EXTREMAL GRAPH THEORY FOR FACE NUMBERS OF FLAG COMPLEXES

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SETUP

A simplicial complex K is **flag** if the faces of K are precisely the cliques in the 1-skeleton $G = K^{(1)}$.

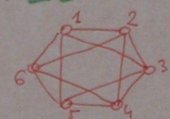
We say $K = \mathrm{Clique}(G)$

$f_i(K) = \#(i+1)$-cliques in G

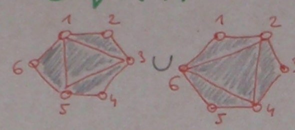
$(f_0(K) = |V(G)|,\ f_1(K) = |E(G)|)$

EXAMPLE

G:



Clique(G):



$\equiv D^2 \cup_{S^1} D^2 \equiv S^2$

PROBLEM

Describe the properties of $f(K)$ for interesting classes of flag complexes K.

As always, spheres are interesting.

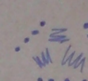
But clique numbers are graph theory.

UPPER BOUNDS

Let **$K(n,s)$** = Turan's balanced s-partite complete graph $K_{\underbrace{\frac{n}{s},\dots,\frac{n}{s}}_{s}}$

$J(n,s)$ = the same with a cycle in each part

$[\mathrm{Clique}(J(n,s)) \equiv \underbrace{S^1 * \dots * S^1}_{s} \equiv S^{2s-1}]$

$K(n,3) =$  $\} \lfloor \frac{n}{3} \rfloor$ or $\lceil \frac{n}{3} \rceil$

$J(n,3) =$ 

THEOREM

Part of conjecture by Nevo, Lutz

[3] If K is a flag homology $(2s-1)$-manifold with n vertices, n is large, and $k \le s$, then

$$f_{k-1}(K) \le f_{k-1}(J(n,s)).$$

[2] If $k=2$ then the conclusion

$$f_1(K) \le f_1(J(n,s)) \approx \frac{s-1}{2s} n^2 + n$$

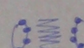
holds under much weaker assumptions on K (weak pseudomanifold + $f_s(K) = O(n^s)$).

Every ridge in exactly two facets.

Conjecture of Gal

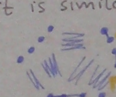
[1] If K is a flag 3-manifold and $f_1(K)$ is close to maximal, precisely

$$\tfrac{1}{4}n^2 + \tfrac{1}{2}n + \tfrac{17}{4} < f_1(K) \le \tfrac{1}{4}n^2 + n$$

and n is large then the 1-skeleton of K is still , with the cycles of not-too-different lengths.

PROOFS

K has n vertices and $G = K^{(1)}$ → upper bound $f_s(K) = O(n^s)$ → If G is as dense as $K(n,s)$ but has only $O(n^s)$ $(s+1)$-cliques then it is similar to $K(n,s)$: → G is in fact "at most" $J(n,s)$



some form of Dehn–Sommerville

Stability in extremal graph theory (Szemerédi regularity, Erdös – Simonovits)

Geometric properties of K (eg. $K_{1,\underbrace{3,\dots,3}_{s}}$ - freeness of G by van Kampen – Flores)

[Big problem: Repeat this in even dimension.]

LOWER BOUNDS

The Local-k-Conjecture (LOC(k))

If G is connected and locally k-connected then

$$|E(G)| \ge (k+1)|V(G)| - C(k)$$

for some universal constant $C(k)$.

The neighborhood $G[N_G(v)]$ of every vertex v is a k-connected graph.

Status:

LOC(1), LOC(2) hold [4]. $k \ge 3$ open.

$C(1) = 3$, $C(2) = 6$

The d-dimensional Flag Lower Bound Conjecture (FLBC(d)) (Gal, Nevo)

If G is a graph such that $\mathrm{Clique}(G)$ is a d-manifold then

$$|E(G)| \ge (2d-1)|V(G)| - 2(d^2-1).$$

Equality holds for the boundary of the d-hyperoctahedron and some of its iterated subdivisions.

Why care?

$\mathrm{LOC}(2d-2) \Rightarrow \mathrm{FLBC}(d)$

by Athanasiadis' results on connectivity of flag pseudomanifolds.

Example:

$\mathrm{LOC}(2) \Rightarrow \mathrm{FLBC}(2)$

[OK, we knew FLBC(2) anyway, since it just says $|E| \ge 3|V| - 6$ for a 2-manifold.]

Better example:

$\mathrm{LOC}(4) \Rightarrow \mathrm{FLBC}(3)$

so, LOC(4) would yield a "combinatorial" proof of the only known case of the Charney–Davis conjecture.

[1] A., Hladký, "Dense flag triangulations of 3-manifolds via extremal graph theory", Trans. AMS, to appear
[2] A., "An upper bound theorem for a class of flag weak pseudomanifolds", arxiv/1303.5603
[3] A., in preparation
[4] A., Adamaszek, Mnich, Schmidt, "A lower bound conjecture for locally highly connected graphs", in preparation