

Algorithmic complexity of finding cross-cycles in flag complexes

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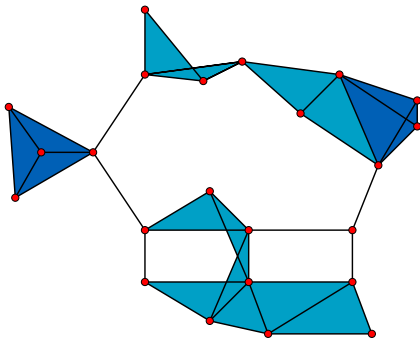


Combinatorial algebraic topology:

- Given an interesting class of combinatorial objects (graphs, posets, lattices, hyperspace arrangements...)
- ... and a construction which assigns to them topological spaces ...
- ... what are the relations between the combinatorial and topological properties?
- Algorithms?

Clique complexes

- If G is a graph, then the **clique complex** $Cl(G)$ is the simplicial complex whose faces are the cliques (complete subgraphs) of G .



Source: Wikipedia

- Geometric example: Vietoris-Rips complexes.
- a.k.a. **flag** complexes

Independence complexes

- If G is a graph, then the **independence complex** $\text{Ind}(G)$ is the simplicial complex whose faces are the independent sets (edge-free sets) of G .



$$\text{Ind}(G) = \text{Cl}(\overline{G}).$$

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- Example: cross-polytope boundaries

$\text{Ind}(e)$



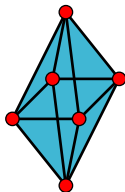
S^0

$\text{Ind}(e \sqcup e)$



$S^1 = S^0 * S^0$

$\text{Ind}(e \sqcup e \sqcup e)$

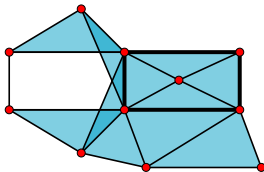


$S^2 = S^0 * S^0 * S^0$

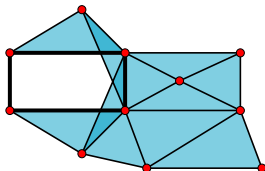
Spherical homology classes

- The homology group $H_k(L)$ “counts k -dim holes” in L .
- The “smallest holes” in flag complexes are given by embedded cross-polytopal spheres:

$$S^{k-1} = \underbrace{S^0 * \dots * S^0}_k \hookrightarrow L \text{ gives an element of } H_{k-1}(L).$$



trivial



non-trivial

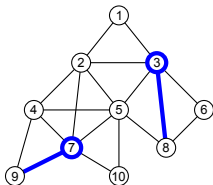
Cross-cycles and matchings

Definition (Cross-cycle, topologically)

A cross-cycle in a flag complex L is a homology class defined by an embedded subcomplex $S^0 * \dots * S^0$, such that at least one face of that subcomplex is a maximal face of L . Such a homology class is non-zero.

Definition (Cross-cycle, combinatorially)

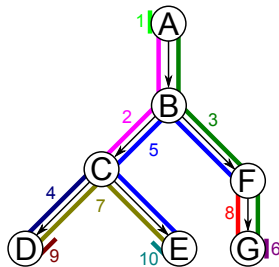
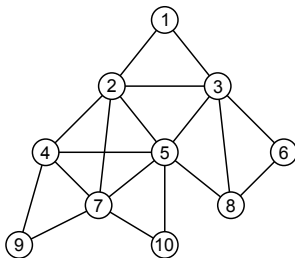
A cross-cycle in $\text{Ind}(G)$ corresponds to an induced matching in G which contains a maximal independent set of G .



Defines an element of $H_1(\text{Ind}(G))$.

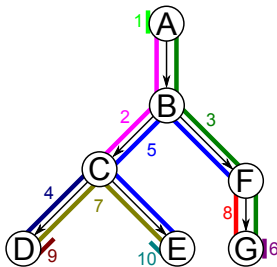
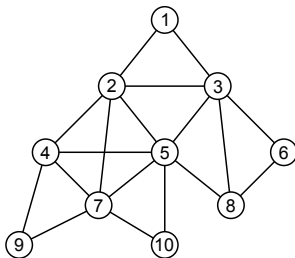
Chordal graphs

- A graph is **chordal** iff it is an intersection graph of subtrees of a fixed tree.



Chordal graphs

- A graph is **chordal** iff it is an intersection graph of subtrees of a fixed tree.



- Extensively studied in algorithmic graph theory.
- Well studied $\text{Ind}(G)$: topologically (Woodroffe, Kawamura, Engström, ...), algebraically (van Tuyl, Engström, Dao).

Theorem (Homology of chordal graphs is combinatorializable)

If G is a chordal graph then the homology group $H_(\text{Ind}(G))$ has a basis consisting of cross-cycles.*

Consequently,

- $H_{k-1}(\text{Ind}(G))$ is non-trivial

if and only if

- G has an induced matching of size k containing a maximal independent set.

Theorem (Poly-time)

The following problem is solvable in polynomial time.

- *Given a chordal graph G , decide if $\text{Ind}(G)$ has trivial homology in all dimensions:*

$$\tilde{H}_i(\text{Ind}(G)) = 0 \text{ for all } i ?$$

Theorem (Hardness)

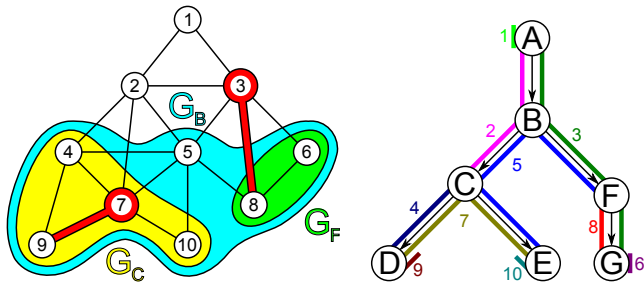
The following problem is NP-complete.

- *Given a chordal graph G and an integer k , decide if*

$$H_k(\text{Ind}(G)) = 0 ?$$

The algorithm for chordal graphs

- A bottom-up recursion in the tree model of G .



- Computes special solutions that can be easily merged over subtrees.
- But no control over size.

Theorem (Homology is hard for flag complexes)

The following problem is NP-hard:

- *Given a graph G and an integer k , decide if*

$$H_k(\text{Cl}(G)) = 0 \quad ?$$

Theorem (Homology is hard for arbitrary simplicial complexes)

The following problem is NP-hard:

- *Given a simplicial complex L , presented as the list of maximal faces (facets), and an integer k , decide if*

$$H_k(L) = 0 \quad ?$$

- Connectivity $\geq k$?

$$H_i(L) = 0 \text{ for } i \leq k.$$

- Homological dimension $\leq k$?

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Thank you!