Graph coloring

Lecture notes, vol. 5

Planar Graphs

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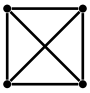
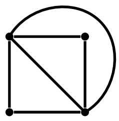
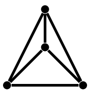
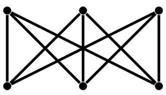
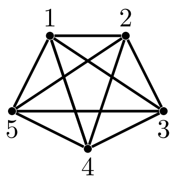
Definition 1. *G is* planar *if it can be drawn on $\mathbb{R}^2$ (the plane) so that edges intersect only at their common endpoints. We call such a drawing an* "embedding" *(some authors say* "drawing"*).*

Example 2.  K_4 not an embedding  embedding (K_4 is planar)

 straight line-embedding

Theorem 3. *(Fáry,1948) If G has an embedding, then it also has one where every edge is a straight line segment.*

Remark 4. G can be treated as a topological space [($CW-$,$\Delta-$,simplicial-) complex]. Then G is planar if (as a topological space) it embeds into $\mathbb{R}^2$ (*embedding* $\equiv$ *continuous*, *injective map*)

Example 5.  $K_{3,3}$ 

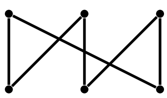
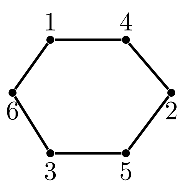
K_5

Observation 6. ***"K_5 is not planar"***

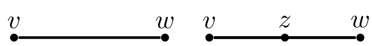
Proof. In any planar embedding the cycle 1-2-3-4-5-1 has to be drawn as a polygon:
We can draw at most 2 non intersecting diagonals inside this polygon.
We can draw at most 2 non intersecting diagonals outside this polygon.
<u>But</u> we have to draw 5 diagonals, so that is impossible. □

Observation 7. ***"$K_{3,3}$ is not planar"***

Proof. The 6-cycle has to be drawn as a polygon.
We need edges: 15,26,34
At most 1 can appear inside
At most 1 can appear outside □

 C_6 

Definition 8.

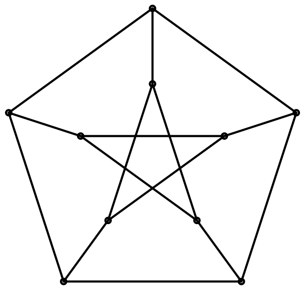
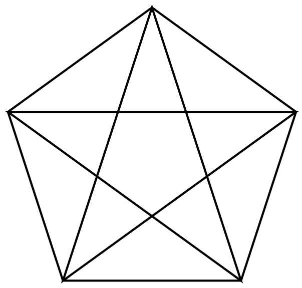
- *An edge subdivision is the replacement* 

where z is a new vertex.
- *An edge contraction is the indentification of the two endpoints of an edge.*
- *H is minor of G if H can be obtained from G by removing edges and contracting edges.*

Theorem 9. *The following are equivalent :* (a) *G is planar*

(b) *G contains no iterated subdivision of K_5 or $K_{3,3}$ as a subgraph (Kuratowski,1930)*

(c) *G has no K_5 or $K_{3,3}$ as a minor (Wagner,1937)*

Example 10.  G=Petersen graph  K_5 minor

Remark 11. If G is planar then G has no K_5 or $K_{3,3}$ subdivision/minor as subgraph

$(a) \implies (b)$, $(a) \implies (c)$ are easy implications

Proof. An embedding of G would contain an embedding of K_5 or $K_{3,3}$ □

Theorem 12. *(The Four-Color Theorem) Every planar graph is 4-colorable.*

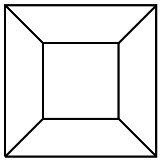
Proof history

- 1800-1850 first mentioned
- 1852 a student of De Morgan conjectured 4-colors are sufficient
- Cayley popularized it a lot
- 1879 Alfred Kempe published a proof
- 1880 Tait had another proof
- 1890 Heawood found an error in Kempe's proof (but proved the 5-color theorem), Petersen found an error in Tait's proof
- 1960 Heesh found a method that could give a proof but involved analysing a huge number of cases
- 1976 Appel, Haken analysed these cases with a computer ($\approx$ 2000 cases)
- 1990 Robertson, Seymour and others gave a new computer-assisted proof ($\approx$ 600 cases)

Definition 13. *A face is any connected component of $\mathbb{R}^2$ after removing the embedded graph.*

Observation 14.
- *There is exactly one unbounded face.*
- *Each face is an open subset of $\mathbb{R}^2$.*

Observation 15. *A graph is planar if and only if it can be embedded in S^2 (the sphere). Suppose G is embedded in S^2.Pick a point of S^2 not in the embedding. Use the stereographic projection to map G onto $\mathbb{R}^2$. Note that in a spherical embedding each face is bounded and homeomorphic to an open disk.*

Example 16. Q_3 as planar graph. 

<u>**Notation**</u> Suppose I have G with a fixed planar embedding (or spherical embedding)
v= # vertices, e= # edges, f= # faces.

Theorem 17. *(Euler's formula) If G is planar and connected, then for any planar embedding of G :*

$$v - e + f = 2.$$

Proof. By induction
If f=1 then G has no cycles, as otherwise any cycle of the graph would seperate $\mathbb{R}^2$ into at ≥ 2 parts. Hence G is a tree, $e = v - 1$ and

$$v - e + f = v - (v - 1) + 1 = 2.$$

If $f \geqslant 2$ then pick an edge $xy \in E(G)$ so that on the two sides of xy we have two different faces of the embedding. Now $G - xy$ is planar, connected and it has $f(G - xy)$=$f(G) - 1$, $e(G - xy)$=$e(G) - 1$, $v(G - xy)$=$v(G)$. The proof follows by induction. □

Euler cared about regular polyhedra in $\mathbb{R}^3$
<u>**Very quick application**</u>: Classification of Platonic solids (regular polytopes).

Definition 18. *A polytope is regular if:*

1. *All vertices have the same degree $k \geqslant 3$,*
2. *All faces are polygons with the same number of sides $l \geqslant 3$.*

Let it have v vertices, e edges, f faces in the spherical embedding.

We have these equations: $\begin{cases} v - e + f = 2 \\ kv = 2e \\ lf = 2e \end{cases}$

and so:
$e(\frac{2}{k} - 1 + \frac{2}{l}) = 2 \implies \frac{2}{k} + \frac{2}{l} = 1 + \frac{2}{e} \implies \frac{1}{k} + \frac{1}{l} = \frac{1}{2} + \frac{2}{e} > \frac{1}{2}$.
This can be satisfied only for $(k, l) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$. For each case we uniquely determine v, e, f.

Corollary 19. *Suppose G has at least three vertices.*

(a) If G is planar then $e \leqslant 3v - 6$

(b) If G is planar and triangle free then $e \leqslant 2v - 4$

Proof. We can assume G is connected. Then $v - e + f = 2$. Count the edges around each face. Each face has length $\geqslant 3$ so we get at least $3f$. But each edge is counted twice, so we get exactly $2e$. That means $2e \geqslant 3f$ or $f \leqslant \frac{2}{3}e$.
$2 = v - e + f \leqslant v - e + \frac{2}{3}e = v - \frac{1}{3}e$
$e \leqslant 3v - 6$
If G is triangle-free then we have a stronger inequality $2e \geqslant 4f$ and continue the same way. □

Observation 20. *This gives another proof of non-planarity of $K_{3,3}$ and K_5*
K_5*:* $v = 5, e = 10 \qquad 10 \nleq 3 \cdot 5 - 6$
$K_{3,3}$*: is triangle-free,* $v = 6, e = 9 \quad 9 \nleq 2 \cdot 6 - 4$