can define Atiyah class $A(E) \in H^1(X;\Omega^1_X\otimes\mathrm{End}(E))$, which agree with $\frac{1}{2\pi i}F_\nabla \in A^2(X;\mathrm{End}(E))$.

Other classes

12.8 Chern character. Let $P \in \mathbb{C}[[M_{n\times n}]]$ be given by the formula:

$$P(B) = \sum_{k=0}^{\infty} \frac{tr(B^k)}{k!}\,,$$

where $B = \frac{i}{2\pi}F_\nabla$. In terms of symmetric functions

$$P(t_1, t_2, \dots, t_n) = \sum_{k=0}^{\infty}\sum_{i=1}^{n} \frac{t_i^k}{k!} = \sum_{k=0}^{\infty} e^{t_i}\,.$$

The resulting characteristic class is denoted by $ch(E)$.

- Properties:
  $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$
  $ch(E_1 \otimes E_2) = ch(E_1) \cup ch(E_2)$

The second identity follows from $e^{a+b} = e^a\, e^b$.

12.9 In general having a formal power series $f[[x]]$ we define an additive characteristic class satisfying:

- $a_f(L) = f(c_1(L))$ for a line bundle L
- $a_f(E_1 \oplus E_2) = a_f(E_1) + f(E_2)$

12.10 Example: if $f[[x]] = e^x$, then $a_f(E) = ch(E)$.

- To express the homogeneous components of $ch(E)$ assume that E is a sum of line bundles L_i, let $x_i = c_1(L_i)$
- $ch(E)_{(0)} = \mathrm{rk}E$
- $ch(E)_{(1)} = x_1 + x_2 + \dots + x_n = c_1(E)$
- $ch(E)_{(2)} = \frac{1}{2}(x_1^2 + x_2^2 + \dots + x_n^2) = \frac{1}{2}(x_1 + x_2 + \dots + x_n)^2 - \sum x_i x_j = \frac{1}{2}c_1^2(E) - c_2(E)$

12.11 For a formal power series $f[[x]]$ we define a multiplicative characteristic class satisfying:

- $m_f(L) = f(c_1(L))$ for a line bundle L
- $m_f(E_1 \oplus E_2) = m_f(E_1) \cup m_f(E_2)$
- Example: if $f[[x]] = 1 + x$, then $m_f(E) = c_\bullet(E)$.

12.12 Todd class: Let

$$f[[x]] = \frac{x}{1-e^{-x}} = \frac{x}{x - x^2/2 + x^3/6\dots} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \dots$$

(the coefficients are $\pm\frac{\text{Bernoulli number}}{n!}$)

- $td(E)_{(0)} = 1$
- $td(E)_{(1)} = \frac{1}{2}c_1(E)$
- $td(E)_{(2)} = \frac{x_1^2}{12} + \frac{x_1x_2}{4} + \frac{x_2^2}{12} = \frac{1}{12}(c_1^2(E) + c_2(E))$ (for degrees ≤ 2 it is enough to perform computation for $\mathrm{rk}E = 2$)
- Alternative description: $td(E)$ is associated to the function on matrices $B \mapsto \frac{\det(B)}{\det(id - e^{-B})}$.

12.13 Hirzebruch-Riemann-Roch (Huybrechts 5.1.1) Let E be a holomorphic vector bundle on a compact manifold X. Then

$$\chi(X; E) = \int_X td(TX) \cup ch(E)\,.$$

12.14 Essentially it is enough to check the equality for $X = \mathbb{P}^n$, $E = \mathcal{O}(k)$.

$$LHS = \dim(\mathbb{C}[x_0, x_1, \ldots, x_n]_k) = \binom{n+k}{k}$$

- Lemma: Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T\mathbb{P}^n \to 0\,.$$

Hence

$$td(T\mathbb{P}^n) = \left(\frac{h}{1-e^{-h}}\right)^{n+1}\,,$$

where $h = [\omega_{FS}] \in H^2(\mathbb{P}^n)$.

-

$$RHS = \int_{\mathbb{P}^n} \left(\frac{h}{1-e^{-h}} e^{kh}\right)^{n+1} =$$

$$= \left[\left(\frac{h}{1-e^{-h}}\right)^{n+1} e^{kh}\right]_{\text{coef of } h^n} = Res_{h=0}\left(\frac{e^{kh}}{(1-e^{-h})^{n+1}}\right) = Res_{h=0}\frac{e^{(k+n+1)h}}{(e^h-1)^{n+1}} = \ldots$$

Let $u = e^h - 1$, $du = e^h dh$

$$\cdots = Res_{u=0}\frac{(u+1)^{n+k}}{u^{n+1}} = [(u+1)^{n+k}]_{\text{coef of } u^n}$$

12.15 Exercise: X = hypersurface of degree d in $\mathbb{P}^n$, $E = \mathcal{O}(k)$:

$$td(TX) = td(T\mathbb{P}^n)/td(\nu_X) = \left(\frac{h}{1-e^{-h}}\right)^{n+1}\frac{1-e^{-dh}}{dh} = \frac{h^n}{d}\frac{1-e^{-dh}}{(1-e^{-h})^{n+1}}$$

...

12.16 If $\dim X = 1$, suppose L is of degree d, i.e. $c_1(L) = d[pt]$ then

$$\chi(X;L) = [(1 + c_1(TX)/2)(1 + c_1(L))]_{(1)} = deg(c_1(TX)/2 + c_1(L)) = \frac{1}{2}\chi_{top(X)} + d = 1 - genus + d\,.$$

12.17 If $\dim X = 2$, $L = \mathcal{O}(D)$, then $c_1(L) = [D]$. Let $c_i = c_i(TX)$

$$\chi(X;L) = [(1 + c_1/2 + \frac{1}{12}(c_1^2 + c_2)(1 + D + D^2/2)]_{(2)} = \deg(\frac{1}{12}(c_1^2 + c_2) + \frac{c_1 \cup D + D^2}{2}$$

$$\chi(X;\mathcal{O}_X) + \frac{c_1 \cdot D + D^2}{2}$$

Using a common notation in algebraic geometry $c_1 = -K_X$

$$\chi(X;L) = \chi(X;\mathcal{O}_X) + \frac{(D-K_X)\cdot D}{2}\,.$$

or with $p_a = -\dim H^1(X;\mathcal{O}_X) = \chi(X;\mathcal{O}_X) - 1$ (arithmetic genus)

$$\chi(X;L) = 1 + p_a + \frac{(D-K_X)\cdot D}{2}\,.$$

12.18 Hirzebruch class: Let

$$f_y(x) = x\frac{1+ye^{-x}}{1-e^{-x}} = (1+y)\frac{x}{1-e^{-x}} - xy = (1+y) + \frac{1}{2}(1-y)x + \frac{1+y}{12}x^2 - \frac{1+y}{720}x^4 + \frac{1+y}{30240}x^6 + \ldots$$

Here y is a parameter or a free variable.

- Exercise by Hirzebruch-Riemann-Roch (Huybrechts Cor. 5.1.4)

$$\int_X \text{Hirzebruch class} = \sum_{p=0}^{n} \chi(X, \Omega_X^p)y^p = \sum_{p,q} h^{p,q}y^p\,.$$

- For $y = -1$ we obtain $\chi_{top}(X)$ the topological Euler characteristic,
  For $y = 0$ we obtain $Td(X) = \chi(X, \mathcal{O}_X)$ Todd genus
  For $y = 1$ we obtain the signature.

13 Positive line bundles

13.1 Riemann-Roch

$$\chi(X;E) = \int_X td(TX) \cup ch(E)\,.$$

Holomorphic invariant = Topological data.

- Goal: compute $\dim \Gamma(X,E) = \dim H^0(X;E)$.
- In general it is not possible (by topological data).
- If $H^k(X;E) = 0$ for $k > 0$, then $\dim \Gamma(X,E) = \chi(X;E)$.
- Hence importance of *vanishing theorems*.

We concentrate on linear bundles

13.2 Linear algebra:

$$\{1\tfrac{1}{2} - \text{linear forms on } V\} \quad \leftrightarrow \quad \Lambda^2 V^*_{\mathbb{R}} \cap \Lambda^{1,1} V^* \subset \Lambda^2(V^* \otimes_{\mathbb{R}} \mathbb{C})\,.$$

13.3 We say that $\omega \in A^{1,1}(X) \cap A^2(X;\mathbb{R}) \subset A^2(X;\mathbb{C})$ is positive, if there exists a hermitian product such that ω is equal to minus its imaginary part $\langle\langle x,y\rangle\rangle = \langle x,y\rangle - i\omega(x,y)$. If locally in some coordinates the hermitian product is given by a matrix $H = [h_{i,j}]$, then

$$\omega = \frac{i}{2}\sum h_{ij} dz_i \wedge \bar{d}z_j\,.$$

13.4 A linear bundle is positive if it admits a connection ∇ such that $\frac{i}{2\pi}F_\nabla$ is a positive (1,1)-form.

13.5 Example of positive bundles: $\mathcal{O}_{\mathbb{P}^n}(k)$, $k > 0$.

13.6 Denote $\Omega_X^{\dim X}$ by K_X, call it the canonical bundle/divisor.

13.7 Kodaira(-Nakano) Vanishing theorem
[Huybrechts Proposition 5.2.2, Griffiths-Harris p 154.]
If L is positive, then $H^i(X; K_X \otimes L)) = 0$ for $i > 0$.
(In algebraic geometry notation $L = \mathcal{O}(D)$. The vanishing theorem reads $H^i(X; K_X + D) = 0$.)
[Dowód na końcu w §14]

13.8 Corollary: Assume L is positive. For any line bundle L' we have vanishing $H^i(X; L^\nu \otimes L') = 0$ for $i > 0$, $\nu >> 0$.

•

$$L^\nu \otimes L' = K_X \otimes (K_X^{-1} \otimes L^\nu \otimes L')\,.$$

The bundle $K_X^{-1} \otimes L^\nu \otimes L'$ has the connection form equal to

$$-F_{\nabla_{K_X}} + \nu F_{\nabla_L} + F_{\nabla_{L'}}\,.$$

For sufficiently large ν it is positive.

- If K_X^{-1} is positive, then vanishing holds already for $\nu = 1$. (Fano manifold, e.g. $\mathbb{P}^n$, Grassmannians, flag varieties.)

13.9 If L is generated by global holomorphic sections (we say ,,globally generated") iff for each $x \in X$ there exists a section $s \in H^0(X;L)$, such that $s(x) \neq 0$.

13.10 Let $s_0, s_2, \dots, s_r$ be the basis of $H^0(X;L)$. Define $\phi : X \to \mathbb{P}^r$ by

$$x \mapsto [s_0 : s_1 : \cdots : s_r]\,.$$

• Coordinate-free construction: If L is globally generated, then for $x \in X$ define a function on the space of the global sections $H^0(X;L)$

$$\Phi(x) \in \mathrm{Hom}(H^0(X;L), L_x) \overset{\text{up to a scalar}}{\simeq} H^0(X;L)^*\,,$$

$$\Phi(x) : s \mapsto s(x) \in L_x \simeq \mathbb{C}\,.$$

• We obtain a natural map

$$\phi : X \to \mathbb{P}(H^0(X;L)^*)\,.$$

• Then $L = \phi^*(\mathcal{O}(1))$ (by tautological identification $H^0(\mathbb{P}(V^*);\mathcal{O}(1)) \simeq V$). Hence L is ,,nonnegative", i.e. L admits a connection such that the associated Hermitian form is nonnegative semi-definite.

• This property is preserved by pull-backs.

13.11 Kodaira embedding theorem.

[Huybrechts Proposition §5.3, Griffiths-Harris p 176.]

If a bundle L is positive, then for $\nu >> 0$ the tensor power L^ν is generated by global sections and the natural map $X \to \mathbb{P}(H^0(X;L^\nu)^*)$ is an embedding. (We only assume that, X is a compact analytic complex manifold, and as a corollary from GAGA we obtain that X is algebraic.)

• Steps of the proof:

a) assume that ϕ_{L^ν} is well defined, i.e. the base locus of L^ν is empty:

$$\forall x \in X \qquad H^0(X;L^\nu) \twoheadrightarrow L^\nu_x = H^0(X;L^b \otimes \mathcal{O}_X/m_x)$$

b) ϕ_{L^ν} separates the points: suppose for $x \in X$ the restriction $H^0(X;L^\nu) \to L^\nu_x \oplus L^\nu_y$ is surjective.

(note b) $\Rightarrow$ a))

b') Equivalently: the restriction

$$H^0(\tilde{X};\tilde{L}^\nu) \to H^0(E;\tilde{L}^\nu_{|E}) = H^0(\tilde{X};\tilde{L}^\nu \otimes \mathcal{O}_{\tilde{X}}/I_E)$$

is surjective, where $\tilde{X} = Bl_x Bl_y X$, $\tilde{L}$ is the pull-back of L to $\tilde{X}$, E the sum of the exceptional divisors, $I_E \simeq \mathcal{O}(-E)$ the ideal sheaf of E. The restriction map is a part of an exact sequence

$$\to H^0(\tilde{X};\tilde{L}^\nu) \to H^0(\tilde{X};\tilde{L}^\nu \otimes \mathcal{O}_{\tilde{X}}/I_E) \to H^1(\tilde{X};\tilde{L}^\nu \otimes \mathcal{O}(-E)) \to$$

obtained from

$$0 \to \mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_E \to 0\,.$$

By vanishing theorem $H^1(\tilde{X};\tilde{L}^\nu \otimes \mathcal{O}(-E)) = 0$ for $\nu >> 0$.

Similarly:

c) ϕ_{L^ν} has nondegenerate differential at x. Equivalently

$$H^0(X;L^\nu \otimes m_x) \to L^\nu_x \otimes T^*_x X = H^0(X;L^\nu \otimes \Omega^1_X \otimes \mathcal{O}_X/m_x)$$

is surjective.

c') the restriction

$$H^0(\tilde{X};\tilde{L}^\nu \otimes I_E) \to H^0(E;\tilde{L}^\nu_{|E} \otimes I_E) = H^0(\tilde{X};\tilde{L}^\nu \otimes \mathcal{O}_{\tilde{X}}/I_E \otimes I_E)$$

is surjective, where $\tilde{X} = Bl_x X$, $\tilde{L}$ where $\tilde{X}$ is the pull-back of L, E is the exceptional divisor, I_E is the ideal sheaf of E. Note that I_E restricted to $E = \mathbb{P}(T_x X)$ is isomorphic to $\mathcal{O}(1)$, hence

$$H^0(E;\tilde{L}^\nu_{|E} \otimes I_E) = L^\nu_x \otimes T^*_x X\,.$$

The restriction map is a part of an exact sequence

$$\to H^0(\tilde{X};\tilde{L}^\nu \otimes I_E) \to H^0(\tilde{X};\tilde{L}^\nu \otimes \mathcal{O}_{\tilde{X}}/I_E \otimes I_E) \to H^1(\tilde{X};\tilde{L}^\nu) \to$$

13.12 Kähler manifolds with integral (up to a scalar) Kähler form are projective. If (X, ω) is a Kähler manifold and $[\omega]$ is of the form $\lambda[\omega']$ for $\lambda \in \mathbb{R}_+$, $[\omega'] \in H^*(X;\mathbb{Z})$, then X embeds into a projective space.

• Proof. It is enough to show that: if $[\omega]$ is integral, then there exist a holomorphic line bundle L with a curvature form equal to $\frac{2\pi}{i}c_1(L)$.

• The short exact sequence

$$\underline{\mathbb{Z}} \to \underline{\mathbb{C}} \to \mathcal{O}_X$$

of sheaves induces a long exact sequence

$$\to Pic(X) = H^1(X, \mathcal{O}_X^*) \stackrel{c_1}{\to} H^2(X;\mathbb{Z}) \stackrel{\iota}{\to} H^2(X;\mathcal{O}_X) \to \, .$$

Assume that $[\omega']$ is integral. We will show that $[\omega']$ lies in the image of c_1, or equivalently it belongs to the kernel of the map ι. The map ι factors as follows

$$H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) \twoheadrightarrow H^{0,2}(X) = H^2(X;\mathcal{O}_X)\, .$$

The second map is induced by the map of shaves $\underline{\mathbb{C}} \to \mathcal{O}_X$.The classes of the type $(1,1)$ lie in the kernel.

• It remains to show, that if $c_1(L) = [\omega]$ then L admits a connection ∇ with

$$\frac{i}{2\pi} F_\nabla = \omega\, .$$

13.13 adjusting the connection. Suppose $[\omega] = c_1(L) \in H^2(X)$, where ω is a real (1,1)-form. Then there exists a connection ∇ such that $\omega = \frac{i}{2\pi} F_\nabla$.

• Proof: locally in a trivialization $F_\nabla = \bar\partial\partial \log(h(z))$. For a different choice of a metric $h' = e^\rho h$. Hence

$$F_{\nabla'} = \bar\partial\partial \log(e^\rho h(z)) = F_\nabla + \bar\partial\partial\rho.$$

We want to find

$$F_{\nabla'} = -2\pi i\omega = F_\nabla + d\beta$$

for a given $\beta \in A^1(X)$. It remains to solve an equation

$$\bar\partial\partial\rho = d\beta\, .$$

Then $h' = e^\rho h$ and ∇' is the required connection.

• We apply $\partial\bar\partial$-lemma [Huy, Cor 3.2.10]: for a given exact form $d\beta$ of the type (1,1), there exists ρ such that $d\beta = \bar\partial\partial\rho$.

• Existence of ρ is the conclusion of $\partial\bar\partial$-lemma *For* $\phi \in A^{p,q}$

$$\phi = d\beta \quad \Rightarrow \quad \exists\gamma \quad \phi = \partial\bar\partial\gamma$$

13.14 Numerical criteria for admitting a positive connection, i.e. ampleness

• Nakai-Moishezon criteion

• Kleiman-Kriterion

[Ten temat już należy do innego przedmiotu.]

---

14 Dowód twierdzenia Kodairy-Nakano o znikaniu

14.1 Jeśli L i K dodatnie, to $L \otimes K$ też. Jeśli $K = L^{\otimes n}$ jest dodatnie, to L też. Obcięcie zachowuje dodatniość.

Poniżej używamy E jako oznaczenie wiązki, bo L jest zarezerwowane na operator Lefschetza. Zakładamy, że X jest wartą rozmairością analityczną.

14.2 Ustalamy metrykę hermitowską na X i E. Definiujemy skręcone harmoniczne

$$\mathcal{H}^{p,q}(E) := \ker(\Delta_E) \subset A^{p,q}(E)\,, \qquad \Delta_E = \bar\partial_E \bar\partial_E^* + \bar\partial_E^* \bar\partial_E$$

14.3 Jeśli X jest zwarta to

$$A^{p,q}(E) = \mathcal{H}^{p,q}(E) \oplus im(\bar\partial_E) + im(\bar\partial_E^*)$$

14.4 Z rozkładu Hodge'a dla $\bar\partial$

$$H^{p,q}(X;E) = \mathcal{H}^{p,q}(E) = ker(\Delta_E)\,.$$

14.5 Mamy
– Operator $L : A^{p,q}(X;E) \to A^{p+1,q+1}(X;E)$ i sprzężony L^*
– $\nabla = \nabla^{1,0} + \bar\partial_E$ oraz tożsamość (u Huybrechtsa nazwana tożsamością Nakano)

$$[L^*, \bar\partial_E] = -i(\nabla^{1,0})^*,$$

która jest uogólnieniem tożsamości Kählera $[L^*, \bar\partial] = -i\partial^*$.

14.6 (Znikanie Kodairy-Nakano) Jeśli E jest dodatnia, to $H^{p,q}(X;E) = H^q(X;\Omega_X^p \otimes E) = 0$ dla $p+q > \dim X$.
Forma krzywizny F_∇ jest typu (1,1), lokalnie

$$F_\nabla = dA = \bar\partial\partial(\log h)\,.$$

Lokalnie dla przekroju η

$$\nabla^{1,0}\eta = \partial\eta + \partial(\log h) \wedge \eta$$

$$\bar\partial_E(\nabla^{1,0}\eta) = (-\partial\bar\partial\eta + \bar\partial(\partial(\log h) \wedge \eta)) = -\partial\bar\partial\eta - \partial(\log h) \wedge \bar\partial\eta + \bar\partial\partial(\log h) \wedge \eta$$
$$\nabla^{1,0}(\bar\partial_E\eta) = \partial\bar\partial\eta + \partial(\log h) \wedge \bar\partial\eta$$

$$\boxed{F_\nabla \wedge \eta = \bar\partial_E \nabla^{1,0}\eta + \nabla^{1,0}\bar\partial_E\eta}$$

Niech $\eta \in \mathcal{H}^{p,q}(E)$,

$$\bar\partial_E\eta = 0, \qquad \bar\partial_E^*\eta = 0$$

Wtedy

$$\boxed{F_\nabla \wedge \eta = \bar\partial_E \nabla^{1,0}\eta}$$

Stąd

$$i\langle L^* F_\nabla \eta, \eta\rangle = i\langle L^* \bar\partial_E \nabla^{1,0}\eta, \eta\rangle \overset{Nakano}{=} i\langle (\bar\partial_E L^* - i(\nabla^{1,0})^*)\nabla^{1,0}\eta, \eta\rangle =$$
$$= i\langle(\bar\partial_E L^*, \eta\rangle + \langle(\nabla^{1,0})^*\nabla^{1,0}\eta, \eta\rangle = i\langle(L^*, \partial_E^*\eta\rangle + \langle\nabla^{1,0}\eta, \nabla^{1,0}\eta\rangle = \langle\nabla^{1,0}\eta, \nabla^{1,0}\eta\rangle \geq 0$$

Podobnie

$$i\langle F_\nabla L^*\eta, \eta\rangle = i\langle(\bar\partial_E\nabla^{1,0} + \nabla^{1,0}\bar\partial_E)L^*\eta, \eta\rangle = i\langle(\nabla^{1,0}\bar\partial_E)L^*\eta, \eta\rangle = i\langle\nabla^{1,0}(L^*\bar\partial_E + i(\nabla^{1,0})^*)\eta, \eta\rangle =$$
$$-\langle\nabla^{1,0}(\nabla^{1,0})^*\eta, \eta\rangle = -\langle(\nabla^{1,0})^*\eta, (\nabla^{1,0})^*\eta\rangle \leq 0$$

Stąd

$$i\langle[L^*, F_\nabla]\eta, \eta\rangle = ||(\nabla^{1,0})^*\eta||^2 + ||\nabla^{1,0}\eta||^2 \geq 0$$

Ale

$$iF_\nabla \wedge - = 2\pi L$$

bo L jest dodatnia, więc $\frac{i}{2\pi}F_\nabla$ jest formą Kählera. Stąd

$$i\langle[L^*, F_\nabla]\eta, \eta\rangle = 2\pi\langle[L^*, L]\eta, \eta\rangle = -2\pi\langle H\eta, \eta\rangle = 2\pi(n-(p+q))||\eta||^2.$$

Jeśli $n-(p+q) \leq 0$, to $i\langle[L^*, F_\nabla]\eta, \eta\rangle \leq 0$. Zatem $||\eta||^2 = 0$.

14.7 Wniosek: jeśli $E^{\otimes n} = \mathcal{O}(1)_X$ dla pewnego zanurzenia $X \subset \mathbb{P}^N$, to $H^k(X, \Omega^n_X \otimes E) = 0$ dla $k > 0$.

14.8 Poprzez dualność Serre'a (lub bezpośrednio, korzystając z tego, że $\frac{i}{2\pi} F_{\nabla_{E^*}} = -\omega$):

$$H^k(X, E^*) = 0$$

dla $k < n$.