

*Root and root finding*  
are concepts familiar to most branches of mathematics.

# Square roots and higher roots of graphs

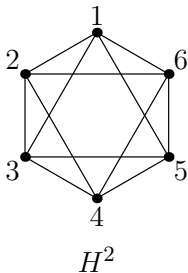
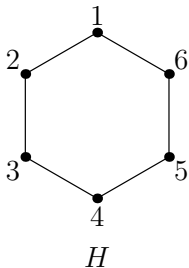
Anna Adamaszek, Michał Adamaszek

DIMAP, University of Warwick, UK

# Powers and roots of graphs

## Definition

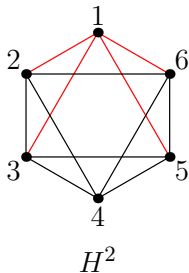
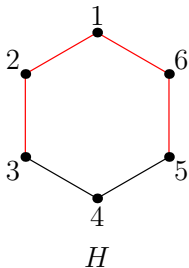
If  $H$  is a graph, its  $r$ -th power  $G = H^r$  is the graph on the same vertex set such that two distinct vertices are adjacent in  $G$  if their distance in  $H$  is at most  $r$ . We call  $H$  the  $r$ -th root of  $G$ .



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# Problems related to graph roots

- Does  $G$  have an  $r$ -th root?
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## Theorem (Motwani, Sudan '94, Le, Nguyen '09)

*It is NP-complete to decide if  $G$  has a square/ $r$ -th root.*

Lin, Skiena '95, Kearney, Corneil '98, Chang, Ko, Lu '06

$r$ -th tree roots can be found in linear time.

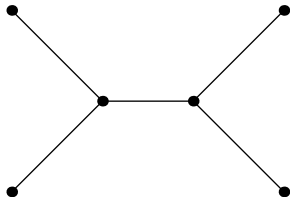
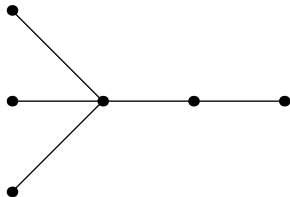


# Tree roots

Lin, Skiena '95, Kearney, Corneil '98, Chang, Ko, Lu '06

$r$ -th tree roots can be found in linear time.

**Warning!** Very non-unique for  $r \geq 3$ !



are both 3-rd roots of the complete graph!

# Large-girth roots

## Definition

The *girth* of a graph is the length of its shortest cycle.

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- Square roots of girth at least 5 ... ????

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## AA, MA '09

- $r$ -th roots of girth at least  $2r + 3$  can be found in poly-time.
- Square roots of girth at least  $r + 2$  are NP-hard to find.

# How it works

**Task.** We know  $B_x = B_r(x)$  for each vertex  $x \in H$ . What can we say about  $H$ ?

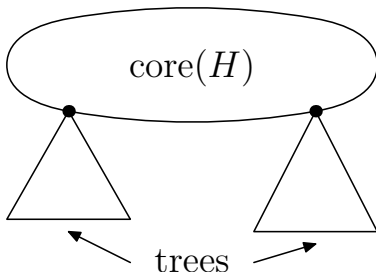
**Assumption.**  $H$  has girth  $\geq 2r + 3$ .

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- Miracle 1.

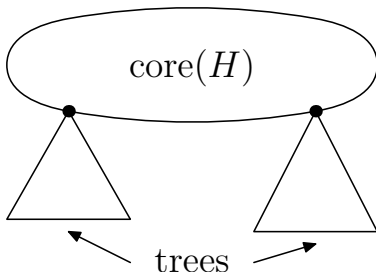


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- Miracle 2. We can compute *all*  $r$ -th roots of the core, and there is linearly many of them.

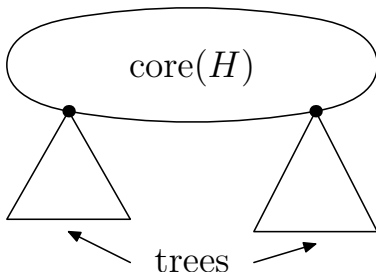


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- Miracle 2. We can compute *all*  $r$ -th roots of the core, and there is linearly many of them.
- Miracle 3. Given each core we can attach the trees in the right places.

# Roots with no leaves

**Task:** We know  $B_x = B_r(x)$  for each vertex  $x \in H$ . What can we say about  $H$ ?

**Assumption:**  $H$  has girth  $\geq 2r + 3$  and no leaves.

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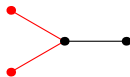


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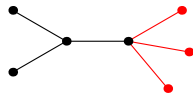
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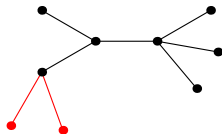
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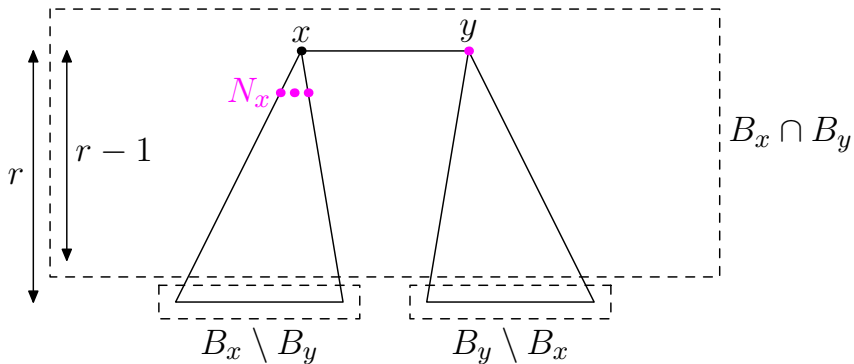
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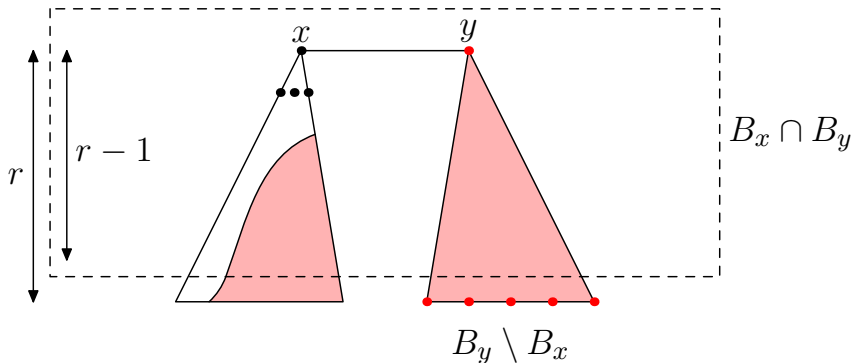
# The submiracle explained



- $B_x \cup B_y$  is a tree
- the leaves are  $B_x \setminus B_y$  and  $B_y \setminus B_x$
- goal: find  $N_x$

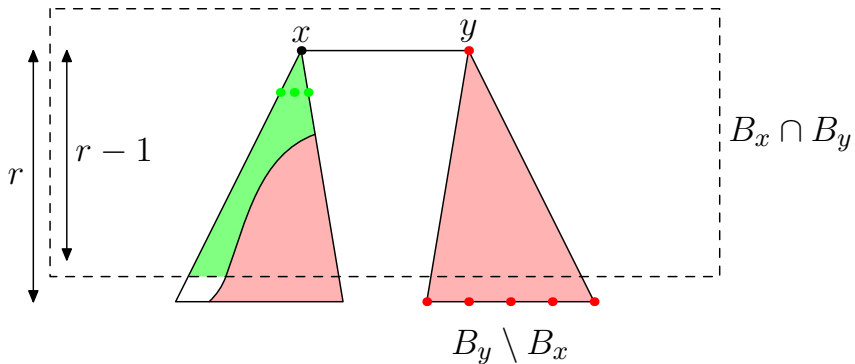


# The submiracle explained



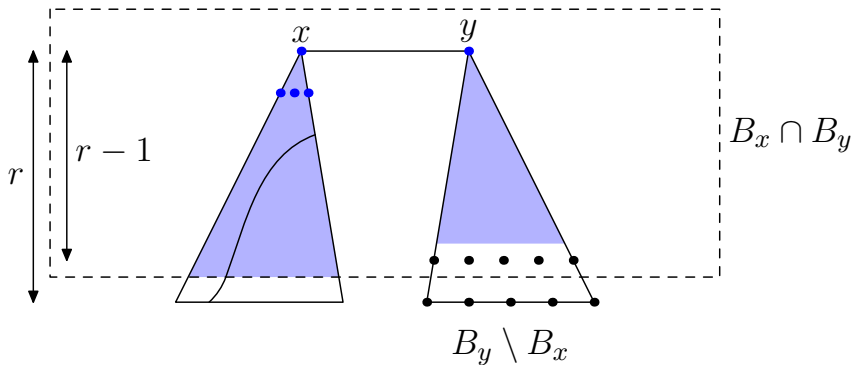
- The set  $\mathcal{R} = \bigcup_{v \in B_y \setminus B_x} B_v$  is marked in red

# The submiracle explained



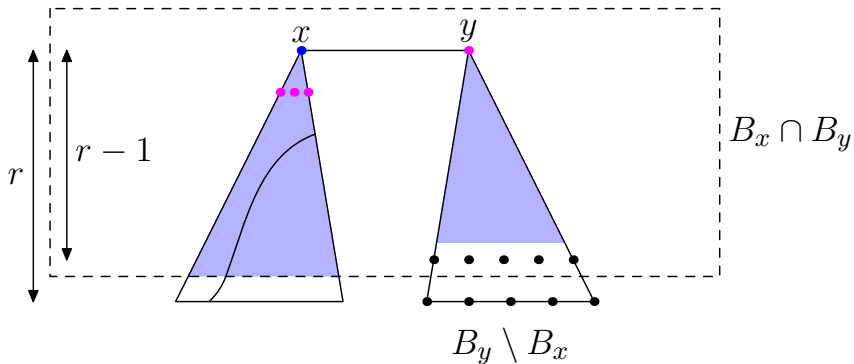
- $\mathcal{R} = \bigcup_{v \in B_y \setminus B_x} B_v$
- $\mathcal{G} = B_x \cap B_y \setminus \mathcal{R} \setminus \{x\}$
- $N_x \subseteq \mathcal{G}$

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- $\mathcal{R} = \bigcup_{v \in B_y \setminus B_x} B_v$
- $\mathcal{G} = B_x \cap B_y \setminus \mathcal{R} \setminus \{x\}$
- $\mathcal{B} = B_x \cap B_y \cap \bigcup_{v \in \mathcal{G}} B_v$

# The submiracle explained



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- $\mathcal{G} = B_x \cap B_y \setminus \mathcal{R} \setminus \{x\}$
- $\mathcal{B} = B_x \cap B_y \cap \bigcup_{v \in \mathcal{G}} B_v$
- $N_x = B_x \cap B_y \cap \bigcap_{v \in \mathcal{B}} B_v \setminus \{x\}$  — neighbours of  $x$

# If we had an algorithm...

Input:  $G$

Output:

a graph  $H$ , of girth at least 5, such that  $H^2 = G$

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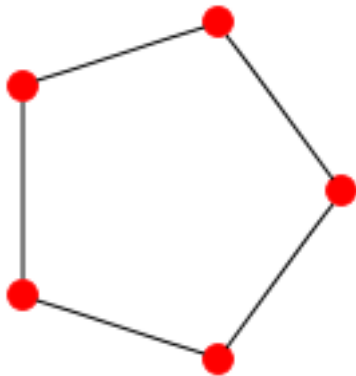
Output:



• a graph  $H$  of diameter 2 and girth 5

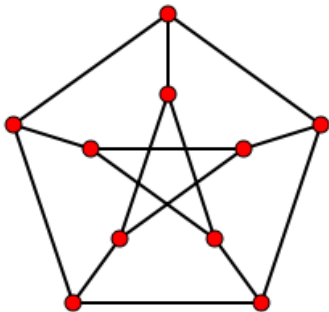


# Cycle $C_5$



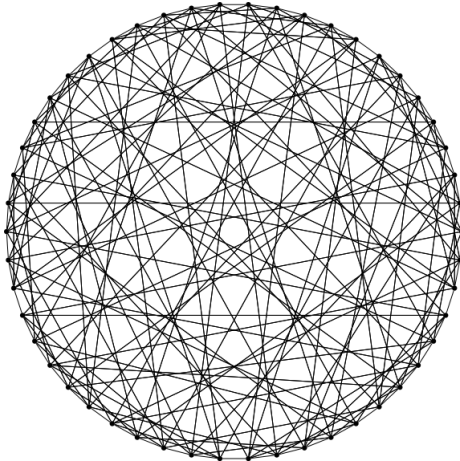
$|V| = 5 = 2^2 + 1$ , 2-regular, girth=5, diam=2

# Petersen graph



$|V| = 10 = 3^2 + 1$ , 3-regular, girth=5, diam=2

# Hoffman-Singleton graph



$|V| = 50 = 7^2 + 1$ , 7-regular, girth=5, diam=2

# A mysterious $(57, 5)$ -cage



$|V| = 3250 = 57^2 + 1$ , 57-regular, girth=5, diam=2

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Input:  $K_{3250}$

Output:

The mysterious  $(57, 5)$ -cage (a 57-regular graph with  $\text{diam}=2$ ,  $\text{girth}=5$  and 3250 vertices).

# Uniqueness of square roots

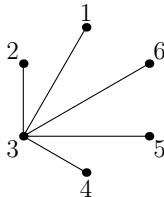
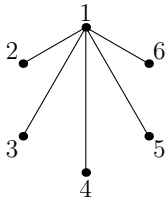
Ross, Harary '60

The **tree** square root of a graph is unique up to iso.

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The square root **of girth  $\geq 7$**  of a graph is unique up to iso.

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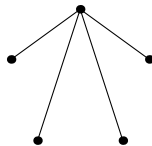
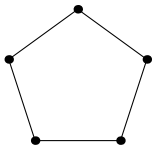
The square root of **girth**  $\geq 7$  of a graph is unique up to iso.

AA, MA '10

The square root of **girth**  $\geq 6$  of a graph is unique up to iso.

Stop!

The square root of **girth**  $\geq 5$  of a graph may not be unique.



# Uniqueness of square roots with no leaves

Levenshtein et.al. '08

The square root of girth  $\geq 7$  and no leaves of a graph is unique.

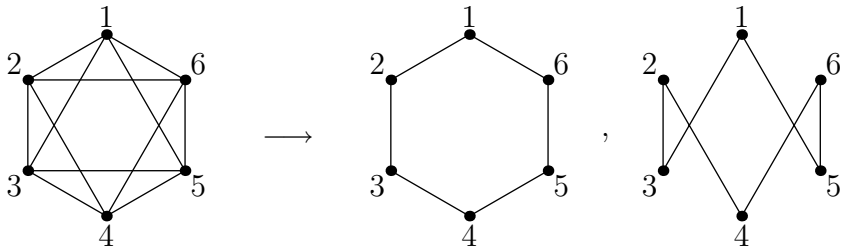
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Stop!

False for **girth  $\geq 6$** .



## OK, what about higher roots?

CONJECTURE, Levenshtein '08

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AA,MA '09; Miracle 2

Each  $G$  has at most  $O(|G|)$   $r$ -th roots of girth  $\geq 2r + 3$  and no leaves.



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