Clique complexes of cycle powers

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DIMAP Workshop on Combinatorics and Graph Theory
April 2011

Introduction

- Graphs

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- Simplicial complexes
  - V, $\mathcal{F} \subset 2^V$ – faces $\quad B \subset A \in \mathcal{F}$ then $B \in \mathcal{F}$

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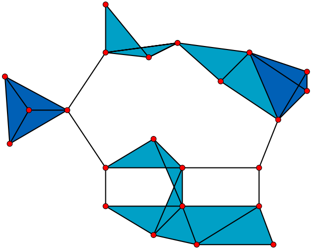

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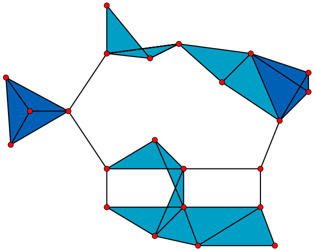

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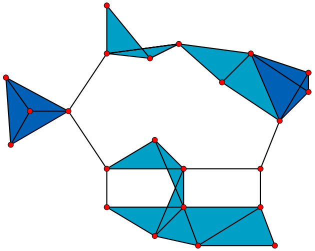
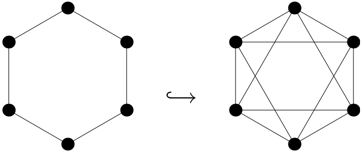


- $\mathrm{Cl}(G)$: $\mathcal{F} =$ cliques of G
- $\mathrm{Ind}(G)$: $\mathcal{F} =$ independent sets of G

Graph powers vs. clique complexes

- The r-th power G^r has edges $uv \in E(G^r)$ whenever

$$\mathrm{dist}_G(u, v) \leq r$$



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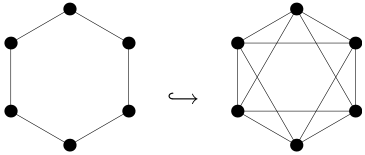

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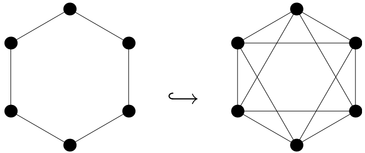

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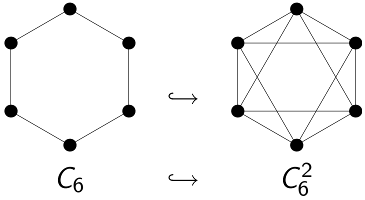

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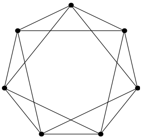
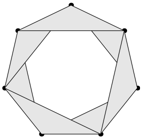
- $\mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2) \hookrightarrow \mathrm{Cl}(G^3) \hookrightarrow \cdots \hookrightarrow \Delta_n$
- $\mathrm{Cl}(G^r)$ is the Vietoris-Rips complex of G

Examples

- $G = C_7^2$





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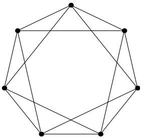

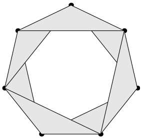

Möbius band $\mathcal{M}$

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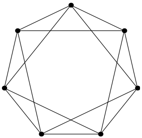

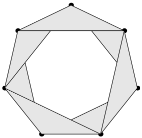

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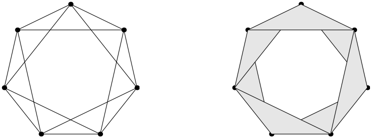

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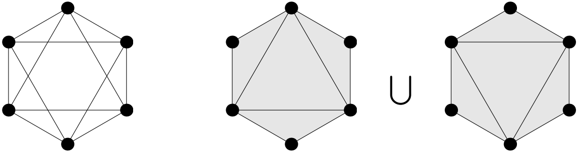

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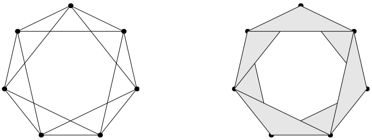

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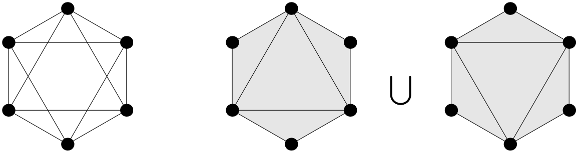

$= S^2$

Universality

Theorem (A.)

For any r and any simplicial complex K there is a graph G with a homotopy equivalence

$$\mathrm{Cl}(G^r) \simeq K$$

Stability (for $r = 2$)

When is the inclusion $\mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2)$ a homotopy equivalence?

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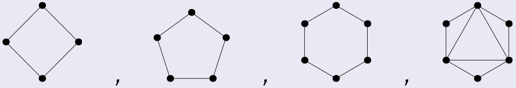
The above condition is satisfied when G is



– free

(it does not have either of the four graphs as an induced subgraph.)

Cycles

Cycles

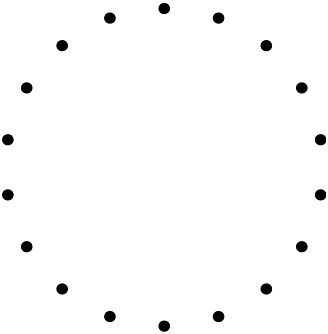
$$\mathrm{Cl}(C_n^r)$$

$r =$	0	1	2	3	4	5	6	7	8	9	10
C_8	$\bigvee^7 S^0$	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$
C_9	$\bigvee^8 S^0$	S^1	S^1	$\bigvee^2 S^2$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
C_{10}	$\bigvee^9 S^0$	S^1	S^1	S^1	S^4	$*$	$*$	$*$	$*$	$*$	$*$
C_{11}	$\bigvee^{10} S^0$	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$
C_{12}	$\bigvee^{11} S^0$	S^1	S^1	S^1	$\bigvee^3 S^2$	S^5	$*$	$*$	$*$	$*$	$*$
C_{13}	$\bigvee^{12} S^0$	S^1	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$
C_{14}	$\bigvee^{13} S^0$	S^1	S^1	S^1	S^1	S^3	S^6	$*$	$*$	$*$	$*$
C_{15}	$\bigvee^{14} S^0$	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$	$\bigvee^2 S^4$	$*$	$*$	$*$	$*$
C_{16}	$\bigvee^{15} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^7	$*$	$*$	$*$
C_{17}	$\bigvee^{16} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$	$*$
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$
C_{19}	$\bigvee^{18} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$
C_{20}	$\bigvee^{19} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^3 S^4$	S^9	$*$
C_{21}	$\bigvee^{20} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^6 S^2$	S^3	$\bigvee^2 S^6$	$*$
C_{22}	$\bigvee^{21} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	S^{10}
C_{23}	$\bigvee^{22} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^7
C_{24}	$\bigvee^{23} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^7 S^2$	S^3	S^5
C_{25}	$\bigvee^{24} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^4 S$

Cycles - Evolution

$$(C_{18})^0$$

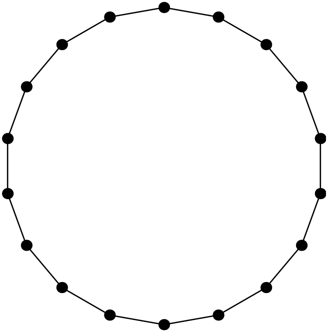
$r =$	0	1	2	3	4	5	6	7	8	9	10
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$



Cycles - Evolution

$$(C_{18})^1 \to S^1$$

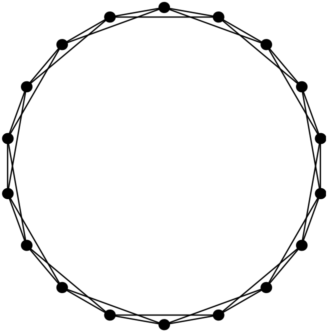
$r =$	0	1	2	3	4	5	6	7	8	9	10
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	*	*



Cycles - Evolution

$$(C_{18})^2 \to S^1$$

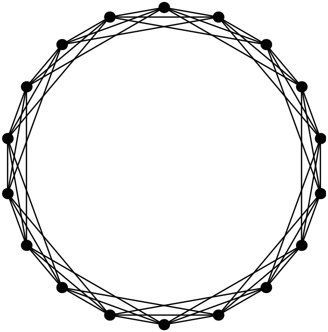
$r =$	0	1	2	3	4	5	6	7	8	9	10
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$



Cycles - Evolution

$$(C_{18})^3 \longrightarrow S^1$$

$r =$	0	1	2	3	4	5	6	7	8	9	10
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$



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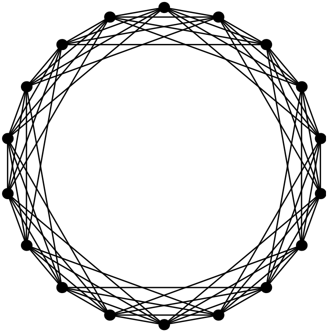
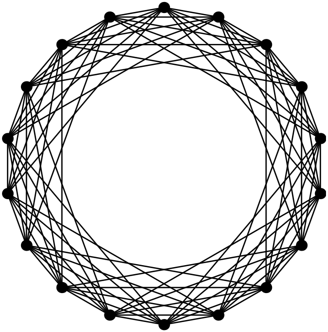
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$



Cycles - Evolution

$$(C_{18})^5 \to S^1$$

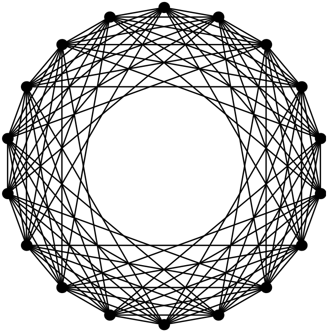
$r =$	0	1	2	3	4	5	6	7	8	9	10
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$



Cycles - Evolution

$$(C_{18})^6 \to \bigvee^5 S^2$$

$r =$	0	1	2	3	4	5	6	7	8	9	10
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$



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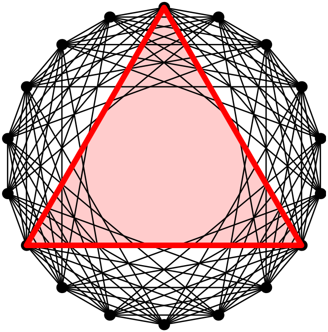
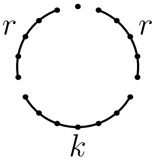


Cycles and their complements

With the parameter

$$k = n - 2r - 1$$



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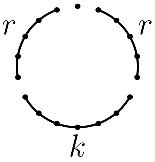
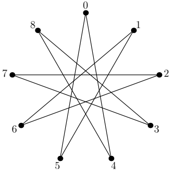
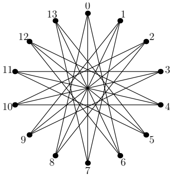


consider the complements $T_{n,k} := \overline{C_n^r}$ known as the *circular complete graphs*.



$T_{9,2} = \overline{C_9^3}$



$T_{14,3} = \overline{C_{14}^5}$

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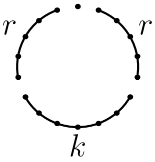

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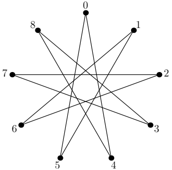

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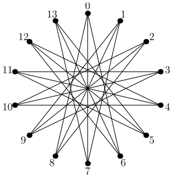

$T_{14,3} = \overline{C_{14}^5}$

$$\mathrm{Cl}(\mathbf{C_n^r}) = \mathrm{Ind}(\mathbf{T_{n,k}})$$

Main result

Theorem (A.)

For reasonable values of n, k the independence complexes of *circular complete graphs* *satisfy*

$$\mathrm{Ind}(T_{n,k}) \simeq \Sigma^2 \mathrm{Ind}(T_{n-2(k+1),k})$$

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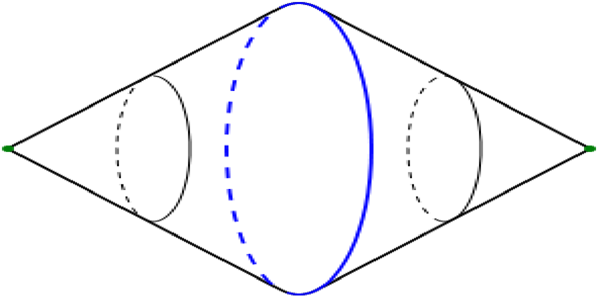

Barmak's method for Ind(triangle-free graphs)

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$$\mathrm{Ind}(T_{n,k}) \simeq \Sigma\, \mathrm{Ind}(S_{n,k}).$$
- The procedure can be repeated for $S_{n,k}$ leading to $T_{n-2(k+1),k}$.

$$\mathrm{Cl}(C_n^r) \simeq \Sigma^2 \mathrm{Cl}(C_{4r-n}^{3r-n}) = \Sigma^2 \mathrm{Cl}(C_{n-2\cdot(2r-n)}^{r-1\cdot(2r-n)})$$

$r =$	0	1	2	3	4	5	6	7	8	9	10
C_8	$\bigvee^7 S^0$	S^1	S^1	S^3	*	*	*	*	*	*	*
C_9	$\bigvee^8 S^0$	S^1	S^1	$\bigvee^2 S^2$	*	*	*	*	*	*	*
C_{10}	$\bigvee^9 S^0$	S^1	S^1	S^1	S^4	*	*	*	*	*	*
C_{11}	$\bigvee^{10} S^0$	S^1	S^1	S^1	S^3	*	*	*	*	*	*
C_{12}	$\bigvee^{11} S^0$	S^1	S^1	S^1	$\bigvee^3 S^2$	S^5	*	*	*	*	*
C_{13}	$\bigvee^{12} S^0$	S^1	S^1	S^1	S^1	S^3	*	*	*	*	*
C_{14}	$\bigvee^{13} S^0$	S^1	S^1	S^1	S^1	S^3	S^6	*	*	*	*
C_{15}	$\bigvee^{14} S^0$	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$	$\bigvee^2 S^4$	*	*	*	*
C_{16}	$\bigvee^{15} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^7	*	*	*
C_{17}	$\bigvee^{16} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^5	*	*	*
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	*	*
C_{19}	$\bigvee^{18} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	*	*
C_{20}	$\bigvee^{19} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^3 S^4$	S^9	*
C_{21}	$\bigvee^{20} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^6 S^2$	S^3	$\bigvee^2 S^6$	*
C_{22}	$\bigvee^{21} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	S^{10}
C_{23}	$\bigvee^{22} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^7
C_{24}	$\bigvee^{23} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^7 S^2$	S^3	S^5
C_{25}	$\bigvee^{24} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^4 S$

Conclusion

$$\mathrm{Cl}(C_n^r) \simeq \Sigma^2\mathrm{Cl}(C_{4r-n}^{3r-n}) = \Sigma^2\mathrm{Cl}(C_{n-2\cdot(2r-n)}^{r-1\cdot(2r-n)})$$

$r =$	0	1	2	3	4	5	6	7	8	9	10
C_8	$\bigvee^7 S^0$	S^1	S^1	S^3	*	*	*	*	*	*	*
C_9	$\bigvee^8 S^0$	S^1	S^1	$\bigvee^2 S^2$	*	*	*	*	*	*	*
C_{10}	$\bigvee^9 S^0$	S^1	S^1	S^1	S^4	*	*	*	*	*	*
C_{11}	$\bigvee^{10} S^0$	S^1	S^1	S^1	S^3	*	*	*	*	*	*
C_{12}	$\bigvee^{11} S^0$	S^1	S^1	S^1	$\bigvee^3 S^2$	S^5	*	*	*	*	*
C_{13}	$\bigvee^{12} S^0$	S^1	S^1	S^1	S^1	S^3	*	*	*	*	*
C_{14}	$\bigvee^{13} S^0$	S^1	S^1	S^1	S^1	S^3	S^6	*	*	*	*
C_{15}	$\bigvee^{14} S^0$	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$	$\bigvee^2 S^4$	*	*	*	*
C_{16}	$\bigvee^{15} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^7	*	*	*
C_{17}	$\bigvee^{16} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^5	*	*	*
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	*	*
C_{19}	$\bigvee^{18} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	*	*
C_{20}	$\bigvee^{19} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^3 S^4$	S^9	*
C_{21}	$\bigvee^{20} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^6 S^2$	S^3	$\bigvee^2 S^6$	*
C_{22}	$\bigvee^{21} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	S^{10}
C_{23}	$\bigvee^{22} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^7
C_{24}	$\bigvee^{23} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^7 S^2$	S^3	S^5
C_{25}	$\bigvee^{24} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^4 S$

Conclusion

$$\mathrm{Cl}(C_n^r) \simeq \Sigma^2\mathrm{Cl}(C_{4r-n}^{3r-n}) = \Sigma^2\mathrm{Cl}(C_{n-2\cdot(2r-n)}^{r-1\cdot(2r-n)})$$

$r =$	0	1	2	3	4	5	6	7	8	9	10
C_8	$\bigvee^7 S^0$	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$
C_9	$\bigvee^8 S^0$	S^1	S^1	$\bigvee^2 S^2$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
C_{10}	$\bigvee^9 S^0$	S^1	S^1	S^1	S^4	$*$	$*$	$*$	$*$	$*$	$*$
C_{11}	$\bigvee^{10} S^0$	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$
C_{12}	$\bigvee^{11} S^0$	S^1	S^1	S^1	$\bigvee^3 S^2$	S^5	$*$	$*$	$*$	$*$	$*$
C_{13}	$\bigvee^{12} S^0$	S^1	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$
C_{14}	$\bigvee^{13} S^0$	S^1	S^1	S^1	S^1	S^3	S^6	$*$	$*$	$*$	$*$
C_{15}	$\bigvee^{14} S^0$	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$	$\bigvee^2 S^4$	$*$	$*$	$*$	$*$
C_{16}	$\bigvee^{15} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^7	$*$	$*$	$*$
C_{17}	$\bigvee^{16} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$	$*$
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$
C_{19}	$\bigvee^{18} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$
C_{20}	$\bigvee^{19} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^3 S^4$	S^9	$*$
C_{21}	$\bigvee^{20} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^6 S^2$	S^3	$\bigvee^2 S^6$	$*$
C_{22}	$\bigvee^{21} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	S^{10}
C_{23}	$\bigvee^{22} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^7
C_{24}	$\bigvee^{23} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^7 S^2$	S^3	S^5
C_{25}	$\bigvee^{24} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^4 S$[illegible]

Conclusion

$$\mathrm{Cl}(C_n^r) \simeq \Sigma^2 \mathrm{Cl}(C_{4r-n}^{3r-n}) = \Sigma^2 \mathrm{Cl}(C_{n-2\cdot(2r-n)}^{r-1\cdot(2r-n)})$$

$r =$	0	1	2	3	4	5	6	7	8	9	10
C_8	$\bigvee^7 S^0$	S^1	S^1	S^3	*	*	*	*	*	*	*
C_9	$\bigvee^8 S^0$	S^1	S^1	$\bigvee^2 S^2$	*	*	*	*	*	*	*
C_{10}	$\bigvee^9 S^0$	S^1	S^1	S^1	S^4	*	*	*	*	*	*
C_{11}	$\bigvee^{10} S^0$	S^1	S^1	S^1	S^3	*	*	*	*	*	*
C_{12}	$\bigvee^{11} S^0$	S^1	S^1	S^1	$\bigvee^3 S^2$	S^5	*	*	*	*	*
C_{13}	$\bigvee^{12} S^0$	S^1	S^1	S^1	S^1	S^3	*	*	*	*	*
C_{14}	$\bigvee^{13} S^0$	S^1	S^1	S^1	S^1	S^3	S^6	*	*	*	*
C_{15}	$\bigvee^{14} S^0$	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$	$\bigvee^2 S^4$	*	*	*	*
C_{16}	$\bigvee^{15} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^7	*	*	*
C_{17}	$\bigvee^{16} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^5	*	*	*
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	*	*
C_{19}	$\bigvee^{18} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	*	*
C_{20}	$\bigvee^{19} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^3 S^4$	S^9	*
C_{21}	$\bigvee^{20} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^6 S^2$	S^3	$\bigvee^2 S^6$	*
C_{22}	$\bigvee^{21} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	S^{10}
C_{23}	$\bigvee^{22} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^7
C_{24}	$\bigvee^{23} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^7 S^2$	S^3	S^5
C_{25}	$\bigvee^{24} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^4 S$