Complex Manifolds 2021/22

Lecture summary. **This is not a replacement for a textbook.**

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The main reference: Daniel Huybrechts, Complex Geometry, An introduction. (Springer 2005)

Also:
Donu Arapura, Algebraic Geometry over the Complex Numbers (Universitext)
Claire Voisin, Hodge Theory and Complex Algebraic (Cambridge Studies in Advanced Mathematics)

1 Introduction

1.1 Definition of complex manifolds

1.2 Projective spaces

1.3 Grassmannians (affine maps, Plücker embedding, $Gr(2,4)$ as a quadric in $\mathbb{P}^5$).

1.4 Hyperplane in $\mathbb{P}^n$ (quadrics, elliptic curves in $\mathbb{P}^2$)

1.5 Complex manifolds as real manifolds are orientable since any linear complex map preserves the distinguished orientation of the underlying real vector space.

1.6 Basic information about topological coverings an induced complex structures: If $f : X \to Y$ is a topological covering, Y has a structure of a complex manifold, then X has a natural structure of a complex manifold and f is a holomorphic.

Curves

1.7 Riemann surfaces (= oriented surfaces with a Riemannian metric) and complex surfaces: each Riemannian surface has a complex structure. Genus of Riemann surface.

• The rotation by 90° in the tangent space allows to introduce a structure of complex vector space. This structure is ,,integrable" i.e. it comes from a structure of a complex manifold (a proof will be later, it follows trivially from Newlander-Nirenberg theorem).

1.8 Riemann uniformization theorem: any complex curve is isomorphic to $\mathbb{P}^1$ or it is a quotient of $\mathbb{C}$ or $\mathbb{D} \simeq \mathbb{H}$.

• Another formulation: any simply connected complex curve is isomorphic to $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D}$. This is a generalization of the Riemann theorem for open subsets in $\mathbb{C}$.

1.9 The automorphism group of $\mathbb{P}^1$ is equal to $PGL_2(\mathbb{C})$. Any complex-analytic automorphism of $\mathbb{P}^1$ is given by a linear formula. (The same statement holds for $\mathbb{P}^n$.)

• Proof: Composing with a linear map we can assume that $f(0) = 0$, $f(\infty) = \infty$. Expanding at infinity we get an estimation $1/|f(z)| < c/|z|$. Hence the function $g(z) = z/f(z)$ is bounded. It has no poles, since at 0 the zero of the denominator cancels out and there are no more zeros of f. Hence by Liouville theorem $g(z)$ is constant.

• Hence each automorphism of $\mathbb{P}^1$ has a fixed point – the eigenvector of the linear map.
Topological proof: there are no nontrivial topological covering $\mathbb{P}^1 \simeq S^2 \to C$ except $C = \mathbb{RP}^2$. But the real projective plane is not orientable, so it cannot be a complex curve.

1.10 Automorphisms of $\mathbb{C}$ are given by affine maps $f(z) = az + b$. There is no fixed points only if $a = 1$.

• The map $f : \mathbb{C} \to \mathbb{C}$ extends to $\mathbb{P}^1$. It is continues at ∞. By Riemann extension theorem it is holomorphic ∞ and we apply (1.9).

1.11 The complex quotients of $\mathbb{C}$ are of the form $\mathbb{C}/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}$.

- The nontrivial discrete subgroups of $\Lambda \subset (\mathbb{C},+) \simeq \mathbb{R}^2$ are of the form $\Lambda = \langle a, b\rangle$ for $b/a \in \mathbb{H}$, (or $\Lambda = \langle a\rangle$). We can restrict our attention to subgroups of the form $\Lambda = \langle 1, \tau\rangle$, $\tau \in \mathbb{H}$.
- The group $PSL_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by homography: $\begin{pmatrix} s & t \\ u & v\end{pmatrix} \cdot z = (sz+t)/(uz+v)$. The subgroup $PSL_2(\mathbb{R})$ preserves $\mathbb{H}$.
- Suppose $\tau, \tau' \in \mathbb{H}$. Then $\mathbb{C}/\langle 1,\tau\rangle \simeq \mathbb{C}/\langle 1, \tau'\rangle$ if and only if τ and τ' belong to the same orbit of $PSL_2(\mathbb{Z})$.
- Proof: We define a function on the set of positively oriented $\mathbb{R}$-bases in $\mathbb{C}$: $\phi(a,b) = b/a = \tau \in \mathbb{H}$. Two positively oriented bases of Λ differ by the operations $(a,b) \mapsto (a, b+a)$ and $(a,b) \mapsto (b,-a)$. The corresponding matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ acting on $\tau = a/b$ generate $PSL_2(\mathbb{Z})$. Thus the function ϕ induces a map

$$\{\text{latices in } \mathbb{C}\}/Aut(\mathbb{C}) \longrightarrow \{\text{isomorphism classes of curves of genus 1}\}$$

Exercise: show that this map is a bijection.
To show that it is a surjection one has to argue, that every curve of genus 1 is a quotient of $\mathbb{C}$, i.e. it is not a quotient of $\mathbb{D}$. It is enough to check that $Aut(\mathbb{D})$ does not contain $\mathbb{Z}^2$.

- There is an embedding $\mathbb{C}/\Lambda \to \mathbb{P}^2$ — see Weierstrass function $\wp$ (exercises).

1.12 The disc automorphisms are the same as the automorphism of the upper hyperplane $\mathbb{H}$: $Aut(\mathbb{H}) = PSL_2(\mathbb{R})$.

- $Aut(\mathbb{D})$ consist of homographies by the Schwartz lemma (we can assume $f(0) = 0$).

1.13 Discrete subgroups of $PSL_2(\mathbb{R})$ are called Fuchsian groups (grupy Fuksa) The curves of higher genera $g > 1$ are quotients $\mathbb{H}/G$ where $G \subset PSL_2(\mathbb{R})$ is Fuchsian and acts without fixed points.

1.14 Read more: [Huybrechts, Complex Geometry, Chapter 2.1]

2 Weierstrass preparation

Local theory: see [§1, Huybrechts].

2.1 Cauchy-Riemann operator $\frac{\partial}{\partial \bar z} = \frac{1}{2}\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ and complex differential $\frac{1}{2}\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$.

2.2 Differentials $dz = dx + idy$ and $d\bar z = dx - idy$.

- For any C^∞ function on $f : \mathbb{C} \to \mathbb{C}$ the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\,.$$

(Hint if $A = \begin{pmatrix} 1 & 1 \\ i & -i\end{pmatrix}$ then $(A^T)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i\end{pmatrix}$.)

2.3 Recollection of theorems for complex analytic functions in one variable

- series expansion
- Cauchy integration formula
- maximum principle
- identity principle
- Liouville theorem

2.4 Residue $res_{z_0}(f) = \frac{1}{2\pi i}\int_{\partial D_{z_0}} f\, dz$, where D_{z_0} is a small disc around z.

- residue is an invariant of the differential form $f(z)dz$. When we change variable $z = g(u)$ then the form $f(z)dz$ should be replaced by

$$g^*(f(z)dz) = f(g(u))g'(u)du\,,$$

i.e. $res_{z_0} f = res_{g^{-1}(z_0)} f \circ g \cdot g'$.

2.5 Residue theorem: for a meromorphic function f (enough to assume: holomorphic away from a discrete set $\{z_1, z_2, \ldots, z_n\}$) on a compact Riemann surface S

$$\sum_k res_{z_k}(f) = 0\,.$$

• Proof from the Stokes theorem: Assume that the discs D_{z_k} for $z \in Sing(f)$ do not intersect:

$$\sum_{z_k}\int_{\partial D_{z_k}} f\,dz = -\int_{\partial(S\setminus\bigcup D_{z_k})} f\,dz = -\int_{S\setminus\bigcup D_{z_k}} d(f\,dz) = -\int_{S\setminus\bigcup D_{z_k}} \frac{\partial f}{\partial \bar z} d\bar z \wedge dz = 0.$$

2.6 A formula for the number of zeros in a disc has a generalization which will be used later. If $f(z) \neq 0$ for $|z| = \epsilon$ then for $\ell \geq 0$ we have

$$\frac{1}{2\pi i}\int_{S_\epsilon} \frac{f'(\xi)}{f(\xi)}\xi^\ell d\xi = \sum_{|\alpha|<\epsilon,\, f(\alpha)=0} \alpha^\ell.$$

Many variables - references to [Huybrechts §1.1]

2.7 Definition: a C^∞ function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if $\partial_{\bar z_k f} = 0$ for $k = 1, 2, \ldots, n$.

2.8 Cauchy integral formula Prop 1.1.2

2.9 Hartogs theorem Prop 1.1.4

2.10 Corollary: zero set of a holomorphic function ($f \not\equiv 0$) has real codimension equal 2 or it is empty.

• Remark: any analytic set (eg zero set of a holomorphic function) is triangulable by Łojasiewicz theorem, so there is no ambiguity with the notion of dimension.

2.11 Weierstrass preparation theorem (Th. 1.1.6).

2.12 Algebraic fact used in the proof: elementary symmetric functions σ_k can be expressed by power sums p_k.

2.13 Weierstrass preparation theorem – division version (Prop 1.1.17).

3 Local ring

3.1 Implicit function theorem (for $f : \mathbb{C} \times \mathbb{C}^{n+1} \to \mathbb{C}$ with $\frac{\partial f}{\partial z_1} \neq 0$) as a special case of Weierstrass theorem

3.2 The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is a unique factorization domain (Prop 1.1.15).

• Any Weierstrass polynomial is indecomposable in $\mathcal{O}_{\mathbb{C}^{n-1},0}[z]$ iff it is indecomposable in $\mathcal{O}_{\mathbb{C}^n,0}$.

3.3 The local ring $\mathcal{O}_{\mathbb{C}^n,0}$ is noetherian (Prop 1.1.18).

• In the course of the proof: first choose $f \in I$ and fix a coordinate system adapted for f. Later you are not allowed to change the coordinate system.

3.4 BUT: If $\emptyset \neq U \subset \mathbb{C}^n$, $n > 0$ then $\mathcal{O}_{\mathbb{C}^n}(U)$ is not noetherian.

• Proof: Any $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ is generated by $I \cap (\mathcal{O}_{\mathbb{C}^{n-1},0}[z])$ and any Weierstrass polynomial $g \in I$ (by division version of WPT).

3.5 Germ of sets and ideals in the local ring:

- The germ of the set $Z(J)$ defined by an ideal $J \subset \mathcal{O}_{\mathbb{C}^n,0}$.

— if $J_1 \subset J_2$ then $Z(J_1) \supset Z(J_2)$

- The ideal of function germs vanishing on the germ of a set $I(X)$. We have:

— if $X_1 \subset X_2$ then $I(X_1) \supset I(X_2)$

3.6 Compositions of Z and I

- $X \subset Z(I(X))$ for any set germ,
- $J \subset I(Z(J))$ for any ideal,
- $X = Z(I(X))$ for analytic set germs (i.e. of the form $X = Z(J)$)

— since $J \subset I(Z(J))$ then $X = Z(J) \supset Z(I(Z(J))) = Z(I(X))$.

- Hilbert nullstellensatz: $I(Z(J) = \sqrt{J}$ (see sketch of a proof in Huybrechts p.20).

3.7 Let $g \in \mathcal{O}_{\mathbb{C}^n,0}$ be indecomposable, then if $f_{|Z(g)} = 0$, then g divides f (Cor. 1.1.9)

- Proof from the division version of Weierstrass preparation theorem
- Key step: if g is indecomposable Weierstrass polynomial, then $g_w(z)$ generically (w/r to w) has distinct roots.

— let K be the quotient field of $\mathcal{O}_{\mathbb{C}^{n-1},0}$. The polynomials $g_w(z)$ and $g'_w(z)$ are coprime (by Gauss lemma), so there exist $\alpha(z), \beta(z) \in K(z)$ such that $\alpha(z)g_w(z) + \beta(z)g'_w(z) = 1$. Passing to $\mathcal{O}_{\mathbb{C}^{n-1},0}$, removing the denominators

$$\tilde{\alpha}(z)g_w(z) + \tilde{\beta}(z)g'_w(z) = \gamma$$

with $0 \neq \gamma \in \mathcal{O}_{\mathbb{C}^{n-1},0}$. At the points where $\gamma(w) \neq 0$ the polynomial g_w does not have multiple roots.

- Corollary: Nullstellensatz for principal ideals.

3.8 The germ of a set is indecomposable (also called irreducible) if and only if $I(X)$ is a prime ideal (Lemma 1.1.28)

4 GAGA

J-P.Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier 6: 1-42, (1956)

See also: Amnon Neeman, Algebraic and analytic geometry. Cambridge University Press (2007)

4.1 For an algebraic manifold X (a scheme in general) we define ,,analytification" X^{an}.

- As a set $X = X^{an}$.
- While X has Zariski topology, X^{an} has classical topology (glued from the open subsets $U \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$). The identity map $\iota : X^{an} \to X$ is continuous (every Zariski open set is open in the classical topology). [Serre §5 Lemma 1]
- Both spaces are ringed. We have distinguished sheaves of rings $\mathcal{O}_X$ (algebraic functions) and $\mathscr{H}_X$ (holomorphic functions), the stalks are local rings. We have a map

$$\theta_X : \iota^{-1}\mathcal{O}_X \to \mathscr{H}_X,$$

i.e. ι extends to a map of ringed spaces. Here ι^{-1} denotes the pull-back of a sheaf. The map θ_X is injective, flat, an isomorphism after completion in $\mathfrak{m}$. [Serre §6, prop 4]

4.2 For an algebraic sheaf $\mathcal{F}$ over an algebraic manifold we define ,,analytification"

$$\mathcal{F}^{an} = \mathscr{H}_X \otimes_{\iota^{-1}\mathcal{O}_X} \iota^{-1}\mathcal{F}\,.$$

Of course $\mathcal{O}_X^{an} = \mathscr{H}_X$. [Serre §9, Prop 10]

4.3 Definition: Let $(Y, \mathscr{R}_Y)$ be a ringed space. The sheaf $\mathscr{R}_Y$–modules $\mathcal{F}$ is coherent iff
1) locally there is a surjective map $(\mathscr{R}_Y^N)_{|U} \to \mathcal{F}_{|U}$ for some N (i.e. $\mathcal{F}$ is locally finitely generated),
2) for any map $(\mathscr{R}_Y^M)_{|U} \to \mathcal{F}_{|U}$ the kernel is finitely generated.

By $Coh(Y)$ we denote the category of coherent sheaves.

- Mind the difference comparing with the definition for algebraic varieties.

4.4 Oka Theorem: $\mathcal{F} = \mathscr{H}_X$ is coherent. (This is not a tautology!) References in [Serre §3 Prop.1]

4.5 Analytification of sheaves is a functor preserving coherent sheaves [Serre §9]

$$(-)^{an} : Sh(X) \to Sh(X^{an})$$

4.6 (Serre) If X is projective, $\mathcal{F}$ coherent then the natural map $H^*(X; \mathcal{F}) \to H^*(X^{an}; \mathcal{F}^{an})$ is an isomorphism. [Serre §12 Th. 1]

- Relative version: Let $f : X \to Y$ be a projective morphism of algebraic varieties. Then f induces a functor

$$f_*^{an} : Coh(X^{an}) \to Coh(Y^{an})$$

and

$$(f_*\mathcal{F})^{an} = f_*^{an}\mathcal{F}^{an}$$
$$(R^k f_*\mathcal{F})^{an} = R^k f_*^{an}\mathcal{F}^{an}$$

If $Y = pt$ then we recover the previous formulation.

4.7 (Serre cont.) If X is a projective variety, then $(-)^{an}$ restricted to $Coh(X)$ is an equivalence of categories.
The above means:
(i) $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to Hom_{\mathscr{H}_X}(\mathcal{F}^{an}, \mathcal{G}^{an})$ is an isomorphism. [Serre §12 Th. 2]
(ii) For any analytic coherent sheaf G there exists an algebraic sheaf $\mathcal{F}$ such that $G \simeq \mathcal{F}^{an}$. [Serre §12 Th. 3]

4.8 The proofs can be reduced to $X = \mathbb{P}^n$. To check the equality $H^*(X; \mathcal{F}) \simeq H^*(X^{an}; \mathcal{F}^{an})$ we can assume (by various cohomology exact sequences) that $\mathcal{F} \simeq \mathcal{O}(m)$.

4.9 For a proof of (i) use the equality of sheaf-Homs

$$(\underline{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))^{an} = \underline{Hom}_{\mathscr{H}_X}(\mathcal{F}^{an}, \mathcal{G}^{an})$$

which holds for algebraic coherent sheaves. Then apply the general principle

$$Hom_Y(F, G) = H^0(Y; \underline{Hom}_Y(F, G)),$$

and apply 4.6.

4.10 For a proof of (ii) have to show that any analytic sheaf F on $X = \mathbb{P}^n$ after tensoring with $\mathscr{H}_X(m)$ for some big m is globally generated, i.e. there exists k and a surjection

$$\mathscr{H}_X^k \to F(m) := F \otimes_{\mathscr{H}_X} \mathscr{H}_X(m),$$

which is equivalent to: for each point $x \in X$

$$\text{Global sections of } F(m) \to F(m)_x$$

is a surjection, [Serre §16 Lemma 8]. Then $F(m) = coker(\mathscr{H}_X^\ell \to \mathscr{H}_X^k)$, thus by (i) it is algebraic, [Serre §17].

4.11 Corollary (Chow Theorem): Any analytic subvariety $\mathbb{P}^n$ is described by a set of polynomial equations..

Differential forms and de Rham cohomology – summary

4.12 Global differential forms on a C^∞-manifold M will be denoted by $A^\bullet(M) = \bigoplus_{k=0}^{\dim M} A^k(M)$. (The notation $\Omega^\bullet(M)$ is reserved for holomorphic forms.)

4.13 $A^\bullet(M)$ is a commutative algebra with gradation $ab = (-1)^{\deg(a)\deg(b)} ba$)

4.14 differential satisfies the Leibniz rule $d(ab) = ad(b) + (-1)^{\deg(a)} b$

4.15 the linear space $A^k(M)$ is the space of the global sections of a sheaf A^k_M.

4.16 $\underline{\mathbb{R}}_M \hookrightarrow A^0_M \to A^1_M \to A^2_M \to \dots$ is a soft (in particular acyclic) resolution of the constant sheaf $\underline{\mathbb{R}}_M$, therefore

$$H^k(A^\bullet(M), d) = H^k(M; \underline{\mathbb{R}}_M) \simeq H^k_{sing}(M; \mathbb{R}) \,.$$

The cohomology groups are denoted by $H^k(M)$, we skip $\mathbb{R}$ in the notation.

4.17 exterior product of forms induces multiplication in cohomology $H^k(M) \times H^\ell(M) \to H^{k+\ell}(M)$

4.18 if M is compact, $n = \dim M$ and M has a chosen orientation, then the integral of n-forms induces a map $\int_M : H^n(M) \to \mathbb{R}$. If M is connected, then $\int_M$ is an isomorphism.

4.19 (Poincaré Duality) if M is compact, oriented of dimension n, then the bilinear form

$$\int_M - \wedge - : H^k(M) \times H^{n-k}(M) \to \mathbb{R}$$

is nondegenerate.

4.20 if M is oriented (not necessarily compact), then we consider cohomology with compact supports

$$H^k_c(M) = H^k(A^\bullet_c(M)).$$

Then

$$\int_M - \wedge - : H^k_c(M) \times H^{n-k}(M) \to \mathbb{R}$$

is defined and it is a nondegenerate 2-linear form.

4.21 Having a Riemannian metric on a compact manifold allows to define **harmonic** forms $\mathcal{H}^k(M)$ (see 4.28). The harmonic forms are closed and the resulting map $\mathcal{H}^k(M) \to H^k(M)$ is an isomorphism. However the product of harmonic forms does not have to be harmonic.

Hodge theory for $\mathbb{C}^\infty$ manifolds

Suppose M is equipped with Riemannian metric, i.e. a scalar product at each tangent space T_xM. Let $n = \dim M$.

4.22 Volume form is denoted by $vol \in A^n(M)$.

4.23 Hodge star: for $x \in M$

$$* : \Lambda^k T^*_x M \to \Lambda^{n-k} T^*_x M$$

It is defined by the property

$$a \wedge *b = \langle a, b\rangle vol$$

for each $a, b \in \Lambda^k T^*_x M$. The Hodge star extends to

$$* : A^k(M) \to A^{n-k}(M)$$

pointwise.

4.24 We have
(i) $*^2 = (-1)^{k(n-k)}$ on k-forms.
(ii) $\langle \alpha, *\beta \rangle = (-1)^{k(n-k)} \langle *\alpha, \beta \rangle$,

4.25 Let's define $d^* = (-1)^{n(k+1)+1} * d* : A^k(M) \to A^{k-1}(M)$.

4.26 For compact manifold M, $a \in A^{k-1}(M)$, $b \in A^k(M)$ we have

$$\langle da, b \rangle_M = \langle a, d^* b \rangle_M$$

We say that d^* is formally adjoint to d.
Proof

$$0 = \int_M d(a \wedge b) = \int_M da \wedge b + (-1)^{k-1} \int_M a \wedge db\,.$$

$$\begin{aligned}
\langle a, d^* b \rangle_M &= \int_M \langle a, (-1)^{d(k+1)+1} * d * b \rangle \\
&= (-1)^{d(k+1)+1} \int_M a \wedge * * d * b && \deg(*d*b) = k-1 \\
&= (-1)^{d(k+1)+1+(k-1)(d-k+1)} \int_M a \wedge d * b && d(k+1)+1+(k-1)(d-k+1) \equiv_2 k \\
&= (-1)^k \int_M a \wedge d * b = \int_M \langle da, b \rangle vol
\end{aligned}$$

4.27 Laplasian on forms is defined by

$$\Delta = dd^* + d^* d$$

It can be interpreted as the ,,super-commutator" $[d, d^*]_s$.

- In general the supercommutator of elements of a graded algebra $A = \bigoplus_{k \in \mathbb{Z}} A^k$ is defined by

$$[\phi, \psi]_s = \phi\psi - (-1)^{k\ell} \psi\phi \qquad \text{if} \qquad \phi \in A^k, \quad \psi \in A^\ell.$$

4.28 Harmonic forms: $\mathcal{H} := ker\Delta$.

5 Hodge theory

5.1 The operator $\Delta = dd^* + d^* d$ is formally self-adjoint

$$(\Delta a, b) = (a, \Delta b).$$

5.2 For a compact oriented C^∞-manifold M the following holds in $A^\bullet(M)$
1) $\mathcal{H} = ker(d) \cap ker(d^*)$
2) $ker(d^*) = im(d)^\perp$, $ker(d) = im(d^*)^\perp$, $ker(\Delta) = im(\Delta)^\perp$,
(hence $\mathcal{H} = ker(d) \cap im(d)^\perp$)
3) the spaces $\mathcal{H}$, $im(d)$ and $im(d^*)$ are perpendicular.

Proof:
1) suppose $a \in ker(\Delta)$:

$$0 = (\Delta a, a) = (dd^* a, a) + (d^* da, a) = (d^* a, d^* a) + (da, da) = ||d^* a||^2 + ||da||^2$$

2) Let $P = d$, d^* of Δ. If $a \in ker(P^*)$ then $0 = (P^* a, b) = (a, Pb)$, hence $a \in im(P)^\perp$.
Conversely, if $a \in im(P)^\perp$, then $0 = (a, PP^* a) = ||P^* a||^2$, so $P^* a = 0$.
3) It remains to show that the spaces $im(d)$ and $im(d^*)$ are perpendicular $(d^* a, db) = (a, d^2 b) = 0$.
(Here we used that $d^2 = 0$, all the rest was an abstract properties of formally adjoint operators.)

5.3 Hodge decomposition

$$\mathcal{A}^\bullet(M) = \underbrace{im(d) \oplus \mathcal{H}}_{ker(d)} \oplus im(d^*).$$

This decomposition is orthogonal.

• The decomposition follows from a general property of elliptic differential operators, which we will not prove. We would have to extend the space of C^∞ forms and consider Sobolev spaces. See [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Cambridge Studies in Advanced Mathematics. Theorem 5.22, p.128-9]. For any *elliptic* operator $P : C^\infty(E) \to C^\infty(F)$

$$C^\infty(E) = ker(P) \oplus P^*(C^\infty(F)) .$$

(Exercise: prove the corresponding statement for a linear map between finite dimensional spaces.)

• In our case $P = \Delta$, $P^* = \Delta$

$$A^\bullet(M) = \mathcal{H} \oplus \Delta(A^\bullet(M)) .$$

Moreover we have

$$im(\Delta) \subset im(d) + im(d^*).$$

But from orthogonality $(im(d) \oplus im(d^*)) \cap \mathcal{H} = 0$, hence

$$im(\Delta) = im(d) \oplus im(d^*).$$

5.4 Corollary 1: $\mathcal{H} \to H^*(M)$ is an isomorphism.
Moreover: if $\Delta(a) = 0$ and $a' = a + db$, then $||a'|| \geq ||a||$.
Any harmonic form is the representative of its cohomology class, which has the smallest norm.

5.5 Corollary 2: Tricky proof of the Poincaré duality: Let $[\alpha] \neq 0 \in H^*(M)$, then there exists a class $[\beta]$ (in the complementary gradation) such that $\int_M \alpha \wedge \beta \neq 0$.

• Proof: let's assume that α is harmonic. Set $\beta = *\alpha$. Then β is harmonic as well ($d(*\alpha) = \pm * d^*(\alpha) = 0$ and $d^*(*\alpha) = \pm * d(\alpha) = 0$). We have

$$\int_M \alpha \wedge *\alpha = \int_M ||\alpha||^2 vol = ||\alpha||^2_M .$$

5.6 Heat equation $\alpha : \mathbb{R}_+ \to A^*(M)$ with the initial condition $\alpha(0) = \alpha$

$$\frac{d}{dt}\alpha(t) = -\Delta\alpha(t) ,$$

see [D. Arapura, Algebraic Geometry over the Complex Numbers] §8

• the solution exists for $t \geq 0$

• $\alpha_H := \lim_{t\to\infty} \alpha(t)$ exists and is a harmonic form.
(Laplasian has nonnegative eigenvalues: if $\Delta(\alpha) = \lambda\alpha$ then

$$\lambda||\alpha|| = (\Delta\alpha, \alpha) = ||d\alpha||^2 + ||d^*\alpha||^2 \geq 0 .$$

hence the limit exists.)

• $\alpha = \alpha_H + \Delta G(\alpha)$, where $G(\alpha) = \int_0^\infty (\alpha(t) - \alpha_H)dt$ is the Green operator $G : \mathcal{H}^\perp \to A^\bullet(M)$.

• Let's check for α being an eigenvector $\Delta\alpha = \lambda\alpha$, $\lambda \neq 0$: The solution is of the form $\alpha(t) = e^{-\lambda t}\alpha$. Then

$$\Delta\left(\int_0^\infty e^{-\lambda t}\alpha dt\right) = \int_0^\infty e^{-\lambda t}\lambda\alpha\, dt = \left(\int_0^\infty e^{-\lambda t}\lambda\, dt\right)\alpha = \alpha .$$

• If $\beta(t)$ is a solution with the initial condition β, then $d\beta(t)$ is a solution with the initial condition $d\beta$ (because $d\Delta = ddd^* + dd^*d = dd^*d = dd^*d + d^*dd = \Delta d$).

• If $\alpha = \alpha_H + d\beta$ then $\alpha_t = \alpha_h + d\beta_t$.

• If $d\alpha = 0$, then $d\alpha_t = 0$ and $[\alpha_t] = [\alpha]$
Proof $\alpha = \alpha_h + d\beta$, $(\alpha_t - \alpha_h)' = -\Delta(d\beta_t) = -d\Delta(\beta_t)$

$1\frac{1}{2}$–linear algebra

5.7 Complex structure on a real vector space is an automorphism I satisfying $I^2 = -id$. It decomposes $V_{\mathbb{C}} := V \otimes \mathbb{C}$ into eigenspaces

$$V_{\mathbb{C}} = V_i + V_{-i}\,.$$

Necessarily $\dim V$ is even and one can find a real basis $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n$ of V, such that $I(e_k) = f_k$, $I(f_k) = -e_k$.

- The vectors $e_k - if_k$ form a basis of V_i: $\quad I(e_k - if_k) = f_k + ie_k = i(e_k - if_k)$
- The vectors $e_k + if_k$ form a basis of V_{-i}: $\quad I(e_k + if_k) = f_k - ie_k = -i(e_k - if_k)$

5.8 We are more concerned about the dual space: $\mathbb{C}$-linear form are said to have the type (1,0)

$$\Lambda^{10}V^* := \{\phi \in Hom_{\mathbb{R}}(V, \mathbb{C}) \mid \phi(Iv) = i\phi(v)\ \}\,,$$

the antilinear forms are said to have the type (0,1)

$$\Lambda^{01}V^* := \{\phi \in Hom_{\mathbb{R}}(V, \mathbb{C}) \mid \phi(Iv) = -i\phi(v)\ \}\,,$$

We have

$$V^* \otimes \mathbb{C} = \Lambda^{10}V^* \oplus \Lambda^{01}V^*\,.$$

5.9 The dual basis is denoted by

$$dx_k := e_k^*\,, \quad dy_k := f_k^*\,.$$

We define

$$dz_k := dx_k + idy_k\,, \quad d\overline{z}_k := dx_k - idy_k\,.$$

The 1-forms dz_k are the basis of $\Lambda^{10}V^*$, and $d\overline{z}_k$'s are the basis of $\Lambda^{01}V^*$.

5.10 We have a $\mathbb{C}$-linear isomorphism $(V^*, I) \overset{\Phi}{\simeq} (\Lambda^{10}V^*, i)$, $\Phi(f)(v) = f(v) - if(Iv))$
$\Phi(If)(v) = f(Iv) - if(I^2v) = f(Iv) + if(v) = i(f(v) - if(Iv)) = i\Phi(f)(v)$

And an anti-linear isomorphism: $(V^*, I) \overset{\psi}{\simeq} (\Lambda^{01}V, i)$, $\Psi(f)(v) = f(v) + if(Iv))$
$\Psi(If)(v) = f(Iv) + if(I^2v) = f(Iv) - if(v) = -i(f(v) + if(Iv)) = -i\Psi(f)(v)$

5.11 The exterior forms of the type (p, q):

$$\Lambda^k V_{\mathbb{C}}^* = \bigoplus_{p+q=k} \Lambda^{pq}\,, \qquad \Lambda^{pq} := \Lambda^p(\Lambda^{10}V^*) \wedge \Lambda^q(\Lambda^{01}V^*)\,.$$

– Conjugation acts on $\Lambda^k(V^* \otimes \mathbb{C}) = (\Lambda^k V^*) \otimes \mathbb{C}$. We have

$$\overline{\Lambda^{pq}} = \Lambda^{qp}.$$

– The operator I acts on $\Lambda^{p,q}V^*$ via multiplication by $i^{(p-q)}$

Hermitian linear algebra

Suppose (V, I) is a real vector space with a complex structure.

5.12 Hermitian product

$$V \otimes V \to \mathbb{C}$$

$$\langle\!\langle v, w \rangle\!\rangle = \langle v, w \rangle - i\omega(v, w)$$

consists of:
– I-invariant scalar product $\langle v, w \rangle$,
– I-symplectic form $\omega(v, w)$
– the scalar product and the symplectic form determine each other $\omega(v, w) = \langle I(v), w \rangle = -\langle v, I(w) \rangle$.

5.13 The volume form is defined as the wedge of an orthonormal (positively oriented) basis vectors of V^*:

- suppose $\dim_{\mathbb{C}}(V) = n$

$$vol = (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n) = (\tfrac{i}{2})^n (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n)$$

- $$\omega = \sum_{k=1}^{n} dx_k \wedge dy_k = \tfrac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k \,.$$

$$\omega^n = n! vol \,.$$

- Remark: the form ω belongs to $\Lambda^2 V^* \cap \Lambda^{11} V^* \subset \Lambda^2 V^*_{\mathbb{C}}$.

5.14 Exercise $\Lambda^{10} \perp \Lambda^{01}$

6 Linear Lefschetz, almost complex [Huybrechts §1.2]

6.1 The Lefschetz operator

$$L(\alpha) := \omega \wedge \alpha \,.$$

- „Linear version of Hard Lefschetz"

$$L^k : \Lambda^{n-k} V \xrightarrow{\simeq} \Lambda^{n+k} V$$

is an isomorphism. (An elementary proof can be given, but it is tedious. The claim follows form $\mathfrak{sl}(2)$–action, to be explained later.)

6.2 Suppose $\dim V = n$. Let us define $H \in End(\Lambda V^*)$ as the multiplication by $k - n$ on $\Lambda^k V$.

- We have $[H, L] = 2L$.
- Proof: $a \in \Lambda^k$: $HLa - LHa = (k + 2 - n)La - (k - n)La = 2La$.

6.3 Let us define the adjoint operator L^*

$$\langle L\alpha, \beta \rangle = \langle \alpha, L^* \beta \rangle$$

L^* lowering the gradation by 2. We have:

- $[H, L] = 2L, \quad [H, L^*] = -2L^*$
- $[L, L^*] = H$.
- Proof: It is enough to check these identities for $\dim V = 1$, because all three operators well behave with respect to the orthogonal direct sum.

6.4 The vector space ΛV^* is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{Z})$:

$$\rho : \mathfrak{sl}_2(\mathbb{Z}) \to End(\Lambda V^*), \qquad \rho(h) = H, \qquad \rho(x) = L, \qquad \rho(y) = L^* \,,$$

where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

6.5 Recollection of representation theory:

- simple representations (not containing proper subrepresentations) are of the form $S_k = Sym^k(\mathbb{C}^2)$ (the same for the theory over $\mathbb{R}$).
- any $\mathfrak{sl}_2$ representation can be decomposed into simple representations
- all eigenvalues of h on a representation W are integers, x^k defines an isomorphism $W_{-k} \to W_k$ $(k \geq 0)$.

- $x : W_k \to W_{k+2}$ is mono for $k < 0$, epi for $k + 2 > 0$
- Lefschetz decomposition: For $k \geq 0$ let us define the primitive subspace

$$P_k = \{w \in W_{-k} \,|\, x^{k+1} w = 0\}.$$

We have

$$W_{-k} = P_k \oplus x P_{k+2} \oplus x^2 P_{k+4} \oplus \dots .$$

6.6 The primitive forms (attention at the gradation shift): for $0 \leq k \leq n$ let us define

$$P^{n-k} = \{\alpha \in \Lambda^{n-k} V^* \mid L^{k+1} \alpha = 0\}$$
$$P^{p,q} = \Lambda^{p,q} \cap P^{p+q}_{\mathbb{C}} .$$

We have

$$P^{n-k}_{\mathbb{C}} = \bigoplus_{p+q=n-k} P^{p,q} .$$

6.7 Lefschetz decomposition of forms:

$$\Lambda^{n-k} V^* = P^{n-k} \oplus L(P^{n-k-2}) \oplus \dots \oplus L^j (P^{n-k-2j}) \oplus \dots .$$

- FURTHER STRATEGY: we know that above analysis applies to any complex vector space with a Hermitian product. Hence it applies to tangent spaces of a complex manifold. We obtain a list operators, decompositions etc in the space of complex-valued differential forms. We will show that this structure survives in the cohomology of a complex projective variety. Instead of projectivity it is enough to assume that the manifold has Kähler structure – a notion defined in terms of differential forms.

6.1 Linear algebra → differential/complex manifolds structure

6.8 An almost complex manifold (M, I) is a pair, where M is a real C^∞-manifold and $I \in End(TM) = TM \otimes T^*M$ a tensor satisfying $I^2 = -id$ (i.e. a complex structure in each T_pM smoothly depending on the point $p \in M$.)

6.9 The complexified space of forms decomposes as a direct sum $A^k(M)_{\mathbb{C}} = \bigoplus_{p+q=k} A^{p,q}(M)$. In general $d_{|A^{10}(M)}$ can have components of the type (0,2). If the almost complex structure comes from a holomorphic local coordinates, then (0,2) component vanishes:

$$d(f\, dz_k) = \sum_{\ell \neq k} \tfrac{df}{dz_\ell} dz_\ell \wedge dz_k + \sum_{\ell} \tfrac{df}{d\bar{z}_\ell} d\bar{z}_\ell \wedge dz_k .$$

6.10 Notation: the eigenspace of I acting on $TM \otimes \mathbb{C}$:

$$T^{10}M = (TM \otimes \mathbb{C})_i , \qquad T^{01}M = (TM \otimes \mathbb{C})_{-i} .$$

- The global sections of the above bundles will be denoted by $\Gamma(T^{10}M)$ and $\Gamma(T^{01}M)$.

6.11 [Huybrechts 2.6.17] The condition $(d\alpha)_{02} = 0$ for $\alpha \in A^{10}(M)$ is equivalent to the *involutivity* condition of the distribution $T^{01}M$:

$$\forall u, v \in \Gamma(T^{01}M) \quad [u, v] \in \Gamma(T^{01}M).$$

- Proof: The condition $d\alpha \in A^{20}(M) \oplus A^{11}(M)$ is equivalent to $d\alpha_{|\Lambda^2 T^{01}M} = 0$.

Suppose $\alpha \in A^{10}(M)$, $u, v \in \Gamma(T^{01}M)$. We compute from the Cartan formula:

$$d\alpha(u, v) = u\alpha(v) - v\alpha(u) - \alpha([u, v]) = \alpha([u, v]) .$$

Hence the condition $d\alpha \in A^{20}(M) \oplus A^{11}(M)$ is equivalent to

$$\alpha([u, v]) = 0 \qquad \text{for } u, v \in \Gamma(T^{01}M).$$

- We allow α to be arbitrary form of the type (1,0) and we conclude

$$d(A^{10}(M)) \subset A^{20}(M) \oplus A^{11}(M) \qquad \iff \qquad \forall u, v \in \Gamma(T^{01}M) \quad [u, v] \in \Gamma(T^{01}M) .$$

6.12 Recollection from the real differential geometry: Let $D \subset TM$ be a subbundle (i.e. a distribution). We say that D is integrable if at every point of M there exist a local coordinates $x_1, x_2, \ldots, x_{\dim M} : U \to \mathbb{R}$ such that $D_{|U}$ is the subbundle of TU consisting of the vector fields tangent to the fibers of the map $(x_1, x_2, \ldots, x_k) : U \to \mathbb{R}^k$, where $k = \dim M - \mathrm{rk} D$ is the codimension of the distribution.

• **Frobenius Theorem**: D is integrable if and only if D is involutive (i.e. if vector fields u, v are sections of D then $[u, v]$ is a section of D).

6.13 Let $P = \frac{1}{2}(iI+id) \in End(TM \otimes \mathbb{C})$ be the projection onto the (0,1)-component. The involutivity condition is equivalent to

$$(***) \qquad \forall u, v \in \Gamma(TM) \quad P[Pu, Pv] = [Pu, Pv].$$

• We transform this identity to obtain the Nijenhuis tensor $N_I \in T^*M \otimes T^*M \otimes \mathbb{C}$

$$N_I(X, Y) = -I^2[X, Y] + I([IX, Y] + [X, IY]) - [IX, IY]\,.$$

• Claim: $N_I = 0$ if and only if (***). This is a formal checking (exercise)

• Exercise: Show that N_I is indeed a tensor (i.e. the map $(X, Y) \mapsto N_I(X, Y)$ is $C^\infty(M)$-bilinear.

6.14 Newlander-Nirenberg theorem [Hyubrechts 2.6.19]: The almost-complex structure I originates from a complex structure on the manifold iff $N_I = 0$.

7

7.1 If M is a complex manifold, then

$$d(A^{p,q}(M)) \subset A^{p+1,q}(M) \oplus A^{p,q+1}(M)$$

$$d = \partial + \overline{\partial}, \quad \partial^2 = 0 = \overline{\partial}^2, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0\,.$$

7.2 Dolbeault complex: for $0 \le p \le \dim_{\mathbb{C}} M$ we have a complex

$$0 \to A^{p,0}(M) \stackrel{\overline{\partial}}{\to} A^{p,1}(M) \stackrel{\overline{\partial}}{\to} \ldots \stackrel{\overline{\partial}}{\to} A^{p,\dim M}(M) \to 0, ,$$

$$ker(\overline{\partial} : A^{p,0}(M) \to A^{p,1}(M)) = \Omega^p(M)\,.$$

Here $\Omega^p(M)$ denotes the form of the type $(p, 0)$ with holomorphic coefficients.

7.3 We define Dolbeault cohomology [Huybrechts 2.6.20]:

$$H^q_{Dol}(M; \Omega^p) := H^q(A^{p,\bullet}(M), \overline{\partial})$$

7.4 Holomorphic Poincaré lemma [Hyubrechts 1.3.7]: the complex of sheaves

$$0 \to \Omega^p_M \to A^{p,0} \to A^{p,1} \to A^{p,2} \to \ldots$$

is exact.

• This means that if $\bar{\partial}\alpha = 0$, $\alpha \in A^{p,q}(U)$, then *locally* there exists β such that $\bar{\partial}\beta = \alpha$, i.e. for each point $p \in U$ there exists $V \subset U$, $p \in V$ and $\beta \in A^{p,q-1}(V)$ such that $\bar{\partial}\beta = \alpha_{|V}$.

7.5 Sheaf cohomology - a summary, see eg [Huybrechts, Appendix B]

7.6 Cohomology with the coefficients in a sheaf $\mathcal{F}$: there are two important construction

- Čech cohomology
- Sheaf cohomology as the derived functor of Γ - taking the global sections.

1) we find a resolution of $\mathcal{F}$, i.e. an exact complex

$$0 \to \mathcal{F} \to I^0 \to I^1 \to I^2 \to \dots$$

with the sheave A^k sufficiently good (acyclic, e.g. injective)
2) we apply the functor of global sections (and cut off the first term)

$$\Gamma(I^0) \to \Gamma(I^1) \to \Gamma(I^2) \to \dots$$

This complex is no longer exact.
3) W compute cohomology:

$$H^k(M;\mathcal{F}) = H^k(\Gamma(I^\bullet))\,.$$

We have $H^0(M;\mathcal{F}) = \Gamma(\mathcal{F})$, because the functor Γ is left-exact.

7.7 In our case, when the base is paracompact any *soft* resolution is acyclic. („Soft" means, that sections defined on a closed set can be extended to global sections.)

- Suppose M is a C^∞-manifold. Any sheaf which is a module over the ring of C^∞-functions is soft.
- The complex of C^∞-forms on C^∞-manifold $A^0 \to A^1 \to A^2 \to \dots$ is a resolution of the sheaf $ker(d : A^0 \to A^1) = \mathbb{R}_M$, the sheaf of locally constant functions.

7.8 The sheaves $A^{p,q}$ are A^0-modules, hence they are soft.

- The Dolbeault complex is a resolution of $\Omega^p_M = ker(\bar\partial : A^{p,0} \to A^{p,1})$
-

$$H^k(M;\Omega^p) = H^k(A^{p,\bullet}(M))$$

i.e the Dolbeault cohomology is the sheaf cohomology in the sense of the homological algebra.

7.9 If M is a complex manifold, then $A^\bullet_{\mathbb{C}} = \bigoplus_{p+q=\bullet} A^{p,q}$ is a resolution of the sheaf $\mathbb{C}_M$.

- For $p \geq 0$ define the Hodge'a filtration (on the sheaf level)

$$F^pA^k = \bigoplus_{p'+q=k,\ p'\geq p} A^{p',q}\,.$$

Claim: $F^pA^\bullet$ is a subcomplex of $A^\bullet$.

- The resulting filtration in cohomology $H^k(M;\mathbb{C}) = H^k\big(A^\bullet(M)_{\mathbb{C}}\big)$

$$F^pH^k(M;\mathbb{C}) = im(H^k(F^pA^\bullet(M)) \to H^k\big(A^\bullet(M)_{\mathbb{C}}\big)\,.$$

7.10 We have

$$F^pA^k/F^{p+1}A^k \simeq A^{p,k-p}\,.$$

- The quotient map is a map of complexes (with a shift of the gradation)

$$(F^pA^\bullet, d) \to (A^{p,\bullet}, \bar\partial)$$

- Passing to cohomology:

$$H^{p+q}((F^pA^\bullet/F^{p+1}A^\bullet)(M)) = H^q(A^{p\bullet}(M)) = H^q(M;\Omega^p_M)\,.$$

The relation between cohomologies of the quotients with cohomology of the entire sheaf is given by the spectral sequence

$$E_1^{p,q} = H^{p+q}(F^pA^\bullet(M)/F^{p+1}A^\bullet(M)) = H^q(M;\Omega^p_M) \ \Rightarrow H^{p+q}(M;\mathbb{C}).$$

Motivation leading to the notion of Čech cohomology :

[B. V. Shabath, Introduction to complex analysis II, Chapter IV].

7.11 Additive Cousin Problem: find a global meromorphic function with prescribed poles. Let $M = \bigcup U_i$ be a covering. On each U_i there is given a meromorphic function f_i. We assume that the differences $g_{ij} = (f_i)_{|U_i \cap U_j} - (f_j)_{|U_i \cap U_j}$ are holomorphic. Does there exist a meromorphic function f on M such that each difference $f_{|U_i} - f_i$ is holomorphic?

7.12 Multiplicative Cousin Problem:
Let $\{U_i\}_{i \in I}$ be a covering of M. On each U_i there is given a meromorphic function f_i. We assume that the quotients $g_{ij} = \frac{(f_i)_{|U_i \cap U_j}}{(f_j)_{|U_i \cap U_j}}$ are holomorphic. Does there exist a meromorphic function f on M such that each quotient $\frac{f_{|U_i}}{f_i}$ is holomorphic?

7.13 The answer is in the language of Čech cohomology. For a covering $\mathscr{U} = \{U_i\}$ the Čecha complex is defined by:

$$\check{C}^k(\mathscr{U}) = \prod_{i_0 < i_1 < \cdots < i_k} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k}) \,.$$

Notation: for a multiindex $I = \{i_0 < i_1 < \cdots < i_k\}$ let $U_I = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k}$. For $\{s_I\} \in \check{C}^{k-1}(\mathscr{U})$ define the differential

$$d(\{s_I\})_J = \sum_{a=1}^{k} (-1)^a (s_{J \setminus j_a})_{|U_J}$$

For example

$$d(\{s_i\})_{j_0,j_1} = (s_{j_1})_{|U_{j_0,j_1}} - (s_{j_0})_{|U_{j_0,j_1}}$$

$$d(\{s_{i_0,i_1}\})_{j_0,j_1,j_2} = s_{j_1,j_2} - s_{j_0,j_2} + s_{j_0,j_1} \qquad \text{restricted to } U_{j_0,j_1,j_2}$$

7.14 Čech cohomology is defined by $\check{H}^k(\mathscr{U};\mathcal{F}) = H^k(\check{C}^\bullet(\mathscr{U};\mathcal{F}), d)$.

7.15 Additive Cousin Problem : Let $\mathcal{F} = \mathcal{O}_M$, the collection of functions $\{g_{i,j}\} \in \check{C}^1(\mathscr{U};\mathcal{O}_M)$ satisfies the cocycle condition:

$$g_{ij} - g_{ik} + g_{jk} = 0.$$

It defines an element of Čech cohomology of the covering $H^1(\{U_i\};\mathcal{O}_M)$. The cohomology class is trivial if the cocycle is a coboundary, i.e. there exists a collection of elements $h_i \in \mathcal{O}_M(U_i)$ such that $g_{ij} = h_j - h_i$.

- the Cousin problem has a solution if and only if the cohomology class $[g_{ij}] = 0$.

Proof: If $g_{ij} = h_j - h_i$, then the meromorphic functions $\tilde{f}_i = f_i + h_i$ agree at the intersections:

$$\tilde{f}_i - \tilde{f}_j = f_i + h_i - f_j + h_j \qquad \text{on } U_i \cap U_j \,.$$

(The converse - exercise.)

7.16 Multiplicative Cousin problem has a positive solution if the cocycle g_i/g_j defines the trivial class in $H^1(\{U_i\};\mathcal{O}^*_M)$.

7.17 Passing to a finer cover defines a map of Čecha cohomology (it does not depend on inscribing function).

7.18 Theorem: If M is paracompact, then

$$\lim_{\substack{\longrightarrow \\ \mathscr{U}}} \check{H}^k(\mathscr{U};\mathcal{F}) \simeq H^k(M;\mathcal{F})$$

(The RHS is in the sense of homological algebra.)

7.19 If the covering is acyclic (i.e. $H^k(U_I;\mathcal{F}) = 0$ for any multiindex I and $k > 0$) then

$$H^k(\{U_i\};\mathcal{F}) \simeq H^k(M;\mathcal{F}).$$

7.20 Sufficient conditions for being acyclic:

- For locally constant sheaves on topological spaces: if all U_I are contractible,
- For coherent sheaves in algebraic geometry: if U_I are affine,
- For coherent sheaves in analytic geometry: if U_I are Stein spaces

Definition $U \subset M$ is Stein if:
– for any pair of points $p, q \in U$ there exists an analytic function $f \in \mathcal{O}_U$ such that $f(p) \neq f(q)$.
– (holomorphic convexity) for any compact set $K \subset U$ the set

$$\bar{K} := \{p \in U \mid \forall f \in \mathcal{O}_U \ |f(p)| \leq sup_{q\in K}|f(q)| \ \}$$

is compact.

7.21 In the cousin problems one can pass to a finer coverings. Since $H^1(\mathbb{P}^n;\mathcal{O}_M) = 0$, so on $\mathbb{P}^n$ the additive Cousin problem has always a positive solution. On curves of positive genus - not always: $genus = \dim H^1(C;\mathcal{O}_C)$.

7.22 Holomorphic Poincaré lemma [Hyubrechts 1.3.7]: 1-dimensional case: Given a form $\alpha = f(z)d\bar{z}$ on a disc. Show that there exists a function $g(z)$ such that $\alpha = \bar{\partial}\bar{g}$.

- i.e. $f = \frac{\partial}{\partial \bar{z}} g$
- Assume that f is defined on the disc of the radius $1+\varepsilon$, let

$$g(z) = \tfrac{1}{2\pi i}\int_{D_\varepsilon} \frac{f(w)}{w-z} dw \wedge d\bar{w}$$

- If f has a support in D_δ, $\delta < 1-\varepsilon$ then

$$g(z) = \tfrac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(w)}{w-z} dw \wedge d\bar{w} = \tfrac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(u+z)}{u} du \wedge d\bar{u}$$

- The above construction of g is a linear mapping: $\textstyle\int : C^\infty(D_{1+\epsilon}) \to C^\infty(D_1)$.
- Analogy with the real case:

— for a real (compactly supported) $f : \mathbb{R} \to \mathbb{R}$ we define the primitive function

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{+\infty} K(t-x)f(t)dt,$$

where

$$K(t) = \begin{cases} 0 \text{ if } t < 0 \\ 1 \text{ if } t \geq 0 \end{cases}, \qquad \text{and} \qquad K'(t) = \delta_0 \,.$$

So the primitive function is expressed by the convolution with K, i.e $F(x) = (K * f)(x)$.
— similarly for complex, compactly supported function $f : \mathbb{C} \to \mathbb{C}$

$$g(z) = (K * f)(z),$$

where $K(z) = \frac{1}{2\pi i}\frac{1}{z}$, which has the property $\frac{\partial}{\partial \bar{z}} K = \delta_0$

7.23 Poincaré lemma in many variables: if $\alpha \in A^{pq}(U)$, $\bar{\partial}\alpha = 0$, then $\alpha = \bar{\partial}\beta$ locally.

- We can assume that $p = 0$.
- We have defined $\textstyle\int : C^\infty(D_{1+\epsilon}) \to C^\infty(D_1)$. It extends to polidiscs, but we have to specify the variable with respect to which we integrate.
- Suppose $\alpha = d\bar{z}_1 \wedge \alpha_1 + \alpha_2$ (α_k with no $d\bar{z}_1$). Define $H_1(\alpha) = \textstyle\int_1(\alpha_1)$ coefficientwise, integration with respect to the first variable.

• Computation for $\dim U = 2$:

$$\alpha = a\,d\bar z_1 + b\,d\bar z_2\,, \qquad H_1(\alpha) = \textstyle\int_1(a)$$

$$\bar\partial H_1(\alpha) = \tfrac{\partial}{\partial \bar z_1}\textstyle\int_1(a)d\bar z_1 + \tfrac{\partial}{\partial \bar z_2}\textstyle\int_1(a)d\bar z_2 = a\,d\bar z_1 + \textstyle\int_1(\tfrac{\partial}{\partial \bar z_2}a)d\bar z_2$$

Since α is closed $\frac{\partial}{\partial \bar z_2}a = \frac{\partial}{\partial \bar z_1}b$

$$\bar\partial H_1(\alpha) = a\,d\bar z_1 + \textstyle\int_1(\tfrac{\partial}{\partial \bar z_1}b)d\bar z_2$$

Hence

$$\tilde\alpha = \alpha - \bar\partial H_1(\alpha) = b\,d\bar z_2 - \textstyle\int_1(\tfrac{\partial}{\partial \bar z_1}b)d\bar z_2 = \tilde b\,d\bar z_2\,,$$

where $\tilde b = b - \int_1(\frac{\partial}{\partial \bar z_1}b)$ is holomorphic with respect to z_1.

• Now we apply the operator $\int$ with respect to the second variable

$$\bar\partial H_2(\tilde\alpha) = \tfrac{\partial}{\partial \bar z_2}\textstyle\int_2(\tilde b)d\bar z_2 d = \tilde b\,d\bar z_2$$

• If

$$\alpha = a\,d\bar z_1 \wedge d\bar z_2\,, \qquad H_1(\alpha) = (\textstyle\int_1 a)dz_2$$

Then

$$\bar\partial H_1(\alpha) = \bar\partial(\textstyle\int_1 a dz_2) = (\tfrac{\partial}{\partial \bar z_1}\textstyle\int_1 a)d\bar z_1 \wedge d\bar z_2 = a\,d\bar z_1 \wedge d\bar z_2$$

7.24 We start with H_1 and write $\tilde\alpha = \alpha - \bar\partial H_k(\alpha)$, then we apply H_2 and so on. In general we prove that after the improvement

$$\alpha := \alpha - \bar\partial H_k(\alpha)$$

we obtain a form which has coefficients holomorphic with respect to $z_1, z_2, \dots, z_k$ and has no $d\bar z_1, d\bar z_2, \dots, d\bar z_k$

8 Hodge theory for Hermitian manifolds

8.1 Hermitian structure on a complex manifold M is a choice of a Hermitian product in each tangent space.

• such structure is a section of $T^*M \otimes \overline{T}^*M$ which is symmetric and positively definite. We assume that it is a C^∞

• real part is a scalar product, the imaginary part - a differential 2-form (which does not have to be closed).

• Hermitian structures exist for paracompact manifolds: we can chose a Hermitian structure locally in maps and glue them using partition of unity.

8.2 We extend Hodge * $\mathbb{C}$-linearly

• If $\dim M = 1$

$$*dz = *(dx+idy) = dy - idx = -i(dx+idy) = -idz\,, \qquad *d\bar z = *(dx-idy) = dy+idx = i(dx-idy) = id\bar z$$

$$*1 = \omega = dx\wedge dy = \tfrac{i}{2}dz\wedge d\bar z\,, \qquad *\omega = 1$$

• In higher dimensions

$$* : \Lambda^{p,q} \xrightarrow{\simeq} \Lambda^{n-q,n-p}$$

$$*dz_I \wedge d\bar z_J = c\,dz_{[n]\setminus J}\wedge d\bar z_{[n]\setminus I}$$

Exercise: compute c.

• Occasionally will appear antilinear star

$$\bar * : \Lambda^{p,q} \xrightarrow{\simeq} \Lambda^{n-p,n-q}, \qquad \bar*(\alpha) = *\bar\alpha = \overline{*\alpha}\,.$$

8.3 We have operators real L, L^*, $H = [L, L^*] = (\deg - n)id$ acting on C^∞-forms $A^*(X)$. The adjoint operator

$$L^* = *^{-1} L * = (-1)^{\deg} * L*$$

. (The sign should be $(-1)^{(\dim_\mathbb{R} M - \deg)\deg}$ but here $\dim_\mathbb{R} TM$ is even). Often in literature L^* is denoted by Λ, but it can be confused with the exterior power). The adjoint operator satisfies $(L\alpha, \beta\rangle = \langle\alpha, L^*\beta)$.

- The complexified operators L, L^*, $H = [L, L^*] = (\deg - n)id$ act on $A^*(X)_\mathbb{C}$. Hence $A^*(X)_\mathbb{C}$ becomes a (infinite dimensional) representation of $\mathfrak{sl}(2)$.
- We take complexification, because we are also interested in the bigradation, available only over $\mathbb{C}$.

8.4 We define operators

$$\partial^* = - * \bar\partial * : A^{p,q}(X) \to A^{p-1,q}(X)\,,$$

$$(p,q) \mapsto (n-q, n-p) \mapsto (n-q, n-p+1) \mapsto (p-1, q)$$

and

$$\bar\partial^* = - * \partial * : A^{p,q}(X) \to A^{p,q-1}(X)\,.$$

We have $d^* = \partial^* + \bar\partial^*$.

- explanation of signs: $d^* = (-1)^{\dim_\mathbb{R} M(\deg + 1) + 1} * d * = - * d*$

8.5 Kähler structure
It can be defined in three equivalent ways:

- Definition 1: locally there exists local coordinates in which $\omega = \frac{i}{2}\sum_k dz_k \wedge d\bar z_k + \mathcal{O}(||x||^2)$.

i.e. in some coordinates the Hermitian metric is the same as for flat the manifold $\mathbb{C}^n$ up to the terms of order 2.

- Definition 2: $d\omega = 0$
- Definition 3: locally $\omega = \partial\bar\partial f$ for some real function $f : U \to \mathbb{R}$.
- Proof 1) $\Rightarrow$ 2) obvious.

8.6 Proof 2) $\Rightarrow$ 1) [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Prop 3.14]

- How to construct good coordinates?

$$\omega = \tfrac{i}{2}\sum_k dz_k \wedge d\bar z_k + \sum_{k,l}(\epsilon^h_{k,l} + \epsilon^a_{k,l}) dz_k \wedge d\bar z_l + \mathcal{O}(|z|^2)$$

where $\epsilon^h_{k,l}$ is a holomorphic linear form, $\epsilon^a_{k,l}$ antiholomorphic liner form.

- $\overline{\epsilon^a_{k,l}} = \epsilon^h_{l,k}$ since ω is real.
- $\frac{\partial}{\partial z_j}\epsilon^h_{k,l} = \frac{\partial}{\partial z_k}\epsilon^h_{j,l}$ since ω is closed

8.7 The most important example: Fubini-Study metric on $\mathbb{P}^n$

$$< \alpha, \beta > = \frac{1}{\pi}\frac{< w, w >< \tilde\alpha, \tilde\beta > - < \tilde\alpha, w >< w, \tilde\beta >}{< w, w >^2},$$

where $\tilde\alpha, \tilde\beta \in T_w\mathbb{C}^{n+1}$ are lifts of the vectors $\alpha, \beta \in T_{[w]}\mathbb{P}^n$.

- in local coordinates on $U_0 = \{z_0 \neq 0\}$

$$\omega = \frac{i}{2\pi}\partial\bar\partial log(1 + \sum_{k=1}^n |w_k|^2)$$

Obviously $d\omega = 0$.

8.8 For $n = 1$

$$\omega = \frac{i}{2\pi}\partial\bar\partial log(1 + |w|^2) = \frac{i}{2\pi}\frac{1}{(1 + w\bar w)^2} dw \wedge d\bar w = \frac{1}{\pi}\frac{1}{(1 + x^2 + y^2)^2} dx \wedge dy$$

- $\int_{\mathbb{P}^1}\omega = 1$, hence $[\omega] \in H^2(\mathbb{P}^1; \mathbb{R})$ is integral, i.e. it comes from $H^2(\mathbb{P}^1; \mathbb{Z})$.

8.9 Corollary: any complex submanifold of $\mathbb{P}^n$ has a Kähler structure.
- $[\omega] = c_1(\mathcal{O}(1)) \in H^2(M;\mathbb{Z})$

8.10 Hodge identities:
i) $[\bar\partial, L] = [\partial, L] = 0$ (since ω is closed)
i') equivalently $[L^*, \partial^*] = [L^*, \bar\partial^*] = 0$
ii) $[\bar\partial^*, L] = i\partial$, $[\partial^*, L] = -i\bar\partial$
ii') equivalently $[L^*, \bar\partial] = -i\partial^*$, $[L^*, \partial] = i\bar\partial^*$ (this is the most difficult, the rest follows)
iii) $[\partial, \bar\partial^*]_s = [\partial^*, \bar\partial]_s = 0$ (i.e $\partial\bar\partial^* + \bar\partial^*\partial = 0$ etc, this is a formal consequence of ii))
iv) $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$ and it commutes with ∂, $\bar\partial$, ∂^*, $\bar\partial^*$, L i L^* (formal algebraic proof)

8.11 Short proof from [C. Voisin, Hodge Theory And Complex Algebraic Geometry I, Prop 6.5].
- Assume according to Definition 1) that ω has a standard form up to the terms of order 2. Therefore in calculations involving only the **first derivatives** at a point we can assume that

$$\omega = \tfrac{i}{2}\sum dz_k \wedge d\bar z_k$$

- We show ii') i.e. $[L^*, \partial] = i\bar\partial^*$. It is enough to check

$$([L^*, \partial](\alpha))_{z=0} = i(\bar\partial^*\alpha)_{z=0}$$

- We decompose $\omega = \sum_k \omega_k$, $\omega_k = \frac{i}{2}dz_k \wedge d\bar z_k$.

The adjoint operator $L_k^* = (\omega_k\wedge)^*$ is expressed by the contraction of differential forms

$$L_k^* = -2i\iota_{\bar v_k}\iota_{v_k}\,,$$

where $v_k = \frac{\partial}{\partial z_k}$, $\bar v_k = \frac{\partial}{\partial \bar z_k}$.
- We decompose $\bar\partial = \sum \bar\partial_k$. The adjoint differentials

$$\partial_k^* = -2\tfrac{\partial}{\partial \bar z_k}\iota_{v_k}\,, \qquad \bar\partial_k^* = -2\tfrac{\partial}{\partial z_k}\iota_{\bar v_k},$$

A sample of check in dim=1

$$\partial^* f\,dz = -*\bar\partial * f\,dz = -*\bar\partial(-if dz) = i * \tfrac{\partial}{\partial \bar z} f d\bar z \wedge dz = -2\tfrac{\partial}{\partial \bar z} f\ * \tfrac{i}{2} dz \wedge d\bar z = -2\tfrac{\partial}{\partial \bar z} f$$

- The operators L_k^* and ∂_ℓ commute for $k \neq \ell$. It remains to check $[L_k^*, \partial_k]$ for $\alpha = f dz_I \wedge d\bar z_J$, considering 4 cases $k \in$ or $\notin I$ or J. For example: suppose $k \notin I$, $k \notin J$

$$[L_k^*, \partial_k] f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J =$$

$$L_k^*\partial_k(f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J) - \partial_k L_k^*(f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J) =$$

$$2i\partial_k(f dz_I \wedge d\bar z_J) =$$

$$2i\tfrac{\partial}{\partial z_k}(f dz_k \wedge dz_I \wedge d\bar z_J) =$$

$$-2i\tfrac{\partial}{\partial z_k}\iota_{\bar v_k}(f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J) =$$

$$i\bar\partial^*(f dz_k \wedge d\bar z_k \wedge dz_I \wedge d\bar z_J).$$

8.12 For a computational proof see Huybrechts.
- The Huybrechts' proof of ii'): an operator $d^c = I^{-1}dI$ is introduced and the adjoint operator $(d^c)^*$

$$d^c = -i(\partial - \bar\partial), \quad (d^c)^* = -*d^c*\,.$$

He shows ii") $[L^*, d] = -(d^c)^*$. The proof is computational, using Lefschetz decomposition into $L^k\alpha$, where α is primitive.

8.13 iii)

$$i[\partial, \bar{\partial}^*] \overset{ii)}{=} [\partial, [L^*, \partial]] = \partial L^* \partial - \partial^2 L^* + L^* \partial^2 - \partial L^* \partial = 0$$

8.14 To show and iv) it is convenient to introduce the language of supercommutators $[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba$. In that notation

$$\Delta_\partial = [\partial, \partial^*] \,.$$

- Leibniz rule, equivalent to the graded Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)} [b, [a, c]] \,.$$

-

$$\Delta_\partial = [\partial^*, \partial] \overset{ii)}{=} i[[L^*, \bar{\partial}], \partial] \overset{Leibniz}{=} i([L^*, \underbrace{[\bar{\partial}, \partial]}_{0}] - [[L^*, \partial], \bar{\partial}]) \overset{ii)}{=} [\bar{\partial}^*, \bar{\partial}] = \Delta_{\bar{\partial}}$$

and from iii) $\Delta = \Delta_\partial + \Delta_{\bar{\partial}}$.

$$[L, \Delta_\partial] = [L, [\partial, \partial^*]] \overset{Leibniz}{=} [\underbrace{[L, \partial]}_{0}, \partial^*] + [\partial, [L, \partial^*]] \overset{ii)}{=} i[\partial, -i\bar{\partial}] = 0$$

9 Cohomology of Kähler manifold

- Corollary $H^*(M) \simeq \mathcal{H}$ is a representation of $\mathfrak{sl}_2(\mathbb{Z})$.
- Practical consequences:

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism (**Hard Lefschetz Theorem**).

- It follows

$$\dim H^k(M) \le \dim H^{k+2}(M) \qquad \text{if } k+1 \le n \,,$$
$$\dim H^k(M) \ge \dim H^{k+2}(M) \qquad \text{if } k+1 \ge n \,.$$

9.1 Hodge decomposition for the operator $\bar{\partial}$

$$\mathcal{A}^{p,q}(M) = \underbrace{im(\bar{\partial}) \oplus \mathcal{H}^{p,q}_{\bar{\partial}}}_{ker(\bar{\partial})} \oplus im(\bar{\partial}^*).$$

$$\bar{\partial} : A^{p,q-1} \to A^{p,q}, \qquad \bar{\partial}^* : A^{p,q+1} \to A^{p,q} \,.$$

-

$$H^q(M; \Omega^p) \simeq \mathcal{H}^{p,q}_{\bar{\partial}} \,,$$

- Since $\Delta_{\bar{\partial}} = \frac{1}{2}\Delta$, we have

$$\mathcal{H}^{p,q}_{\bar{\partial}} = \mathcal{H}^{p,q} \,,$$
$$\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p} \,, \qquad *\mathcal{H}^{p,q} = \mathcal{H}^{n-q,n-p} \,.$$

-

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \,.$$

9.2 Hodge decomposition in cohomology (not talking about harmonic forms)

- Recall the Hodge filtration

$$F^p A^\bullet(M) = A^{\ge p, \bullet}(M)$$

and the induced filtration in cohomology

$$F^p H^*(M) = im\big(H^*(F^p A^\bullet(M) \to H^*(M)\big)$$

This definition is independent from the metric.

- We have

$$H^{p,q}(M) = F^pH^{p+q}(M)) \cap \overline{F^qH^{p+q}(M)}\,.$$

$$H^k(M) = \bigoplus_{p+q=k} H^{p,q}(M)\,,$$

9.3 Let $h^{p,q} = \dim H^{p,q}(M)$.

- Hard Lefschetz implies inequalities

$$h^{p,q} \le \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \le n\,,$$

$$h^{p,q} \ge \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \ge n\,,$$

- The symmetries $h^{p,q} = h^{n-p,n-q} = h^{q,p}$ are organized in the ,,Hodge diamond"
- For example for $n = 3$

$$\begin{array}{ccccccc} & & & h^{33} & & & \\ & & h^{32} & & h^{23} & & \\ & h^{31} & & h^{22} & & h^{13} & \\ h^{30} & & h^{21} & & h^{12} & & h^{03} \\ & h^{20} & & h^{11} & & h^{02} & \\ & & h^{10} & & h^{01} & & \\ & & & h^{00} & & & \end{array} \quad = \quad \begin{array}{ccccccc} & & & 1 & & & \\ & & \spadesuit & & \spadesuit & & \\ & \diamondsuit & & \clubsuit & & \diamondsuit & \\ \square & & \heartsuit & & \heartsuit & & \square \\ & \diamondsuit & & \clubsuit & & \diamondsuit & \\ & & \spadesuit & & \spadesuit & & \\ & & & 1 & & & \end{array}$$

- Hard Lefschetz implies inequalities

$$h^{p,q} \le \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \le n\,,$$

$$h^{p,q} \ge \dim h^{p+1,q+1} \qquad \text{if } p+q+1 \ge n\,,$$

9.4 Moreover

– If $k = n - (p+q) \ge 0$ then $L^k : H^{p,q}(M) \to H^{p+k,q+k}(M)$ is an isomorphism

– If $p + q \le n$ then

$$H^{p,q}(M) = P^{p,q}(M) \oplus L(P^{p-1,q-1}(M) \oplus L^2(P^{p-2,q-2}(M) \oplus \ldots$$

9.5 Generalities about spectral sequence: if $C^\bullet$ is a complex with decreasing filtration

$$C^\bullet = F^0C^\bullet \supset F^1C^\bullet \supset F^2C^\bullet \supset \ldots,$$

then one wishes to relate cohomologies $H^*(F^pC^\bullet/F^{p+1}C^\bullet)$ with $H^*(C^\bullet)$.

- There exists a spectral sequence (under some boundness of degree assumptions)

$$E_0^{p,q} = F^pC^{p+q}/F^{p+1}C^{p+q}, \qquad E_1^{p,q} = H^{p+q}(F^pC^\bullet/F^{p+1}C^\bullet)\,, \qquad \ldots.$$

- There exists a sequence of tables $E_r^{p,q}$ with differentials of degree $(1-r, r)$, such that

1) $H^*(E_r^{\bullet,\bullet}) = E_{r+1}^{\bullet,\bullet}$

2) $E_\infty^{p,q} = F^pH^{p+q}(\mathbb{C}^\bullet)/F^{p+1}H^{p+q}(\mathbb{C}^\bullet)$

9.6 For the total complex of the bicomplex $A^{p,q}(M)$ with the Hodge filtration $F^pA^\bullet(M) = A^{\ge p,\bullet}(M)$ the resulting spectral sequence is called the **Frölicher spectral sequence**.

- Corollary: If M Kähler and compact, then the Frölicher spectral sequence degenerates on E_1, i.e.

$$E_1^{p,q} = H^q(M;\Omega^p) = E_\infty^{p,q}\,.$$

(the higher differentials vanish).

9.7 Corollary: Suppose M Kähler and compact: if $\alpha \in \Omega^p(M)$ then $\partial\alpha = 0$.

- Holomorphic implies closed.
- This is a generalization of: global holomorphic function is constant.

9.8 We say that M is Calabi-Yau if $\Omega^n \simeq \mathcal{O}_M$
(according to more restrictive definitions it is assumed additionally $H^0(M, \Omega^p) = 0$ for $0 < p < n$)
The Hodge diamond looks like this ($n = 3$)

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \clubsuit & & 0 & \\ 1 & & \heartsuit & & \heartsuit & & 1 \\ & 0 & & \clubsuit & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

- We say that M^* is a cohomological mirror of M if $h^{pq}(M^*) = h^{n-p,q}(M)$.
- For 3-manifolds this means $h^{12}(M^*) = h^{11}(M)$ i $h^{11}(M^*) = h^{12}(M)$.
- Problem how to find M^*.

9.9 Serre duality: the exterior product

$$\wedge : \Omega^p \times \Omega^q \to \Omega^{p+q}$$

defines a bilinear map

$$H^k(M; \Omega^p) \times H^\ell(M; \Omega^q) \to H^{k+l}(M; \Omega^{p+q}) .$$

If $k + \ell = p + q = n$ we obtain compose it with the integral $\int : H^n(M; \Omega^n) \simeq H^{2n}(M; \mathbb{C}) \to \mathbb{C}$.

- By Poincaré duality this form is nondegenerate

$$H^k(M; \Omega^p) \simeq H^{n-k}(M; \Omega^{n-p})^* .$$

- More generally: we have a nondegenerate form

$$H^k(M; E) \times H^{n-k}(M; E^* \otimes \Omega^n) \to H^n(M; \Omega^n) \to \mathbb{C}$$

for a locally free sheaf E. In particular for $\Omega^p = E$:

$$\Omega^{n-p} \simeq \underline{Hom}(\Omega^p, \Omega^n) = (\Omega^p)^* \otimes \Omega^n$$

and we recover the previous formula.

9.10 Signature

- For oriented compact C^∞-manifold M of dimension $4m$ the intersection pairing in $H^{2m}(M; \mathbb{R})$ is symmetric and nondegenerate. Its signature is called the signature of M, denoted $sgn(M)$ or $\sigma(M)$.
- Instead the real intersection form we consider $H^*(M; \mathbb{C})$ with the hermitian form. The resulting signature is the same.

9.11 Hodge'a-Riemann relations [Huybrechts 3.3.15]: Define the hermitian form $B(\alpha, \beta)$ on $H^k(M)$ as:

$$B(\alpha, \beta) = \int_M \alpha \wedge \bar{\beta} \wedge \omega^{n-k} .$$

This form is symmetric or antisymmetric depending on the parity of k

$$B(\alpha, \beta) = (-1)^k \overline{B(\alpha, \beta)} .$$

• It is nondegenerate: for $\alpha \in H^k(M)$ there exists $\beta \in H^k(M)$ such that $B(\alpha,\beta) \neq 0$.
— Let $\gamma \in H^{2n-k}(M)$ such that $\int_M \alpha \wedge \bar\gamma \neq 0$ (e.g. $\gamma = \bar{*}\alpha$)
— By Hard Lefshetz $\gamma = L^{n-k}\beta$ for some $\beta \in H^k(M)$

$$B(\alpha,\beta) = \int_M \alpha \wedge \bar\gamma \neq 0\,.$$

• The pairing B restricted to $H^{p,q}(M)$ is non degenerate. The form $\gamma = \bar{*}\alpha$ is of the type $(n-q, n-p)$, hence $L^{k-n}\gamma$ is of the type $(n-q-n+k, n-p-n+k) = (p,q)$.

9.12 Antisymmetric forms over $\mathbb{C}$ can be turned into symmetric:
• if ϕ jest antisymmetric, i.e.

$$\phi(a,b) = -\overline{\phi(b,a)}\,,$$

then $\psi(a,b) := i\phi(a,b)$ is symmetric.
• If hermitian form ψ is symmetric then $\psi(a,a) = \overline{\psi(a,a)}$, hence $\psi(a,a) \in \mathbb{R}$
• We say that such form is positive definite if

$$\psi(a,a) > 0 \qquad \text{for } a \neq 0\,.$$

9.13 Theorem [Hodge-Riemann relations]: Let $k = p+q$. The form

$$i^{p-q} \cdot (-1)^{k(k-1)/2} B(\alpha,\beta)$$

restricted to the primitive space

$$P^{p,q}(M) = P^k(M) \cap H^{p,q}(M)$$

is symmetric and positive definite.

9.14 Proof reduces to calculations for $\Lambda^\bullet\mathbb{C}^n$: one has to check the sign of the form B_0 restricted to $P^{p,q} \subset \Lambda^{p,q} \subset \Lambda(\mathbb{C}^n)^* \otimes \mathbb{C}$. Here B_0 is defined by the formula

$$\alpha \wedge \bar\beta \wedge \omega^{n-k} = B_0(\alpha,\beta) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n\,.$$

• We check the following identity for $\alpha \in P^k$:

$$(***) \quad L^{n-k}\alpha = (-1)^{\frac{k(k-1)}{2}} (n-k)! * I(\alpha)\,,$$

where I is the complex structure acting on $\Lambda(\mathbb{C}^n)^* \otimes_{\mathbb{R}} \mathbb{C}$.
• We show inductively

$$L^j\alpha = (-1)^{\frac{k(k-1)}{2}} \frac{j!}{(n-k-j)!} * L^{n-k-j} I(\alpha).$$

• Having $(***)$:

$$\alpha \wedge \bar\alpha \wedge \omega^{n-k} = \alpha \wedge L^{n-k}(\bar\alpha) = \alpha \wedge (-1)^{\frac{k(k-1)}{2}} (n-k)! * I(\alpha) =$$

$$= (-1)^{\frac{k(k-1)}{2}} \alpha \wedge *(n-k)! I(\alpha) = i^{q-p}(-1)^{\frac{k(k-1)}{2}} (n-k)! < \alpha, \alpha > vol$$

9.15 Corollary [Huybrechts 3.3.18]: Let $n = 2m$. Then M is a real manifold of dimension $4m$. The intersection form in the middle dimension $2m$ is symmetric. It coincides with $B(\alpha,\beta)$.
• The signature of M is defined as the signature of the intersection form $H^{2m}(M)$ is equal to

$$\sum_{p+q \leq m,\ 2|p+q} (-1)^{\frac{k(k-1)+p-q}{2}} \dim(P^{p,q}(M))$$

- We have equality $\dim P^{p,q}(M) = h^{p,q} - h^{p-1,q-1}$. Using symmetries of Hodge diamond $h^{p,q} = h^{q,p} = h^{n-p,n-q}$ we obtain a formula for the signature

$$sgn(M) = \sum_{p,q=0,\ 2|p+q}^{\dim(M)} (-1)^p h^{p,q} \,.$$

- Example: let $n = 4$: we sum up the terms for which $p - q$ is even:

$$sgn(M) = \begin{array}{ccccc} +p^{4,0} & -p^{3,1} & +p^{2,2} & -p^{1,3} & +p^{0,4} \\ & +p^{2,0} & -p^{1,1} & +p^{0,2} & \\ & & +p^{0,0} & & \end{array}$$

$$= \begin{array}{ccccc} +h^{4,0} & -h^{3,1} + h^{2,0} & +h^{2,2} - h^{1,1} & -h^{1,3} + h^{0,2} & +h^{0,4} \\ & +h^{2,0} & -h^{1,1} + h^{0,0} & +h^{0,2} & \\ & & +h^{0,0} & & \end{array}$$

$$= \begin{array}{ccccc} & & +h^{4,4} & & \\ & +h^{4,2} & -h^{3,3} & +h^{2,4} & \\ +h^{4,0} & -h^{3,1} & +h^{2,2} & -h^{1,3} & +h^{0,4} \\ & +h^{2,0} & -h^{1,1} & +h^{0,2} & \\ & & +h^{0,0} & & \end{array}$$

- We can neglect the remaining summands with $p + q$ odd, since $(-1)^q h^{p,q}$ cancels with $(-1)^p h^{q,p}$

$$sgn(M) = \sum_{p,q=0}^{\dim(M)} (-1)^p h^{p,q}$$

Further we can transform the formula:

$$sgn(M) = \sum_{p,q=0}^{\dim(M)} (-1)^q h^{p,q} = \sum_{p=0}^{\dim M} \chi(M; \Omega^p) \,.$$

- Example: For the connected surfaces the intersection form is of the type $(2h^{2,0} + 1, h^{1,1} - 1)$.

10 Application of Hard Lefschetz to topology of maps

10.1 Given a topological fibration $M \hookrightarrow Z \twoheadrightarrow B$.

- Serra spectral sequence

$$E_2^{p,q} = H^p(B; \mathcal{H}^q(M)) \Rightarrow H^{p+q}(Z) \,.$$

- if B is simply-connected, then the coefficient system $\mathcal{H}^q(M)$ is constant and $E_2^{p,q} \simeq H^p(B) \otimes H^q(M)$.

10.2 Theorem [Blanchard, Deligne]: Suppose there exist a class $a \in H^2(Z)$, such that $a_{|M}$ satisfies the conclusion of the Hard Lefschetz, i.e. for some $n \in \mathbb{N}$ and for every $k \in \mathbb{N}$

$$L^k = - \cap a_{|M}^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism. Then $E_2 = E_\infty$, hence

$$H^*(Z) \simeq H^*(B) \otimes H^*(M)$$

as $H^*(B)$-module.

- Proof. The Lefschetz map $L := [a]\cup$ acts on E_*^{**}. We show that the differentials d_r acting on E_r for $r \geq 2$ vanish on primitive cohomology $H^*(B) \otimes P^*(M)$. By multiplicative structure of the spectral sequences and from $d_r(a_{|M}) = 0$ we get the conclusion.

• For $\alpha \in H^p(B) \otimes P^q(M) \subset E_r^{p,q}$ we have $d_r\alpha \in E_r^{p+r,q-r+1}$. Suppose $n-k = q-r+1$. Then $n+k = n+(n-q+r-1) = 2n-q+r-1$

$$L^{n-q+r-1} : E_r^{p+r,q-r+1} \overset{\simeq}{\to} E_r^{p+r,2n-q+r-1}$$

We show that $L^{n-q+r-1}d_r\alpha = 0$:

$$L^{n-q+r-1}d_r\alpha = d_r L^{n-q+r-1}\alpha = d_r L^{r-2}L^{n-q+1}\alpha = 0$$

for $r \geq 2$, since $P^q(M) = ker(L^{n-q+1} : H^q(M) \to H^{2n-q+2}(M))$.

10.3 Morse theory for C^∞-manifolds [Milnor – Morse theory, 1963]

• Def: Morse function $f : M \to \mathbb{R}$ is a proper smooth function such that if $Df(p) = 0$ for $p \in M$ then $D^2 f(p)$ is nondegenerate. Additionally we assume that for each critical value there exist only one critical point of f (critical values of distinct points do not collide).

• $ind(p)$= the index of a critical point = the number of minuses after diagonalization of $D^2 f(p)$.

• for $t \in \mathbb{R}$ let

$$M_{\leq t} = \{p \in M \mid f(p) \leq t\} \,.$$

• Theorem:
1) If there is no critical value in the interval $[a, b]$, then the inclusion $M_{\leq a} \subset M_{\leq b}$ is a homotopy equivalence
2) If $f(p) = c \in [a, b]$ is the only one critical value in the interval $[a, b]$ then $M_{\leq b}$ is homeomorphic to $M_{\leq a}$ with attached $I^{ind(p)} \times I^{n-ind(p)}$ along $\partial I^{ind(p)} \times I^{n-ind(p)}$, (up to homotopy we attach a cell of the dimension $ind(p)$).

10.4 Suppose $M \subset \mathbb{R}^N$ is a compact submanifold, let $f_q(x) = dist(q, x)^2$ for a fixed $q \in \mathbb{R}^N \setminus M$.

• For almost all $q \in \mathbb{R}^N$ the function f_q is Morse.

• Assume that $q = 0$, $p = (a, 0, \dots, 0)$ with $a \in \mathbb{R}_+$, $T_pM = \{x_{n+1} = x_{n+2} = \dots = 0\}$; then M locally a graph of a function $g = (g_1, g_2, \dots g_{N-n} : \mathbb{R}^n \to \mathbb{R}^{N-n}$, $g_1(0) = a$, $g_k(0) = 0$ for $k > 1$, $Dg(0) = 0$;

• then

$$f_q(x) = a^2 + \sum_{i=1}^{n} x_i^2 + (a + g_1(x_1, \dots, x_n))^2 + \sum_{j=2}^{N-n} g_j(x_1, \dots, x_n)^2 = \sum_{i=1}^{n} x_i^2 + 2aQ(x_1, \dots, x_n) + \mathcal{O}(||x||^4) \,,$$

where Q is a quadratic form of g_1, hence

$$D^2 f_q(p) = 2(I + 2aQ) \,.$$

Therefore

$$ind(p) = \#\{\lambda \in spec\, Q \mid \lambda < -\tfrac{1}{2a}\}$$

10.5

$$ind(p) = \#\{\lambda \in spec\, Q \mid \lambda < -\tfrac{1}{2a}\}$$

• Lemma: If $M \subset \mathbb{C}^N$ is a complex submanifold, $q \notin M$ and p is a critical point of f_q, then

$$index(p) \leq \dim_{\mathbb{C}}(M)$$

• Proof: we assume as before that $q = 0$, $p = (a, 0, \dots, 0)$, $a \in \mathbb{R}_+ \subset \mathbb{C}$.

• Very easy algebraic lemma: Suppose Q is a nondegenerate quadratic form on $\mathbb{C}^n$. If v is an eigenvector of the real part $re(Q)$ with the eigenvalue λ, then iv is an eigenvector with the eigenvalue $-\lambda$. Hence the eigenvalues are symmetrically distributed with respect to 0.

• Corollary $2(I + a\, re(D^2 g(0)) = 2(I + 2a\, re(Q))$ is at most $\frac{1}{2}\dim_{\mathbb{R}}(M)$.

10.6 If $M \subset \mathbb{C}^N$ is a complex submanifold of the complex dimension n, then M has the homotopy type of n-dimensional CW-complex. Hence $H^k(M;R) = 0$ for $k > n$ (with coefficient in any ring R).

10.7 „Weak Lefschetz" aka „Lefschetz hyperplane theorem" [Milnor, Morse Theory §7]: If $X \subset \mathbb{P}^N$ is a complex submanifold of dimension n, $Y = X \cap \mathbb{P}^{N-1}$, then $H_{n-k}(X \setminus Y) \simeq H^k(X,Y) = 0$ for $k < n$.

- By the long exact sequence of the pair $i^* : H^k(X) \to H^k(Y)$ is an isomorphism for $k < n-1$, monomorphism for $k = n-1$.
- Similarly: $H^{n-k}(X \setminus Y) \simeq H_k(X,Y) = 0$ for $k < n$. Hence $i_* : H_k(Y) \to H_k(X)$ is an isomorphism for $k < n-1$, epimorphism for $k = n-1$.
- Moreover $i_* : \pi_1(Y) \to \pi_1(X)$ is an isomorphism if $2 < n$, epimorphism if $2 = n$.

10.8 If $X \subset \mathbb{P}^N$, and M is a smooth hypersurface of degree d, then $M \cap X \simeq \iota(X) \cap H$, where $\iota : \mathbb{P}^N \to \mathbb{P}(Sym^d(\mathbb{C}^{N+1}))$ is the Veronese embedding and H is a linear hypersurface in $\mathbb{P}(Sym^d(\mathbb{C}^{N+1}))$.

- Hence for complete intersection $X \subset \mathbb{P}^N$ we have information about all Betti numbers, except the middle one:

$$X = X_{N-n} \subset X_{N-n-1} \subset \cdots \subset X_{N-1} \subset X_N = \mathbb{P}^N$$

$\dim(X_i) = N - i$, since $k < n < \dim(X_i)$ for $i < N - n$, we have isomorphisms $H^k(X_i) \simeq H^k(x_{i+1})$.

$$H^k(X) = \begin{cases} \mathbb{Z} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases}$$

for $k < n$, and from Poincaré duality $H^k(X) \simeq H_{2n-k}(X)$ we get the same result for $n > k$.

- **Corollary**: If $X \subset \mathbb{P}^N$ is a complete intersection then $H^*(X;\mathbb{Q})$ is spanned by $[\omega^k]$ for $* \neq \dim X$.

10.9 Exercise: compute $\dim(H^n(Q_n))$ for a nonsingular quadric $Q_n \subset \mathbb{P}^{n+1}$.

11 Picard-Lefschetz theory

See e.g.:
[Klaus Lamotke], The topology of complex projective varieties after S. Lefschetz. Topology 20 (1981), no. 1, 15-51

[Arapura §13.3] Donu Arapura, Algebraic Geometry over the Complex Numbers, Springer Universitext 2012

[H. Żołądek, The Monodromy Group, Springer 2006]

11.1 Monodromy: Given a topological fibration (with the property of lifting homotopy, if F is of the type of CW-complex, then Serre fibration is enough; if practice we deal with locally trivial fibration)

$$p : E \to B$$

Let $F_b = p^{-1}(b)$ for $b \in B$. Let us fix a base point b. We have a well defined map

$$\mu : \pi_1(B, b) \to [F_b, F_b]$$

where $[-,-]$ denotes the homotopy classes.

- The image of μ belongs to the group of homotopy equivalences.
- The resulting map for the homology:

$$\pi_1(B, b) \to Aut(H_*(F_b))\,.$$

11.2 Lefschetz pencil: let $X \subset \mathbb{P}^N$, A projective subspace of codimension 2.

- The hyperplanes $H \subset \mathbb{P}^N$ containing $A = \mathbb{P}(\widehat{A})$ are parameterized by $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^{N+1}/\widehat{A})$.

Notation: for $\lambda \in \mathbb{P}^1$ the corresponding hyperplane is denoted by H_λ.

- Let's define the coincidence variety

$$\mathcal{X} = \{(x, \lambda) \in X \times \mathbb{P}^1 \mid x \in H_\lambda\}.$$

There are projections

$$X \xleftarrow{q} \mathcal{X} \xrightarrow{p} \mathbb{P}^1.$$

We say that $\mathcal{X}$ is a Lefschetz pencil if $\mathcal{X}$ is smooth and the projection to $\mathbb{P}^1$ has only simple (Morse type) singularities (locally $\sum_{i=1}^n z_i^2$) and in each fiber there is at most one singular point.

- Lemma: for a generic choice of A we obtain a Lefschetz pencil.
- Exercise: $\mathcal{X} = Bl_{X\cap A}X$.

11.3 Let $S \subset \mathbb{P}^1$ be the set of critical values of the projection onto $\mathbb{P}^1$, let $U = \mathbb{P}^1 \setminus S$ be the complement. Then

$$p : \mathcal{X}_U = p^{-1}(U) \to U$$

is a fibration. Fix $\nu \in U$. The fundamental group $\pi = \pi_1(U, \nu)$ acts on the cohomology of the fiber $\mathcal{X}_\nu = p^{-1}(\nu) = X \cap H_\nu = Y$.

11.4 The group $\pi_1(U, \nu)$ is free, generated by the loops nonintersecting $\gamma_1, \gamma_2, \dots, \gamma_k$. Each γ_k is circling around a single singular value of p. The only relation we have $\gamma_1\gamma_2 \dots \gamma_k = 1$.

11.5 Picard-Lefschetz Theorem:

- Define the *invariant cycles*:

$$I := im(\iota^* : H^{n-1}(X) \to H^{n-1}(Y)),$$

and *vanishing cycles*

$$V := ker(\iota_* : H^{n-1}(Y) \to H^{n+1}(X)) \simeq ker(\iota_* : H_{n-1}(Y) \to H_{n-1}(X)).$$

Then

(i) $H^{n-1}(Y) = I \oplus V$ and this is an orthogonal sum with respect to the intersection form.

(ii) Vanishing cycles V are generated by the cycles δ_i (to be defined later), each such cycle is associated to a critical point of $p : \mathcal{X} \to \mathbb{P}^1$ and a loop γ_i encircling that point.

(iii) **Picard–Lefschetz formula** describes the action of the loop γ_i on $H^{n-1}(Y)$:

$$\gamma_i(\alpha) = \alpha + (-1)^{n(n+1)/2}(\alpha, \delta_i)\delta_i,$$

(iv) The invariant cycles are is exactly the subspace invariant with respect to the monodromy action:

$$I = H^{n-1}(Y)^\pi$$

(the inclusion ,,$\subset$" is easy, the opposite inclusion is equivalent to Hard Lefschetz)

(v) the representation $\pi = \pi_1(U)$ on V is simple (does not contain any proper subrepresentation).

- Remark: The subspace generated by $\{\delta_i\}$ can be described not using the particular choices of the individual generators of π: it is spanned by $\gamma(\alpha) - \alpha$, $\gamma \in \pi$

11.6 Proof of (i): The composition $\iota_*\iota^* : H^{n-1}(X) \to H^{n+1}(X)$ is the multiplication by $[\omega]$. By Hard Lefschetz it is an isomorphism.

- Hence ι_* is epi, ι^* mono.
- $$H^{n-1}(Y) = im(\iota^*) \oplus ker(\iota_*) = I \oplus V.$$
- Moreover from the identity $(i^*a \cup b, [Y]) = (a \cup i_*b, [X])$ (where $\cup$ denotes multiplication in cohomology) we obtain that I and V are perpendicular with respect to the intersection form.

11.7 General facts about monodromy for fibrations over S^1

11.8 Assume that $F \subset E \overset{p}{\to} B = S^1$ is a topological fibration (Hurewicz). If you assume that it is locally trivial then there exists a homeomorphism $\mu : F \to F$, such that

$$E = F \times [0,1]/\sim , \qquad (x,0) \sim (\mu(x),1)\,.$$

11.9 The Wang exact sequence

$$\to H_{k+1}(E) \to H_k(F) \overset{\mu_*-1}{\to} H_k(F) \to H_k(E) \to$$

is obtained from the exact sequence of the pair (E, E_+), where $E_+ = p^{-1}([0,1/2])$:

$$\to H_{k+1}(E) \to H_{k+1}(E, E_+) \to H_k(E_+) \to H_k(E) \to$$

- Cohomological version

$$\to H^k(E) \to H^k(F) \overset{\mu^*-1}{\to} H^k(F) \to H^{k+1}(E) \to$$

- From the Wang exact sequence we have

$$im(H^k(E) \to H^k(F)) = H^k(F)^{\pi_1(S^1)}$$

11.10 Milnor fibration: Let $f : \mathbb{C}^n \to \mathbb{C}$ be a germ at 0 of a holomorphic function, $Y = \{f = 0\}$. Let $S_\epsilon \simeq S^{2n-1} \subset \mathbb{C}^n$ be a sufficiently small sphere. Define

$$p = f/|f| : S^{2n-1}_\epsilon \setminus Y \to S^1 \subset \mathbb{C}\,.$$

- There is an alternative, topologically equivalent fibration

$$f : B_\epsilon \cap f^{-1}(S^1_\delta) \to S^1\,,$$

$0 < \delta << \epsilon << 1$.
- Theorem [Milnor]: If 0 is an isolated singular point of f, then the fiber of the Milnor fiber has the homotopy type of the bouquet spheres of the dimension $n-1$.

11.11 Local topology of the complex Morse singularity $\sum z_i^2$. Fix $\epsilon \in \mathbb{R}$:
- The set $Y_\epsilon = \{\sum z_i^2 = \epsilon\}$ is homeomorphic to the sphere TS^{n-1}.

$$\begin{aligned}
Y_\epsilon &= \{\sum x_i y_i = 0,\ \sum x_i^2 - y_i^2 = \epsilon\}\,,\\
Y_\epsilon &= \{(x,y) = 0,\ ||x||^2 = \epsilon + ||y||^2\}\,,\\
Y_\epsilon &= \left\{x \perp y = 0,\ \| \tfrac{x}{\sqrt{\epsilon + ||y||^2}} \|^2 = 1\right\}\,,
\end{aligned}$$

.

On the other hand the tangent space to the sphere

$$TS^{n-1} = \{u \perp y,\ ||u|| = 1\}$$

Sending

$$(x,y) \mapsto (u = \tfrac{x}{\sqrt{\epsilon + ||y||^2}}, y)$$

is a homeomorphism.
- Note that this homeomorphis change the orientation by the sign $(-1)^{n(n+1)/2}$.

11.12 Let

$$\delta = [S^{n-1}] \in H_{n-1}(TS^{n-1}),\ \ell \in H_{n-1}(TS^{n-1}, TS^{n-1} \setminus \{0\}) \qquad \ell - \text{the class of the fiber } TS^{n-1}.$$

The monodromy acts as follows

$$\mu_*(\ell) = \ell + \delta$$

• Precisely, we have the *variation map*

$$var = \mu_* - 1 : H_{n-1}(Y_\epsilon, \partial Y_\epsilon) \to H_{n-1}(Y_\epsilon).$$

$$var(\ell) = \delta.$$

11.13 We come back to the situation of the Lefschetz pencil:

$$\mathcal{X}_{|U} \longrightarrow U = \mathbb{P}^1 \setminus S \ni \lambda.$$

• We fix a $\nu \in U$.

• The loop γ_i starts at ν and circles around the singular value $\lambda_i \in S$. We identify the nonsingular fiber in a close neighbourhood of $\mathcal{X}_{\lambda_i'} \simeq \mathcal{X}_\nu$ by means of choosing a path joining ν with λ_i.

11.14 Picard Lefschetz formula The monodromy acting on $\alpha \in H_{n-1}(\mathcal{X}_\nu) = H_{n-1}(Y) \simeq H^{n-1}(Y)$:

$$\mu_i(\alpha) = \alpha + (-1)^{n(n+1)/2}(\alpha, \delta_i)\delta_i,$$

where δ_i is the class of a small sphere in the fiber $\mathcal{X}_{\lambda_i}$.

• Here, if the number $k = (\alpha, \delta_i)$, then

$$H_{n-1}(\mathcal{X}_\nu) \simeq H_{n-1}(\mathcal{X}_{\lambda_i}) \to H_{n-1}(\mathcal{X}_{\lambda_i}, \mathcal{X}_{\lambda_i} \setminus S_\epsilon) \simeq H_{n-1}(TS^{n-1}, TS^{n-1} \setminus \{0\})$$

$$\alpha \mapsto k\,\ell.$$

11.15 Picard-Lefschetz package – so far we know

1. $I = im(\iota^* : H^{n-1}(X) \to H^{n-1}(Y)) \subset Inv := H^{n-1}(Y)^\pi$
2. $Van = span\{\delta_i \mid p_i \in S\} \subset V := ker(\iota_* : H^{n-1}(Y) \to H^{n+1}(X))$
3. From the Picard-Lefschetz formula: $Inv = Van^\perp$
4. We have proven so far $H^{n-1}(Y) = I \oplus V$, $I^\perp = V$.

11.16 [Lamotke] Homological proof: $Inv = I$ (from that $Van = V$ follows):

• Decompose $\mathbb{P}^1$ into disks: $\mathbb{P}^1 = D_+ \cup D_-$, and we assume that all the critical values of λ_i belong to D_+ and assume that the distinguished regular value $\nu \in D_+ \cap D_-$.

• $\mathcal{X}_\pm = p^{-1}(D_\pm)$, $\mathcal{X}_+ \cap \mathcal{X}_- = \partial\mathcal{X}_- = \partial\mathcal{X}_+ \simeq S^1$

• Let $\mathcal{Z}$ = inverse image of $Z = X \cap A$ with respect to $q : \mathcal{X} \to X$

• We have fixed a regular value $\nu \in \mathbb{P}^1$. The corresponding fiber in $\mathcal{X}$ is denoted by $\mathcal{X}_\nu$ and $Y = X \cap H_\nu$.

• **The fundamental lemma**:

$$H_{n-1}(\mathcal{X}_+, \mathcal{X}_\nu) = \mathbb{Z}^r$$

where $r = |S|$ is the number of critical points of p.
Proof: The pair $(\mathcal{X}_+, \mathcal{X}_\nu)$ has the same homology as

$$\left(\bigvee_k (p^{-1}(D_k) \cap B_\epsilon), \bigvee_k (\mathcal{X}_{\lambda_k'} \cap B_\epsilon)\right),$$

where D_k is a small disk around λ_k and $\lambda_k' \in \partial D_k$.

11.17 Since $\mathcal{X} \setminus \mathcal{Z} \cup \mathcal{X}_\nu \simeq X \setminus Y$

$$H_*(\mathcal{X}, \mathcal{Z} \cup \mathcal{X}_\nu) \simeq H_*(X, Y)$$

By excision

$$H_k(\mathcal{X}, \mathcal{X}_+) \simeq H_k(\mathcal{X}_-, \partial\mathcal{X}_-) \simeq H_k(\mathcal{X}_\nu \times (D^2, S^1)) \simeq H_{k-2}(\mathcal{X}_\nu) = H_{k-2}(Y)$$

Similarly:

$$H_k(\mathcal{X}, \mathcal{Z} \cup \mathcal{X}_+) \simeq H_k(\mathcal{X}_-, (\mathcal{Z} \cap \mathcal{X}_-) \cup \partial\mathcal{X}_-) \simeq H_k((X_\nu, \mathcal{Z} \cap X_\nu) \times (D^2, S^2)) \simeq H_{k-2}(Y, Z)$$

- The exact sequence of the triple $(\mathcal{X}, \mathcal{Z} \cup \mathcal{X}_+, \mathcal{Z} \cup \mathcal{X}_\nu)$:

$$\begin{array}{ccccc} H_n(\mathcal{Z} \cup \mathcal{X}_+, \mathcal{Z} \cup \mathcal{X}_\nu) & \to & H_n(\mathcal{X}, \mathcal{Z} \cup \mathcal{X}_\nu) & \to & H_n(\mathcal{X}, \mathcal{Z} \cup \mathcal{X}_+) \\ \| & & \| & & \| \\ \mathbb{Z}^r = H_n(\mathcal{X}_+, \mathcal{X}_\nu) & & H_n(X, Y) & & H_{n-2}(Y, Z) = 0 \quad (\text{ from HLT }). \end{array}$$

- Corollary: The cycles δ_i, $i = 1, \ldots, r$ span $V = im(H_n(X, Y) \to H_{n-1}(Y))$, hence

$$\boxed{V = Van}\,.$$

□

11.18 By the same method one proves the bouquet theorem for Milnor fibration of an isolated singularity $f : \mathbb{C}^n \to \mathbb{C}$. The fiber has the homotopy type $\bigvee_r S^{n-1}$, where r is a number of critical points of f.

11.19 An example of a family of elliptic curves in $\mathbb{P}^2$ with the monodromy, which is not semisimple:

$$z_0 z_2^2 = 4z_1^3 + (t-3)z_0^2 z_1 + (s-1)z_0^3\,.$$

It has matrix in a suitable basis: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. [Barth-Hulek-Van de Ven, Compact Complex Surfaces V §8]

12 New year lecture

12.1 $\partial\bar{\partial}$–lemma: if $\partial\alpha = 0$ i $\alpha = \bar{\partial}\beta$, then there exists γ such that $\alpha = \partial\bar{\partial}\gamma$.

Proof: Assume that $\alpha \in A^{p,q}$, $\partial\alpha = 0$, $\alpha = \bar{\partial}\beta$:

$$\alpha = \bar{\partial}\beta = \bar{\partial}(\beta_h + \partial\gamma + \partial^*\delta) = \bar{\partial}\partial\gamma + \bar{\partial}\partial^*\delta$$

We show that $\bar{\partial}\partial^*\delta = 0$. Since

$$0 = \partial\alpha = \partial\bar{\partial}\partial^*\delta\,,$$

$$0 = (\partial\bar{\partial}\partial^*\delta, \bar{\partial}\delta) = (\bar{\partial}\partial^*\delta, \partial^*\bar{\partial}\delta) = -||\bar{\partial}\partial^*\delta||^2$$

12.2 Parallel argument dd^c–lemma. Let

$$d^c = I^{-1}dI = i\partial - i\bar{\partial}$$

$$d^c = -i(\partial - \bar{\partial}), \quad (d^c)^* = -*d^c*$$

(we easily check that $[d^c, d^*] = 0$).

- Using Hodge decomposition for d^c (obtained by conjugating the decomposition for d).

$$d^c\alpha = 0 \quad \text{and} \quad \alpha = d\beta \quad \text{then} \quad \exists\gamma\ \alpha = dd^c\gamma\,.$$

- In the proof of $\partial\bar{\partial}$-lemma replace ∂ by d and $\bar{\partial}$ by d^c.

Formality

12.3 CDGA's quasiisomorphism and formality [Huybrechts §3A]

12.4 Theorem: Kähler manifolds have formal $A^\bullet(M)$.
Let $\mathcal{Z}^\bullet_{d^c}(M) = ker(d^c)$. We have a zig-zag map

$$A^\bullet(M) \leftarrow \mathcal{Z}^\bullet_{d^c}(M) \rightarrow H^\bullet_{d^c}(M)\,.$$

12.5 – The inclusion $\mathcal{Z}^\bullet_{d^c}(M) \subset \mathcal{A}^\bullet(M)$ is a quasiisomorphism:
epi, since $\mathcal{H} \subset \mathcal{Z}^\bullet_{d^c}(M)$,
mono: if $d^c\alpha = 0$, $\alpha = d\beta$ then $\alpha = dd^c\gamma$ by dd^c-lemma (the form $\beta' = d^c\gamma \in \mathcal{Z}^\bullet$)
– The differential in $(H^*_{d^c}(M), d)$ is trivial:
Let $[\beta] \in H^*_{d^c}(M)$ (by definition $d^c\beta = 0$). Let $\alpha = d\beta$. Also $d^c\alpha = 0$. Hence there exists γ such that $\alpha = d^c d\gamma$. So $[\alpha] = 0$.
– Let $H^*_{d^c}(M) = ker(d_c)/im(d_c)$, the projection $(\mathcal{Z}^\bullet_{d^c}(M), d) \rightarrow (H^*_{d^c}(M), d)$ is a quasiisomorphism.
epi: since there are harmonic representatives,
mono: suppose $d^c\alpha = 0$, $d\alpha = 0$. The condition $[\alpha] = 0 \in H^*_{d^c}(M)$ means exactly that $\alpha = d^c\beta$. Then $\alpha = dd^c\gamma$. Hence $[\alpha] = 0 \in H^*(\mathbb{Z}_{d^c}(M), d)$.

12.6 Corollary: for Kähler manifolds the hidden cohomological structures (like Massey products) vanish.
(See the triple Massey product $([\alpha], [\beta], [\gamma])$ — def in Huybrechts 3.A.31)
♠

Vector bundles and Chern classes

12.7 Let $Vect^1(X)$ denotes the set of isomorphism classes of (topological) complex linear bundles X. There are various definitions of the first Chern class.

- Axiomatic definition, see Milnor-Stasheff:

a)	$c_1 : Vect^1(-) \rightarrow H^2(-, \mathbb{Z})$ is a natural transformation of functors $Top \rightarrow GrAb$
b)	$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$
c)	$c_1(\mathcal{O}_{\mathbb{P}^1}(1)) =$ the distinguished generator $[pt] \in H^2(\mathbb{P}^1)$

- The identification $Vect^1(X) = [X, \mathbb{P}^\infty] = [X, K(\mathbb{Z}, 2)] = H^2(X; \mathbb{Z})$,
where $[-,-]$ denotes the set of homotopy classes of maps.
- Via the differential in the long exact sequence of cohomologies associated to the short exact sequence of sheaves

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow C(X, \mathbb{C}) \overset{\exp}{\rightarrow} C(X, \mathbb{C}^*) \rightarrow 0$$

$$0 = H^1(X, C(X, \mathbb{C})) \rightarrow H^1(X, C(X, \mathbb{C}^*)) \overset{\simeq}{\rightarrow} H^2(X; 2\pi i\mathbb{Z}) \rightarrow H^2(X, C(X, \mathbb{C})) = 0$$

Note, that we have an identification $Vect^1(X) = \check{H}^1(X, H^1(X, C(X, \mathbb{C}^*))$.
- via the obstruction theory: the obstruction to the existence of a nonzero section belongs to

$$H^2(X; \pi_1(\mathbb{C}^*)) \simeq H^2(X; \mathbb{Z})$$

- $c_1(L) = [zero\ section] \in H^2(L) \simeq H^2(X)$
- a definition via connection (when X is a manifold) $\frac{i}{2\pi}[F_\nabla] = \frac{i}{2\pi}[\partial\bar{\partial}\log h] \in H^2(X; \mathbb{C})$, which we discuss below following [Huybrechs §4].

12.8 Every complex vector bundle admits a Hermitian structure (i.e. a hermitian product on fibers, continously/smoothly changing from a fiber to fiber).

12.9 Connection on a vector bundle over C^∞-manifold:
- a linear map $\nabla : C^\infty(X; E) \rightarrow C^\infty(X; T^*_X \otimes E) =: A^1_X(E)$ satisfying the Leibniz rule

$$\nabla(fs) = df \otimes (s) + fs\,.$$

12.10 Let ∇ and ∇' be two connections. The difference $\nabla - \nabla'$ is A^1_X linear. [Huybrechts 4.2.3]

- Locally, every connection is of the form $\nabla = d + a$, where $a \in A^1(X, End(E))$.
- If ∇ is a connection and $a \in A^1(X, End(E)) = C^\infty(X, T^*X \otimes End(E))$, then $\nabla + a$ is a connection.
- Affine combination of connections $t\nabla_1 + (1-t)\nabla_2$ is a connection.
- Applying a partition of unity associated to the trivializing atlas of E we glue together local connections and obtain a global one.
- The space of connections is isomorphic to $A^1(X, End(E))$. (But no connection is distinguished.)

13 Connections concordant with structures, Chern classes

13.1 Suppose E is a hermitian bundle. A connection is Hermitian if

$$d < s_1, s_2 > = < \nabla s_1, s_2 > + < s_1, \nabla s_2 > .$$

(again the Leibniz formula) [Huybrechts 4.2.9]

13.2 Let V be a Hermitian vector space. By $End(V,h)$ denote the endomorphism a satisfying

$$< a(v), w > + < v, a(w) > = 0 .$$

If $V = \mathbb{C}^n$ with the standard hermitian product, then $End(V,h) = \mathfrak{u}_n = \{A \in M_{n\times n}(\mathbb{C}) \mid A + \bar{A}^T = 0\}$.

- For a Hermitian vector bundle $End(V,h)$ is a real vector bundle of the dimension $= \dim(\mathfrak{u}_{\mathrm{rk}\,E})$.
- As before we prove that the space of Hermitian connection is a real vector space isomorphic to $A^1(X, End(E,h))$. (But no connection is distinguished.)

13.3 If $\mathrm{rk}E = 1$**.** Then $End(E,h) \simeq \mathbb{R}$.

13.4 Suppose X is a complex manifold, E is a holomorphic bundle (the transition functions are holomorphic). Let $A^k(X,E) = \Gamma(A^k_X \otimes E)$, $A^k(X,E) = \bigoplus_{p+q=k} A^{p,q}(X,E)$. The operator $\bar\partial$ is well defined

$$\bar\partial_E : A^{p,q}(X,E) \to A^{p,q+1}(X,E) .$$

Warning: the operator ∂ does not commute with the transition functions. Thus ∂_E is not defined, unless the transition functions are locally constant.

13.5 The connection decomposes into components $\nabla^{10} + \nabla^{01}$. We say that ∇ is compatible with the complex structure if $\nabla^{0,1} = \bar\partial_E$.

13.6 The space of connections compatible with complex structure is isomorphic to $A^{1,0}(X, End(E))$.

13.7 Theorem [Huybrechts 4.2.14]: For a Hermitian holomorphic bundle there exists exactly one connection compatible with the complex structure.

- In local coordinates: let H be the matrix of the Hermitian product, $\nabla = d + A$, (we identify A locally with a matrix, we call it connection matrix)

$$A \in M_{n\times n}(A^{1,0}(X)) , \qquad H \in M_{n\times n}(C^\infty(X)) , \qquad \bar H = H^T , \qquad n = \mathrm{rk}(E).$$

- the Hermitian condition reads

$$dH = A^T H + H\overline{A} .$$

Hence

$$\partial H = A^T H ,$$

so

$$A = \bar H^{-1} \partial(\bar H) .$$

- If $n = 1$, $H = [h]$. then $a = \partial \log(h)$.

13.8 We extend the connection using Leibniz formula to obtain the operator $\nabla_E : A^k(X,E) \to A^{k+1}(X,E)$.

13.9 Connection on E induces a connection on $End(E)$

$$(\nabla f)(s) := \nabla(f(s)) - f\nabla s = [\nabla, f]s\,.$$

13.10 Theorem: The curvature $F_\nabla = \nabla^2 : A^0(E) \to A^2(E)$ is $A^0(X)$-linear, hence it defines a section o the bundle $\Lambda^2 T^*X \otimes End(E))$.

13.11 Locally in the matrix notation

$$F_\nabla = dA + A \wedge A \in M_{n\times n}(A^2(X)).$$

13.12 For a line bundle $E = L = \mathbb{C} \times X$: we have $End(L) = \mathbb{C}$ and $A \wedge A = 0$ (since A is a 1×1 matrix). Then $H = [h]$, $h : X \to \mathbb{R}$

$$F_\nabla = dA = d\partial \log(h) = \bar\partial\partial \log(h)\,.$$

13.13 Example $L = \mathcal{O}(-1)$ on $\mathbb{P}^n$, i.e. the tautological bundle, $L \subset \mathbb{C}^{n+1} \times \mathbb{P}^n$ has the induced Hermitian structure from the trivial bundle $\mathbb{C}^{n+1}$:

$$F_\nabla = d\partial \log(||s||^2) = -\partial\bar\partial \log(||s||^2),$$

where v is any section of L,

$$\frac{i}{2\pi} F_\nabla = -\omega_{FS}\,.$$

- For example on the chart $\{z_0 \neq 0\} \simeq \mathbb{C}^n$ there is a section

$$s([1 : z_1 : \dots : z_n]) = (1 : z_1 : \dots : z_n)\,,$$

hence

$$F_\nabla = -\partial\bar\partial \log(1 + ||z||^2)\,.$$

- Note: $c_1(\mathcal{O}(-1)) = -[\omega_{FS}]$.

Chern classes

13.14 Bianchi identity:

$$\nabla(F_\nabla) = 0 \in A^3(X, End(E)).$$

Locally for $\nabla = d + A$ we have

$$dF_\nabla = d(dA + A \wedge A) = dA \wedge A - A \wedge dA = [dA, A] = [F_\nabla, A]\,.$$

Moreover

$$\nabla(F_\nabla) = dF_\nabla + [A, F_\nabla] = 0\,.$$

13.15 Theorem: If $E = L$ is a line bundle on $\mathbb{C}^\infty$-manifold, then the class $c(L) := \frac{i}{2\pi}[F_\nabla] \in H^2(X;\mathbb{C})$ does not depend on the choice of the connection.

- Proof: for two connections ∇_0, ∇_1 define a connection on $X \times \mathbb{R}$ by the formula $\tilde\nabla = t\nabla_1 + (1-t)\nabla_0$. Inclusions $i_0, i_1 : X \to X \times \mathbb{R}$ on submanifolds $t = 0$ and $t = 1$ are homotopic, so $[F_{\nabla_1}] = i_1^*[F_{\tilde\nabla}] = i_0^*[F_{\tilde\nabla}] = [F_{\nabla_0}]$.

13.16 The class $c(L)$ satisfies the axioms of $c_1(L)$.

Chern classes of the bundles of arbitrary rank

Huybrechts §4.4

13.17 Theorem: For any polynomial map $P : End(\mathbb{C}^n) \to \mathbb{C}$ which is invariant with respect to conjugation the form $P(\nabla_E^2) \in A^{2\deg(P)}(X)$ is closed.

• Lemma (see Milnor-Stasheff, Appendix C, p.297) For $X = (x_{ij})_{i,j}$ define the matrix $P'(X) = (\frac{\partial P}{\partial x_{ji}})_{i,j}$ (note, that the indices i, j are exchanged). We have:
(1) $dP(X) = tr(P'(X) \cdot dX)$.
(2) if P is Ad-invariant, then the matrices $P'(X)$ and X commute.

Proof:
(1) easy
(2) $P((I + tE_{ij})X) = P(X(I + tE_{ij}))$, hence
$\sum_k x_{i,k} \frac{\partial P}{\partial x_{jk}} = \sum_k \frac{\partial P}{\partial x_{ki}} x_{k,j}$

• Proof of theorem:

$$dP(F_\nabla) = tr(P'(F_\nabla)dF_\nabla) = tr(P'(F_\nabla)[F_\nabla, A]) = tr(P'(F_\nabla) \wedge F_\nabla \wedge A - P'(F_\nabla) \wedge A \wedge F_\nabla) =$$

$$= tr(F_\nabla \wedge (P'(F_\nabla) \wedge A) - (P'(F_\nabla) \wedge A) \wedge F_\nabla) = tr([F_\nabla, P'(F_\nabla) \wedge A]) = 0\,.$$

13.18 Remark: the map

$$\mathbb{C}[M_{n\times n}(\mathbb{C})]^{GL_n} \to \mathbb{C}[\text{diagonal matrices}]^{\Sigma_n} = \mathbb{C}[\sigma_1, \sigma_2, \ldots, \sigma_n]$$

is an isomorphism. If P is Ad-invariant, then it can be expressed by the coefficients of the characteristic polynomial. Equivalently, $P(A)$ is a symmetric function in eigenvalues of A.

13.19 The 2-form $P(F_\nabla)$ defines a cohomology class, which does not depend on the connection. For $P = \left(\frac{i}{2\pi}\right)^k \sigma_k$, ($(-1)^k\sigma_k$ is (rk$E - k$)-th coefficient of the characteristic polynomial) the resulting forms represent the Chern classes.

• Verification of axioms:
1) Functoriality (the connection can be pulled back)
2) Whitney formula

$$c(E_1 \oplus E_2) = c(E_1)c(E_2)\,,$$

where $c(E) = 1 + c_1(E) + \cdots + c_{rk(E)}(E)$. By splitting principle we can assume that both E_1 and E_2 are sums of line bundles and the connection is of the product form.
3) Normalization $c_1(\mathcal{O}(-1)) = -[\omega_{FS}]$, by the definition of the Fubini-Study form.

13.20 Corollary: The differential forms obtained by the above constructions are integral (i.e. come from $H^*(X;\mathbb{Z})$.

13.21 Generalities about characteristic classes:

• Let

$$Vect^n(X) = \{\text{isomorphism classes of } n\text{-dimensional complex vector bundles over } X\}$$

• Def: a characteristic class on n-dimensional bundle is a transformation of functors $hTop \to Sets$

$$Vect^n(-) \quad \longrightarrow \quad H^*(-)\,.$$

Since $Vect^n(-)$ is representable,

$$Vect^n(X) = \{\text{homotopy classes } f : X \to Grass_n(\mathbb{C}^\infty)\}$$

for finite CW-complexes, by Yoneda lemma

$$\{\text{Characteristic classes of } n\text{-bundles}\} = H^*(Grass_n(\mathbb{C}^\infty)) = \mathbb{Z}[c_1, c_2, \ldots, c_n]\,.$$

• More generally, for a compact Lie group (or a reductive algebraic group) G, for cohomology with coefficients in $\mathbb{C}$:
Let $Bun^G(X)$ be the set of isomorphism classes of G-bundles over X. This functor $hTop \to Set$ is representable by BG, thus

$$\{\text{Characteristic classes of } G\text{-bundles}\}_{\mathbb{C}} = H^*(BG;\mathbb{C}) = \mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W .$$

by Borel theorem. Here $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{t}$ the Lie algebra of the maximal torus, and $W = NT/T$ is the Weyl group.

• For $G = GL_n$ we have

$$\mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[t_1, t_2, \dots t_n]^{\Sigma_n} ,$$

the ring of symmetric functions in n variables.

14

14.1 Suppose X is complex manifold, L a holomorphic line bundle with a Hermitian structure. Then $F_\nabla = \bar\partial\partial \log h$ is a (1,1)-form.

• If X is Kähler manifold, then $c_1(L) \in H^{1,1}(X) \cap image(H^2(X;\mathbb{Z}))$.

14.2 Theorem: Let X be a Kähler manifold, E a holomorphic bundle, then $c_k(E) \in H^{2k}(X;\mathbb{C})$ is represented by a (k,k)-form.

• Splitting principle: Let $p : Fl(E) \to X$ be a bundle of flag spaces over X. It is constructed as follows: Let $P \to X$ be the associated $GL_n(\mathbb{C})$-principal bundle (such that $E = P \times_{GL_n(\mathbb{C})} \mathbb{C}^n$). Then

$$Fl(E) = P \times_{GL_n(\mathbb{C})} Fl(n) = P \times_{GL_n(\mathbb{C})} GL_n(\mathbb{C})/B_n = P/B_n .$$

Here B_n is the group of upper-triangular matrices. The fiber over $x \in X$ is equal to $Fl(E_x) \simeq Fl(n)$ — the flag variety which parametrizes $0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n = E_x$ $\dim(V_i) = i$.

Let $L_i = V_i/V_{i-1}$. The Hermitian structure defines an isomorphism $V_i = L_i \oplus V_{i-1}$. (Note: This isomorphism is not holomorphic.)

Topologically $p^*E = \bigoplus_{i=1}^{\mathrm{rk}E} L_i$. The Chern classes are topological invariants, hence

$$c_\bullet(p^*E) = \prod_{i=1}^{\mathrm{rk}E}(1 + c_1(L_i))$$

thus $c_k(p^*E)$ is of the type (k,k).

Fact: $p^* : H^*(X) \hookrightarrow H^*(Fl(E))$ is a monomorphism. Moreover it preserves types. Conclusion: $c_k(E)$ is of the type (k,k).

14.3 Huybrechts 4.2.18: in general one can define Atiyah class $A(E) \in H^1(X;\Omega^1_X \otimes End(E))$, which agree with $\frac{1}{2\pi i}F_\nabla \in A^2(X; End(E))$.

14.4 Chern character. Let $P \in \mathbb{C}[[M_{n\times n}]]$ be given by the formula:

$$P(B) = \sum_{k=0}^{\infty} \frac{tr(B^k)}{k!} ,$$

where[1] $B = \frac{1}{2\pi i}F_\nabla$. In terms of symmetric functions

$$P(t_1, t_2, \dots, t_n) = \sum_{k=0}^{\infty}\sum_{i=1}^{n} \frac{t_i^k}{k!} = \sum_{k=0}^{\infty} e^{t_i} .$$

The resulting characteristic class is denoted by $ch(E)$.

• Properties:
$ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$
$ch(E_1 \otimes E_2) = ch(E_1) \cup ch(E_2)$
The second identity follows from $e^{a+b} = e^a\, e^b$.

[1] It is convenient to divide by the factor $2\pi i$.

14.5 In general having a formal power series $f[[x]]$ we define an additive characteristic class satisfying:

- $a_f(L) = f(c_1(L))$ for a line bundle L
- $a_f(E_1 \oplus E_2) = a_f(E_1) + f(E_2)$

14.6 Example: if $f[[x]] = e^x$, then $a_f(E) = ch(E)$.

- To express the homogeneous components of $ch(E)$ assume that E is a sum of line bundles L_i, let $x_i = c_1(L_i)$
- $ch(E)_{(0)} = \mathrm{rk} E$
- $ch(E)_{(1)} = x_1 + x_2 + \cdots + x_n = c_1(E)$
- $ch(E)_{(2)} = \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_n^2) = \frac{1}{2}(x_1 + x_2 + \cdots + x_n)^2 - \sum x_i x_j = \frac{1}{2}c_1^2(E) - c_2(E)$

14.7 For a formal power series $f[[x]]$ we define a multiplicative characteristic class satisfying:

- $m_f(L) = f(c_1(L))$ for a line bundle L
- $m_f(E_1 \oplus E_2) = m_f(E_1) \cup m_f(E_2)$
- Example: if $f[[x]] = 1 + x$, then $m_f(E) = c_\bullet(E)$.

14.8 Todd class: Let

$$f[[x]] = \frac{x}{1 - e^{-x}} = \frac{x}{x - x^2/2 + x^3/6 \dots} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} + \dots$$

- $td(E)$ is associated to the function on matrices $B \mapsto \frac{\det(B)}{\det(id - e^{-B})}$.
- $td(E)_{(0)} = 1$
- $td(E)_{(1)} = \frac{1}{2}c_1(E)$
- $td(E)_{(2)} = \frac{x_1^2}{12} + \frac{x_1 x_2}{4} + \frac{x_2^2}{12} = \frac{1}{12}(c_1^2(E) + c_2(E))$ (it is enough to perform computation for $\mathrm{rk} E = 2$)

14.9 Hirzebruch class: Let

$$f_y(x) = x\frac{1 + ye^{-x}}{1 - e^{-x}} = (1 + y)\frac{x}{1 - e^{-x}} - xy = (1 + y) + \frac{1}{2}(1 - y)x + \frac{1 + y}{12}x^2 - \frac{1 + y}{720}x^4 + \frac{1 + y}{30240}x^6 + \dots$$

Here y is a parameter or a free variable.

- Exercise by Hirzebruch-Riemann-Roch (Huybrechts Cor. 5.1.4)

$$\int_X \text{Hirzebruch class} = \sum_{p=0}^{n} \chi(X, \Omega_X^p) y^p = \sum_{p,q} h^{p,q} y^p \,.$$

- For $y = -1$ we obtain $\chi_{top}(X)$ the topological Euler characteristic,
  For $y = 0$ we obtain $Td(X) = \chi(X, \mathcal{O}_X)$ Todd genus
  For $y = 1$ we obtain the signature.

14.10 Hirzebruch-Riemann-Roch (Huybrechts 5.1.1) Let E be a holomorphic vector bundle on a compact manifold X. Then

$$\chi(X; E) = \int_X td(TX) \cup ch(E) \,.$$

14.11 Example X = hypersurface of degree d in $\mathbb{P}^n$, $E = \mathcal{O}(k)$:

$$td(TX) = td(T\mathbb{P}^n)/td(\nu_X) = \left(\frac{h}{1 - e^{-h}}\right)^{n+1} \frac{1 - e^{-dh}}{dh} = \frac{h^n}{d} \frac{1 - e^{-dh}}{(1 - e^{-h})^{n+1}}$$

$$\int_X td(TX) \cup ch(L) = \int_{\mathbb{P}^n} dh \cup td(TX) \cup ch(L) = \left[h^{n-1} \frac{1 - e^{-dh}}{(1 - e^{-h})^{n+1}} e^{kh} \right]_{\text{coeff of } h^n}$$

$$= Res_{h=0} \frac{(1 - e^{-dh})e^{kh}}{(1 - e^{-h})^{n+1}} \overset{u=1-e^{-h}}{=} Res_{u=0} \frac{(1 - (1 - u)^d)(1 - u)^{-k}}{(1 - u)u^{n+1}}$$

(since $dr = du/(1 - u)$).

14.12 If $\dim X = 1$, suppose L is of degree d, i.e. $c_1(L) = d[pt]$ then

$$\chi(X; L) = [(1 + c_1(TX)/2)(1 + c_1(L))]_{(1)} = deg(c_1(TX)/2 + c_1(L)) = \frac{1}{2}\chi_{top(X)} + d = 1 - genus + d\,.$$

14.13 If $\dim X = 2$, $L = \mathcal{O}(D)$, then $c_1(L) = [D]$. Let $c_i = c_i(TX)$

$$\chi(X; L) = [(1 + c_1/2 + \frac{1}{12}(c_1^2 + c_2)(1 + D + D^2/2)]_{(2)} = \deg(\frac{1}{12}(c_1^2 + c_2) + \frac{c_1 \cup D + D^2}{2}$$

$$\chi(X; \mathcal{O}_X) + \frac{c_1 \cdot D + D^2}{2}$$

Using a common notation in algebraic geometry $c_1 = -K_X$

$$\chi(X; L) = \chi(X; \mathcal{O}_X) + \frac{(D - K_X) \cdot D}{2}\,.$$

or with $p_a = -\dim H^1(X; \mathcal{O}_X) = \chi(X; \mathcal{O}_X) - 1$ (arithmetic genus)

$$\chi(X; L) = 1 + p_a + \frac{(D - K_X) \cdot D}{2}\,.$$

Positive line bundles

14.14 We say that $\omega \in A^{1,1}(X) \cap A^2(X; \mathbb{R}) \subset A^2(X; \mathbb{C})$ is positive, if there exists a hermitian product such that ω is equal to minus its imaginary part $\langle\langle x, y\rangle\rangle = \langle x, y\rangle - i\omega(x, y)$. If locally in some coordinates the hermitian product is given by a matrix $H = [h_{i,j}]$, then

$$\omega = \frac{i}{2} \sum h_{ij} dz_i \wedge \bar{d}z_j\,.$$

14.15 A linear bundle is positive if it admits a connection ∇ such that $\frac{i}{2\pi} F_\nabla$ is a positive (1,1)-form.

14.16 Example of positive bundles: $\mathcal{O}_{\mathbb{P}^n}(k)$, $k > 0$.

14.17 Kodaira-Nakano Vanishing theorem: If $L = \mathcal{O}(D)$ is positive, then[2] $H^i(X; K_X + D) = 0$ for $i > 0$, where $K_X + D$ denotes the sheaf of homomorphic sections of the bundle $\Omega_X^{\dim X} \otimes L$.

14.18 If L is generated by global holomorphic sections (we say ,,globally generated") then we have a natural map

$$\Phi : X \to \mathbb{P}(H^0(X; L)^*)\,.$$

• Suppose $U \subset X$ is a set on which L is trivial, and we can identify the value $s(x)$ with a complex number. For $x \in X$ define a functional on the space of the global sections $A^0(X; L) = H^0(X; L)$

$$\Phi(x) : s \mapsto s(x) \in L_x \simeq \mathbb{C}, .$$

• Then $L = \phi^*(\mathcal{O}(1))$ (by tautological identification $H^0(\mathbb{P}(V^*); \mathcal{O}(1)) \simeq V$). Hence L is ,,nonnegative", i.e. L admits a connection such that the associated Hermitian form is nonnegative semi-definite.

• This property is preserved by pull-backs.

14.19 Kodaira embedding theorem: If a bundle L is positive, then for $\nu >> 0$ it is generated by global sections and the natural map $X \to \mathbb{P}(H^0(X; L^\nu)^*)$ is an embedding. (We only assume that, X is a compact analytic complex manifold, and as a corollary from GAGA we obtain that X is algebraic.)

[2] Notation $H^i(X; K_X + D) = H^i(X; \Omega_X^{\dim X} \otimes \mathcal{O}(D))$.

14.20 If (X, ω) is a Kähler manifold and $[\omega]$ is of the form $\lambda[\omega']$ for $\lambda \in \mathbb{R}_+$, $[\omega'] \in H^*(X; \mathbb{Z})$, then X embedds into a projective space.

- Proof. It is enough to show that $[\omega'] = c_1(L)$ for a holomorphic line bundle.
- The short exact sequence

$$\underline{\mathbb{Z}} \to \underline{\mathbb{C}} \to \mathcal{O}_X$$

of sheaves induces a long exact sequence

$$\to Pic(X) = H^1(X, \mathcal{O}_X^*) \stackrel{c_1}{\to} H^2(X; \mathbb{Z}) \stackrel{\iota}{\to} H^2(X; \mathcal{O}_X) = H^{0,2}(X) \to \ .$$

Assume that $[\omega']$ is integral. We will show that $[\omega']$ lies in the image of c_1, or equivalently it belongs to the kernel of the map ι. The map ι factors as follows

$$H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) \twoheadrightarrow H^{0,2}(X)\,.$$

The second map is induced by the map of shaves $\underline{\mathbb{C}} \to \mathcal{O}_X$ covered by a map of resolutions $A_X^{p,q} \twoheadrightarrow A_X^{0,p}$. The classes of the type $(1,1)$ lie in the kernel.