Clique complexes vs. Graph powers

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Combinatorial algebraic topology

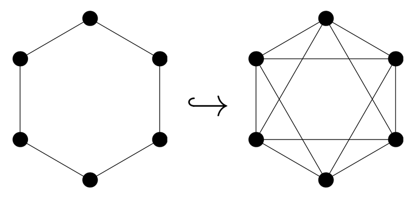
Graphs and their powers

Let G be a finite connected graph. The r-th *distance power* of G is a graph G^r with the same vertex set in which two vertices are adjacent if their distance in G is at most r.
We have a sequence of graph inclusions

$$G \hookrightarrow G^2 \hookrightarrow G^3 \hookrightarrow \cdots$$

which stabilize at the complete graph.

Example. For the 6-cycle the inclusion $C_6 \hookrightarrow C_6^2$ is

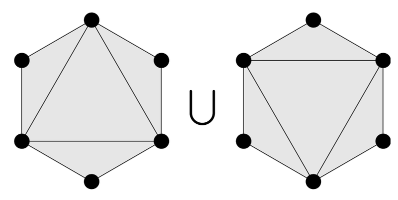


Clique complexes

Various simplicial constructions encode topologically the global combinatorial structure of G, for example:

- The *clique complex* $\mathrm{Cl}(G)$ is a simplicial complex whose faces are the complete subgraphs of G.
- The *independence complex* $\mathrm{Ind}(G)$ is a simplicial complex whose faces are the independent sets of G.

Example. The clique complex $\mathrm{Cl}(C_6^2)$ is the union



of two disks glued along the boundary, that is S^2.

Bringing the two together

The space $\mathrm{Cl}(G^r)$ is the *Vietoris-Rips complex* of subsets of diameter at most r in G.

- Problem 1. What are the interesting features of the spaces $\mathrm{Cl}(G^r)$ for higher r?
- Problem 2. What are the interesting features of the induced inclusions

$$\mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2) \hookrightarrow \mathrm{Cl}(G^3) \hookrightarrow \cdots \quad ?$$

Warm-up exercise. Prove that the induced map of fundamental groups

$$\pi_1(\mathrm{Cl}(G)) \to \pi_1(\mathrm{Cl}(G^r))$$

is *surjective*.

Stability

When is $i : \mathrm{Cl}(G) \hookrightarrow \mathrm{Cl}(G^2)$ a homotopy equivalence?

The most general condition we provide is

Every maximal clique in G^2 (i.e. every maximal face of $\mathrm{Cl}(G^2)$) has the form $N_G[v]$ (it is spanned by a vertex v of G together with all its neighbours in G).

Sketch of proof. The simplices $N_G[v]$ form a covering of $\mathrm{Cl}(G^2)$ and all their nonempty intersections $X_{v_1,\dots,v_k} = N_G[v_1] \cap \cdots \cap N_G[v_k]$ are contractible. One shows that $i^{-1}(X_{v_1,\dots,v_k})$ are also contractible (in fact cones) and a Mayer-Vietoris type argument shows i is a weak equivalence.

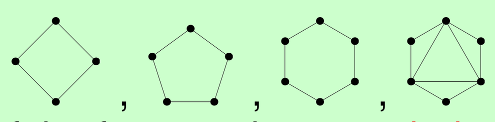
The condition is violated for example by the cliques $\{i, i+2, i+4\}$ in C_6^2 above.

Some special cases of this result include:

G has girth at least 7.

More generally: for any r, if G has girth at least $3r+1$ then $\mathrm{Cl}(G^r)$ *collapses* to $\mathrm{Cl}(G)$ which, since G is triangle-free, is just G.

G is



– free

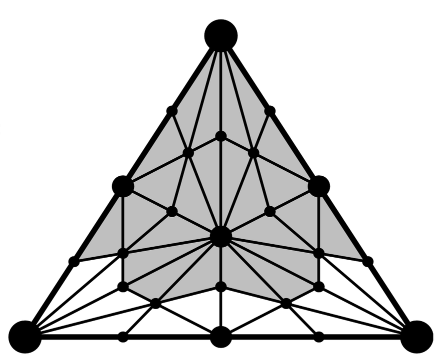
(it does not have either of the four graphs as an *induced* subgraph.)

Universality

For every $r \geq 1$ and every finite simplicial complex K there exists a graph G with a homotopy equivalence

$$\mathrm{Cl}(G^r) \simeq K.$$

- A folklore result when $r = 1$, even with a homeomorphism $K \equiv \mathrm{Cl}(G)$ for $G = (\mathrm{bd}K)^{(1)}$, the 1-skeleton of the *barycentric subdivision* of K.
- For $r \geq 2$ we use $G = (\mathrm{bd}^s K)^{(1)}$, the 1-skeleton of a large iterated subdivision (roughly $s = O(\log r)$).
- Cover $\mathrm{Cl}(G^r)$ with subcomplexes $\mathrm{Cl}((G_v)^r)$ where G_v consists of the vertices which belong to the open star of v in K (see fig.).
- Dochtermann proved G_v is *dismantlable*. It follows also for $(G_v)^r$, so the covering complexes $\mathrm{Cl}((G_v)^r)$ are contractible. An analysis of the intersections of G_v and the *nerve lemma* do the rest.



Another representation, which is off by just some 2-cells, comes from an equivalence

$$\mathrm{Cl}((\mathrm{sd}^{(r-1)}G)^r) \simeq \mathrm{Cl}(G) \vee \bigvee^{(r-1)\mathrm{t}(G)} S^2$$

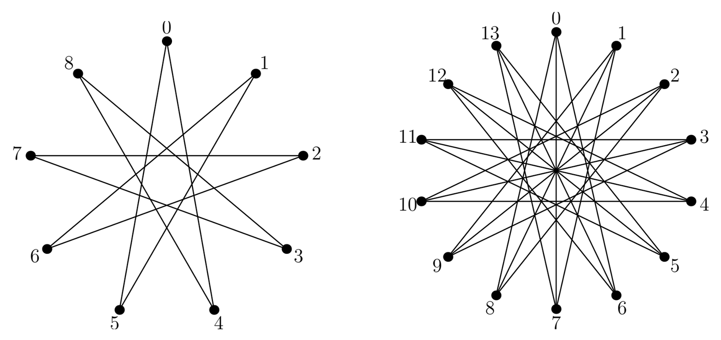
- $\mathrm{t}(G)$ – number of triangles in G
- $\mathrm{sd}^{(r-1)}G$ – an edge-subdivision of G by $r-1$ new vertices on each edge

The first example: cycles

We want to know the homotopy types of

$\mathrm{Cl}(\mathbf{C_n^r})$

With the parameter $k = n - 2r - 1$ consider the complements $T_{n,k} = \overline{C_n^r}$ known as the *circular complete graphs*.



$T_{9,2} = \overline{C_9^3}$ $T_{14,3} = \overline{C_{14}^5}$

So the equivalent question is to identify

$\mathrm{Ind}(\mathbf{T_{n,k}})$

Since $T_{n,k}$ are triangle-free we can exhibit their independence complexes $\mathrm{Ind}(T_{n,k})$ as suspensions:

1) Every maximal independent set contains 0 or its neighbour:

$$\mathrm{Ind}(T_{n,k}) = \mathrm{st}(0) \cup \bigcup_{w \in N(0)} \mathrm{st}(w).$$

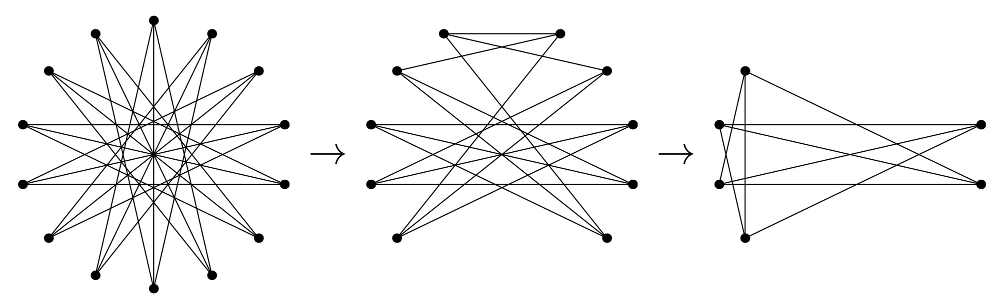
2) Since the vertex 0 is not in any triangle, $N(0)$ is a simplex in $\mathrm{Ind}(T_{n,k})$ and both summands are contractible (by a result of Barmak). Therefore

$$\mathrm{Ind}(T_{n,k}) \simeq \Sigma K, \text{where } K = \mathrm{st}(0) \cap \bigcup_{w \in N(0)} \mathrm{st}(w)$$

3) The constraints defining K can be encoded in the independence complex of some graph $S_{n,k}$.

4) Steps 1), 2), 3) can be repeated for the new graph $S_{n,k}$. The outcome can be identified with $T_{n-2(k+1),k}$.

For example, this process for $T_{14,3}$ is



$T_{14,3}$ → $S_{14,3}$ → $T_{6,3}$

In general:

For $n \geq 3k+3$ the *independence complexes* of *circular complete graphs* satisfy

$$\mathrm{Ind}(T_{n,k}) \simeq \Sigma^2 \mathrm{Ind}(T_{n-2(k+1),k}).$$

The answer

For $\frac{n}{3} \leq r < \frac{n}{2}$ the *clique complexes* $\mathrm{Cl}(C_n^r)$ of *cycle powers* satisfy

$$\mathrm{Cl}(C_n^r) \simeq \Sigma^2 \mathrm{Cl}(C_{4r-n}^{3r-n}) = \Sigma^2 \mathrm{Cl}(C_{n-2\cdot(n-2r)}^{r-1\cdot(n-2r)}).$$

It means that all $\mathrm{Cl}(C_n^r)$ are generated by the double suspension operator Σ^2, acting along lines of slope $(2,1)$, as shown by the arrows ⟶ below.

For any $n \geq 3$ and $0 \leq r < \frac{n}{2}$

$$\mathrm{Cl}(C_n^r) \simeq \begin{cases} \bigvee^{n-2r-1} S^{2l} & \text{if } r = \frac{l}{2l+1}n \\ S^{2l+1} & \text{if } \frac{l}{2l+1}n < r < \frac{l+1}{2l+3}n \end{cases} \text{ for some } l \geq 0.$$

$r=$	0	1	2	3	4	5	6	7	8	9	10	11
C_8	$\bigvee^7 S^0$	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
C_9	$\bigvee^8 S^0$	S^1	S^1	$\bigvee^2 S^2$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
C_{10}	$\bigvee^9 S^0$	S^1	S^1	S^1	S^4	$*$	$*$	$*$	$*$	$*$	$*$	$*$
C_{11}	$\bigvee^{10} S^0$	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$
C_{12}	$\bigvee^{11} S^0$	S^1	S^1	S^1	$\bigvee^3 S^2$	S^5	$*$	$*$	$*$	$*$	$*$	$*$
C_{13}	$\bigvee^{12} S^0$	S^1	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$
C_{14}	$\bigvee^{13} S^0$	S^1	S^1	S^1	S^1	S^3	S^6	$*$	$*$	$*$	$*$	$*$
C_{15}	$\bigvee^{14} S^0$	S^1	S^1	S^1	S^1	$\bigvee^4 S^2$	$\bigvee^2 S^4$	$*$	$*$	$*$	$*$	$*$
C_{16}	$\bigvee^{15} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^7	$*$	$*$	$*$	$*$
C_{17}	$\bigvee^{16} S^0$	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$	$*$	$*$
C_{18}	$\bigvee^{17} S^0$	S^1	S^1	S^1	S^1	S^1	$\bigvee^5 S^2$	S^3	S^8	$*$	$*$	$*$
C_{19}	$\bigvee^{18} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$	$*$
C_{20}	$\bigvee^{19} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^3 S^4$	S^9	$*$	$*$
C_{21}	$\bigvee^{20} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^6 S^2$	S^3	$\bigvee^2 S^6$	$*$	$*$
C_{22}	$\bigvee^{21} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	S^{10}	$*$
C_{23}	$\bigvee^{22} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^7	$*$
C_{24}	$\bigvee^{23} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\bigvee^7 S^2$	S^3	S^5	S^{11}
C_{25}	$\bigvee^{24} S^0$	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\bigvee^4 S^4$	S^7

$r = \frac{n}{3}$ $r = \frac{n}{2}$

All these entries are S^1 because of stability.

All these entries are $*$ (contractible) because C_n^r is complete.

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