ON A LOWER BOUND FOR THE CONNECTIVITY OF THE INDEPENDENCE COMPLEX OF A GRAPH

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Abstract. Aharoni, Berger and Ziv proposed a function which is a lower bound for the connectivity of the independence complex of a graph. They conjectured that this bound is optimal for every graph. We give two different arguments which show that the conjecture is false.

Given a finite simple graph G, its independence complex I_G is defined as the simplicial complex whose vertices are the vertices of G and whose simplices are the independent sets of G. The topology of independence complexes has been studied by a number of authors. In particular, the connectivity of independence complexes has shown to be of interest in the study of Tverberg graphs [5, Theorem 2.2], independent systems of representatives [3, Theorem 2.1] and other important problems.

In [3], Aharoni, Berger and Ziv proposed a function ψ defined on graphs which is a lower bound for the connectivity of I_G and conjectured that this bound is optimal. No explicit proof of this bound is given in that article, although the corresponding bound for the homological connectivity follows immediately from a result of Meshulam [9, Claim 3.1]. Moreover, a homological version of the conjecture has been considered, as well as reformulations taking into account the existence of counterexamples in which the independence complex is simply-connected or not [2].

In this note we give an explicit proof of the fact that $\psi(G)$ is a lower bound for the connectivity of I_G, we prove that the conjecture is true in the cases where I_G is not simply-connected or where $\psi(G) \leq 1$, we show that there exist counterexamples to the conjecture with $\psi(G) = 2$, and that there are counterexamples in which $\psi(G)$ and the connectivity of I_G take arbitrary values l, k with $3 \leq l < k$.

The connectivity $\text{conn}(X)$ of a topological space X is usually defined as follows: $\text{conn}(\emptyset) = -2$, $\text{conn}(X) = k$ if $\pi_i(X) = 0$ for every $0 \leq i \leq k$ and $\pi_{k+1}(X) \neq 0$, and $\text{conn}(X) = \infty$ if $\pi_i(X) = 0$ for every $i \geq 0$. The homological connectivity $\text{conn}_H(X)$ is defined in the same way replacing the homotopy groups $\pi_i(X)$ by the reduced homology groups with integer coefficients $\tilde{H}_i(X)$. In this context, however, in order to keep the notation of [3], we will use the shifted versions

$$\eta(X) = \text{conn}(X) + 2, \quad \eta_H(X) = \text{conn}_H(X) + 2.$$

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With this notation, X is non-empty when $\eta(X) \geq 1$, path-connected if $\eta(X) \geq 2$ and simply-connected when $\eta(X) \geq 3$. By the Hurewicz theorem, connectivity and homological connectivity coincide for simply-connected spaces, while in general $\eta(X) \leq \eta_H(X)$.

All the graphs considered in this note are finite and simple (undirected, loopless and without multiple edges). If e is an edge of a graph G, $G-e$ denotes the subgraph obtained by removing the edge e and $G\backslash e$ is the subgraph obtained by removing the endpoints of e and all neighbours of each of those endpoints. We denote by $E(G)$ the set of edges of G.

Consider the function ψ defined for all finite simple graphs G with values in $\{0, 1, \dots, \infty\}$, as follows

$$\psi(G) = \begin{cases} 0 & \text{if } G = \emptyset \\ \infty & \text{if } G \neq \emptyset \text{ is discrete} \\ \max_{e \in E(G)}\{\min\{\psi(G-e), \psi(G \setminus e) + 1\}\} & \text{otherwise} \end{cases}$$

The join $K * L$ of two simplicial complexes K and L is the simplicial complex with simplices $\sigma \sqcup \tau$ for $\sigma \in K$ and $\tau \in L$. The (unreduced) suspension ΣK is the join of K with a 0-dimensional complex of two vertices.

If e is an edge of a graph G, we also consider e as a 1-dimensional simplicial complex and by $\dot{e}$ we denote the 0-dimensional simplicial complex whose vertices are the endpoints of e. Meshulam [9] observed that $I_{G-e} = I_G \cup (e * I_{G\backslash e})$ and that $I_G \cap (e * I_{G\backslash e}) = \dot{e} * I_{G\backslash e} = \Sigma I_{G\backslash e}$.

Theorem 1. *For any graph G, $\psi(G) \leq \eta(I_G)$.*

Proof. We prove first that $\psi(G) \leq \eta_H(I_G)$. This part of the proof is implicit in [3]. The inequality is trivial for discrete graphs. Assume then that G is non-discrete and let $e \in E(G)$ be such that $\psi(G) = \min\{\psi(G-e), \psi(G \setminus e) + 1\}$. By induction $\psi(G-e) \leq \eta_H(I_{G-e}) = \text{conn}_H(I_{G-e}) + 2$ and $\psi(G \setminus e) \leq \eta_H(I_{G\backslash e}) = \text{conn}_H(I_{G\backslash e}) + 2$, and therefore $\tilde{H}_i(I_{G-e}) = 0$ for every $0 \leq i \leq \psi(G) - 2$ and $\tilde{H}_i(I_{G\backslash e}) = 0$ for every $0 \leq i \leq \psi(G) - 3$.

Following [9], since $\tilde{H}_i(\dot{e} * I_{G\backslash e}) = \tilde{H}_{i-1}(I_{G\backslash e})$ and since $e * I_{G\backslash e}$ is contractible, the Mayer-Vietoris sequence for the triple $(I_{G-e}; I_G, e * I_{G\backslash e})$ gives a long exact sequence

$$\dots \to \tilde{H}_{i-1}(I_{G\backslash e}) \to \tilde{H}_i(I_G) \to \tilde{H}_i(I_{G-e}) \to \tilde{H}_{i-2}(I_{G\backslash e}) \to \dots.$$

We deduce then that $\tilde{H}_i(I_G) = 0$ for every $0 \leq i \leq \psi(G) - 2$ or, in other words, that $\psi(G) \leq \eta_H(I_G)$.

To prove the theorem it suffices to show that the condition $\psi(G) \geq 3$ implies that I_G is simply-connected. If G is discrete, I_G is a simplex. Otherwise, by definition of ψ, there exists an edge e such that

$$\psi(G-e) \geq 3 \quad \text{and} \quad \psi(G \setminus e) \geq 2.$$

By induction I_{G-e} is simply-connected and since $\eta_H(I_{G\backslash e}) \geq \psi(G \setminus e) \geq 2$, $I_{G\backslash e}$ is connected. The suspension $\dot{e} * I_{G\backslash e}$ is then simply-connected and by van Kampen's theorem $\pi_1(I_{G-e})$ is the free product of $\pi_1(I_G)$ and $\pi_1(e * I_{G\backslash e})$. Since $e * I_{G\backslash e}$ is contractible, $\pi_1(I_G) = \pi_1(I_{G-e}) = 0$. □

In [3, Conjecture 2.4] it was conjectured that $\psi(G) = \eta(I_G)$. This has been confirmed for some classes of graphs, e.g. chordal graphs [8], but, as we will show, it is not true in general. In view of Theorem 1 it is clear that the homological version of the conjecture, i.e. the equation $\psi(G) = \eta_H(I_G)$, does not hold in general since $\eta_H(I_G)$ can be strictly greater than $\eta(I_G)$. This follows from the existence of a finite connected complex K with non-trivial fundamental group but such that $H_1(K) = 0$ and the well-known fact that for every finite simplicial complex K

there is a graph G with I_G homeomorphic to K, for instance the complement graph of the 1-skeleton of the barycentric subdivision of K.

Proposition 2. *Let G be a graph.*

a) *If $\psi(G) \in \{0,1\}$, then $\psi(G) = \eta(I_G)$.*
b) *If I_G is not simply-connected, then $\psi(G) = \eta(I_G)$.*

Proof. It is easy to see that $\psi(G) = 0$ if and only if G is empty, so the only non-trivial case of a) is $\psi(G) = 1$.

Since the 1-skeleton of I_G is the complement $\overline{G}$ of G, we have that $\eta(I_G) = 1$ if and only if $\overline{G}$ is disconnected. We will prove, by induction on the number of edges in G, that if $\psi(G) = 1$ then $\overline{G}$ is disconnected. By definition of ψ, G is non-discrete and for every edge e of G we have

$$\psi(G - e) = 1 \text{ or } G \setminus e \text{ is empty.}$$

If there exists an edge $e \in G$ such that $\psi(G-e) = 1$ then, by induction, $\overline{G-e}$ is disconnected and therefore so is $\overline{G}$. It suffices then to consider the case when for every edge $e \in G$ the graph $G \setminus e$ is empty. Translating this into a statement about complements we see that $\overline{G}$ has the following property:

$$\text{for every pair of non-adjacent vertices } x, y \text{ we have } N(x) \cap N(y) = \emptyset,$$

where $N(v)$ is the neighbourhood of v. It is easy to see that this property characterizes precisely the graphs in which every connected component is a clique. Since $\overline{G}$ is not a clique itself, it must be disconnected, as we wanted to show.

To prove b) note that if I_G is not simply-connected, then $\psi(G) \leq \eta(I_G) \leq 2$ by Theorem 1, and the result follows from part a). □

We now prove that the conjecture is not true. The first argument we show is not constructive and reduces to the fact that it is algorithmically undecidable whether $\eta(I_G) \geq 3$ or $\eta(I_G) \leq 2$ for a given graph G, while $\psi(G)$ is a computable function of G.

Proposition 3. *There exists a graph G with $\psi(G) = 2$ and $\eta(I_G) \geq 3$.*

Proof. The truth of the implication

$$\text{if } \psi(G) = 2 \text{ then } \eta(I_G) = 2$$

together with Theorem 1 and Proposition 2 would provide an algorithm (Turing machine) capable of determining if a given finite simplicial complex K is simply-connected. The algorithm would just find a graph G with I_G homeomorphic to K and check if $\psi(G) \geq 3$. However it is known that there can be no such algorithm. It is a consequence of the non-existence of an algorithm to determine whether a group Γ given by a finite presentation is trivial or not [1, 10] and a construction that associates to each presentation of Γ a finite 2-dimensional complex with fundamental group isomorphic to Γ (see [6] for example). □

We will give more explicit counterexamples to the conjecture, all of them different from the one shown in Proposition 3. Their construction requires the next observation in which $G \sqcup H$ denotes the disjoint union of graphs G and H.

Lemma 4. *For any graphs G and H we have $\psi(G \sqcup H) = \psi(G) + \psi(H)$.*

Proof. The result holds when both G and H are discrete. The general case now follows by induction on the number of edges in $G \sqcup H$. For every $e \in E(G)$ we have $(G \sqcup H) - e = (G - e) \sqcup H$ and $(G \sqcup H) \setminus e = (G \setminus e) \sqcup H$. If G is non-discrete, then by induction

$$\begin{aligned} &\max_{e\in E(G)} \big\{ \min\{\psi((G \sqcup H) - e), \psi((G \sqcup H) \setminus e) + 1\}\big\} = \\ = &\max_{e\in E(G)} \big\{ \min\{\psi((G - e) \sqcup H), \psi((G \setminus e) \sqcup H) + 1\}\big\} = \\ = &\max_{e\in E(G)} \big\{ \min\{\psi(G - e), \psi(G \setminus e) + 1\}\big\} + \psi(H) = \\ = &\ \psi(G) + \psi(H). \end{aligned}$$

The same equation holds if H is non-discrete and the maximum is taken over the edges $e \in E(H)$. Then the result follows. □

The lemma also follows immediately from the interpretation of $\psi(G)$ as the maximal value achievable in a certain two-player game (see [3, p.257]).

Note that for any graphs G and H we have $I_{G\sqcup H} = I_G * I_H$. In particular, if $H = e$ is just an edge, $I_{G\sqcup e} = \Sigma I_G$. Note also that $\psi(e) = 1$. Recall that for complexes K and L, the suspension $\Sigma(K \vee L)$ of the wedge between K and L is homotopy equivalent to the wedge $\Sigma K \vee \Sigma L$.

Proposition 5. *For any $l, k \in \{3, 4, \ldots, \infty\}$ with $l \leq k$ there exists a graph G such that $\psi(G) = l$ and $\eta(I_G) = k$.*

Proof. The case $l = \infty$ is trivial. Assume then that l is finite. Note that if G is such that $\psi(G) = l$ and $\eta(I_G) = k \geq 3$, then $\psi(G \sqcup e) = \psi(G) + \psi(e) = l + 1$ by Lemma 4, and $\eta(I_{G\sqcup e}) = \eta(\Sigma I_G) = \eta_H(\Sigma I_G) = \eta_H(I_G) + 1 = \eta(I_G) + 1 = k + 1$. Therefore, it suffices to prove the case $l = 3$.

Let K be an acyclic finite simplicial complex with non-trivial fundamental group, i.e. with the properties

$$\pi_1(K) \neq 0, \quad \tilde{H}_i(K) = 0 \text{ for all } i.$$

(Such K can be obtained for example by triangulating the two-dimensional CW-complex of [7, Example 2.38]). Note that the suspension ΣK is simply-connected and acyclic, hence contractible.

Assume first that k is finite. Since every finite simplicial complex can be realized, up to homeomorphism, as an independence complex of some graph, we can choose a graph H such that we have a homeomorphism

$$I_H \cong K \vee S^{k-2}.$$

Since $\eta(K \vee S^{k-2}) = 2$, we have $\psi(H) = 2$ by Proposition 2.

Let $G = H \sqcup e$. Then $I_G = \Sigma I_H$ is homotopy equivalent to $\Sigma K \vee S^{k-1}$, which in turn is homotopy equivalent to S^{k-1} since ΣK is contractible. It follows that $\eta(I_G) = k$. On the other hand $\psi(G) = \psi(H) + \psi(e) = 3$ by Lemma 4. Therefore G has the desired property.

For the remaining case $l = 3$, $k = \infty$, we consider a graph H such that $I_H \cong K$ and define $G = H \sqcup e$. Then $I_G \cong \Sigma K$ is contractible and $\psi(G) = 3$. □

Still, the study of the conjecture in special cases and for particular classes of graphs is an interesting problem and the bound provided by Theorem 1 can be useful even when it is not sharp.

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