Graph coloring

Lecture notes, vol. 12

Chromatic numbers of cube-like graphs.

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We are about to prove an exponential lower bound $\chi(\mathbb{R}^d) \geq c_1^d$ on the chromatic number of $\mathbb{R}^d$. To this end we introduced modified cube graphs $Q_d(u)$ with vertices $\overline{x} = (x_1, \ldots, x_d)$, $x_i \in \{0,1\}$ and edges between $\overline{x}$ and $\overline{y}$ whenever $\overline{x}$ and $\overline{y}$ differ in exactly u places. (Throughout we will use the overbar $\overline{x}$ to denote vectors). In the natural geometric embedding of the cube these edges all have the same Euclidean length $\sqrt{u}$, therefore $Q_d(u)$ are unit distance graphs in $\mathbb{R}^d$ and $\chi(\mathbb{R}^d) \geq \chi(Q_d(u))$.

The graphs $Q_d(u)$ give pretty good lower bounds on $\chi(\mathbb{R}^d)$ already for small d. Here are results which can be verified using the Sage code we wrote in the exercises:

- $\chi(Q_5(2)) = 8$. Consequently, $\chi(\mathbb{R}^5) \geq 8$. The best known lower bound is 9.
- $\alpha(Q_{10}(4)) = 40$ (this will take about 20min in Sage). Consequently

$$\chi(\mathbb{R}^{10}) \geq \chi(Q_{10}(4)) \geq \frac{|V(Q_{10}(4))|}{\alpha(Q_{10}(4))} = \frac{2^{10}}{40} = 25.6,$$

  that is $\chi(\mathbb{R}^d) \geq 26$. This is the best known bound!

In order to prove some lower bounds valid for all d we need to add a further complication to $Q_d(u)$.

Definition 1. *The graph $Q_d(u,s) \subseteq Q_d(u)$ is the subgraph of $Q_d(u)$ induced by the vertices with exactly s coordinates equal to* 1*. Precisely:*

$$V(Q_d(u,s)) = \{\overline{x} = (x_1, \ldots, x_d) \ : \ x_i \in \{0,1\}, \ \sum_{i=1}^{d} x_i = s\}$$

and $\overline{x}$ and $\overline{y}$ are adjacent in $Q_d(u,s)$ iff they differ in exactly u positions.

Example 2. $Q_3(2,1)$ has vertex set $\{001, 010, 100\}$ and it is isomorphic to K_3.

As in the computational examples above, it is usually easier to say something about the independence number α than directly about the chromatic number χ. Our main theorem, which we will prove in the next part of the lecture, is the following.

Theorem 3. *If p is a prime then*

$$\alpha(Q_d(2p, 2p-1)) \leq \binom{d}{0} + \binom{d}{1} + \cdots + \binom{d}{p-1}.$$

We will prove this theorem in a moment. Let us just note that the condition "p is a prime" suggests that this fact is somewhat algebraic in nature. For now, let us see what this theorem buys us when it comes to chromatic numbers.

Theorem 4. *We have $\chi(\mathbb{R}^d) \geq 1.05^d$ for sufficiently large d.*

Proof. For any prime $p \leq d/2$ we have

$$\chi(\mathbb{R}^d) \geq \chi(Q_d(2p)) \geq \chi(Q_d(2p, 2p-1)) \geq \frac{|V(Q_d(2p, 2p-1))|}{\alpha(Q_d(2p, 2p-1))} \geq \frac{\binom{d}{2p-1}}{p\binom{d}{p-1}}$$

where in the last step we used the inequality of Theorem 3 and the observation $|V(Q_d(u,s))| = \binom{d}{s}$.

Intuitively, the last fraction will be maximized if the binomial coefficient $\binom{d}{2p-1}$ is close to the middle of the d-th row of the Pascal triangle, that is when $p \approx d/4$. Since we can only use p primes, we resort to a classical number-theoretic result of Czebyschev: every interval $[n, 2n]$ contains a prime. That allows us to choose a prime p such that $\frac{d}{8} \leq p \leq \frac{d}{4}$. By carefully cancelling common factors in the binomial coefficients we obtain:

$$\chi(\mathbb{R}^d) \geq \frac{1}{p} \cdot \frac{d-p+1}{2p-1} \cdot \frac{d-p}{2p-2} \cdots \frac{d-2p+2}{p}.$$

Under the condition $d \geq 4p$ each of the last p factors is $\geq \frac{3}{2}$, so:

$$\chi(\mathbb{R}^d) \geq \frac{1}{p}\Big(\frac{3}{2}\Big)^p \geq \frac{4}{d}\Big(\Big(\frac{3}{2}\Big)^{\frac{1}{8}}\Big)^d \geq \frac{4}{d} \cdot 1.051^d \geq 1.05^d$$

where the last inequality holds for sufficiently large d. □

Proof of Theorem 3

Before jumping to the proof, let us review two combinatorial methods of proving inequalities like $A \leq B$, where A, B are some combinatorially defined quantities.

Method 1 — set comparison. If a set of size B contains a subset of size A then $A \leq B$.

Example 5. We will show that $\binom{n}{k} \leq 2^n$. The family of all subsets of $\{1, \ldots, n\}$ has size 2^n, and it contains the family of all k-element subsets, the latter of size $\binom{n}{k}$. Our inequality follows.

That was an easy and completely standard argument. Our next method is also based on an elementary observation in linear algebra.

Method 2 — vector space comparison. If a vector space of dimension B contains A linearly independent vectors then $A \leq B$.

This may seem like an overkill, but it is actually a useful strategy in many otherwise complicated situations (like our Theorem 3). Here is an example of how the method works: the (rather classical) problem known as Odd–Town.

Example 6. n people participate in m clubs. Every club has an odd number of members, and every two clubs have an even number of common members. Prove that $m \leq n$.

First let's note that we may have $m = n$, for example when every person forms its own one-element club.

To solve the problem, encode the clubs $C_1, \ldots, C_m$ via "membership vectors" $\overline{c_1}, \ldots, \overline{c_m}$ of length n, where

$$(\overline{c_i})_j = \begin{cases} 1 & \text{if person } j \text{ belongs to club } i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$. If we write $\langle \overline{x}, \overline{y} \rangle = \sum_i x_i y_i$ for the standard inner product, then

$$\begin{aligned} \langle \overline{c_i}, \overline{c_k} \rangle &= \text{number of common members of } C_i \text{ and } C_k, \\ \langle \overline{c_i}, \overline{c_i} \rangle &= \text{number of members of } \ C_i. \end{aligned}$$

We will show that $\overline{c_1}, \ldots, \overline{c_m}$ are linearly independent. Suppose, for a contradiction, that it is not true. Then we have a linear relation

$$\sum_i a_i \overline{c_i} = 0$$

where not all a_i are zero. Since the coordinates of $\overline{c_i}$ are integers, we can assume that all $a_i \in \mathbb{Z}$ and moreover $\gcd(a_1, \ldots, a_m) = 1$. In particular, a_k is odd for some k. Now:

$$0 = \langle \sum_i a_i \overline{c_i}, \overline{c_k} \rangle = a_k \langle \overline{c_k}, \overline{c_k} \rangle + \sum_{i \neq k} a_i \langle \overline{c_i}, \overline{c_k} \rangle$$

which is a contradiction, because $a_k \langle \overline{c_k}, \overline{c_k} \rangle$ is odd, while all the other terms are even.

We showed that $\overline{c_1}, \ldots, \overline{c_m}$ are linearly independent vectors in $\mathbb{R}^n$. It follows that $m \leq n$.

Very similar arguments will now appear in the proof Theorem 3.

Proof of Theorem 3. As always, we write $\langle \overline{x}, \overline{y} \rangle = \sum_{i=1}^d x_i y_i$. Let $\overline{x}$ and $\overline{y}$ be two different vertices of $Q_d(2p, 2p-1)$. Using the fact that both $\overline{x}$ and $\overline{y}$ have exactly $2p-1$ coordinates equal to 1, we easily get

$$|\{j \ : \ x_j \neq y_j\}| = 2(2p-1-|\{j \ : \ x_j = y_j = 1\}|) = 2(2p-1-\langle \overline{x}, \overline{y} \rangle),$$

hence

$$\langle \overline{x}, \overline{y} \rangle = 2p-1-\frac{1}{2}|\{j \ : \ x_j \neq y_j\}|.$$

Now if $\overline{x}$ and $\overline{y}$ are adjacent in $Q_d(2p, 2p-1)$ then they differ in exactly $2p$ places, and we get $\langle \overline{x}, \overline{y} \rangle = 2p-1-p = p-1$. Otherwise we get some other inner product between 0 and $2p-2$ (because $\overline{x} \neq \overline{y}$). The upshot is that

$$\langle \overline{x}, \overline{y} \rangle \begin{cases} = p-1 & \text{if } \overline{xy} \in E(Q_d(2p, 2p-1)), \\ \not\equiv p-1 \pmod{p} & \text{if } \overline{xy} \notin E(Q_d(2p, 2p-1)). \end{cases}$$

Moreover $\langle \overline{x}, \overline{x} \rangle = 2p-1$ for all $\overline{x}$.

Take any independent set I in $Q_d(2p, 2p-1)$. For any $\overline{x} \in I$ consider the function $f_{\overline{x}} : \{0,1\}^d \to \mathbb{R}$ defined for $\overline{t} = (t_1, \ldots, t_d)$ by the formula

$$f_{\overline{x}}(\overline{t}) = \langle \overline{x}, \overline{t} \rangle^{\underline{p-1}}$$

(recall that $z^{\underline{p-1}} = z(z-1)\cdots(z-(p-2))$ is the falling factorial). The functions $f_{\overline{x}}$ are naturally elements of the $\mathbb{R}$-vector space of all functions $\{0,1\}^d \to \mathbb{R}$. Let us check that the set $\{f_{\overline{x}}\}_{\overline{x} \in I}$ is linearly independent in that space. If not, then we would have a linear relation

$$\sum_{\overline{x} \in I} a_{\overline{x}} f_{\overline{x}} = 0$$

for $a_{\overline{x}}$ not all zero. As in the example before, we can assume that $a_{\overline{x}} \in \mathbb{Z}$ and $\gcd(a_{\overline{x}}) = 1$. In particular, some $a_{\overline{x_0}}$ is not divisible by p. We have

$$0 = \sum_{\overline{x} \in I} a_{\overline{x}} f_{\overline{x}}(\overline{x_0}) = a_{\overline{x_0}} \langle \overline{x_0}, \overline{x_0} \rangle^{\underline{p-1}} + \sum_{I \ni \overline{x} \neq \overline{x_0}} a_{\overline{x}} \langle \overline{x}, \overline{x_0} \rangle^{\underline{p-1}}.$$

We have $\langle \overline{x_0}, \overline{x_0} \rangle^{\underline{p-1}} = (2p-1)(2p-2)\cdots(p+1) \not\equiv 0 \pmod{p}$. Here we use that p is a prime! Since I is an independent set, each $\langle \overline{x}, \overline{x_0} \rangle$ is different from $p-1 \pmod{p}$, hence one of the factors in the falling factorial formula for $\langle \overline{x}, \overline{x_0} \rangle^{\underline{p-1}}$ is divisible by p. That is a contradiction, since all the terms in the formula above are now divisible by p except for the first one.

We would now like to know $\dim(\operatorname{span}\{f_{\overline{x}}\}_{\overline{x} \in I})$. A more explicit representation of $f_{\overline{x}}$

$$f_{\overline{x}}(t_1, \ldots, t_d) = \Big(\sum x_i t_i\Big)\Big(\sum x_i t_i - 1\Big)\cdots\Big(\sum x_i t_i - (p-2)\Big)$$

reveals, after opening the brackets, that $f_{\overline{x}}$ is a linear combination of monomials of degree at most $p-1$ in the d variables $t_1, \ldots, t_d$. Since $t_i \in \{0,1\}$, we have $t_i^2 = t_i$, so $f_{\overline{x}}$ is in fact equal to a linear combination of square-free monomials of degree at most $p-1$ in d variables. The dimension of the vector space of such functions is $\binom{d}{0} + \cdots + \binom{d}{p-1}$, where $\binom{d}{i}$ is the number of square-free monomials of degree i (that is, products of i out of d variables).

To conclude, $\{f_{\overline{x}}\}_{\overline{x} \in I}$ is a set of linearly independent vectors in a vector space of dimension $\binom{d}{0} + \cdots + \binom{d}{p-1}$, which means that $|I| \leq \binom{d}{0} + \cdots + \binom{d}{p-1}$, as we wanted to prove. $\square$

Remark 7. The book *Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra* by Jiří Matoušek is a recommended source if you are interested in algebraic tools in combinatorics (preliminary version from the author's homepage `http://kam.mff.cuni.cz/~matousek/stml-53-matousek-1.pdf`). The proof above followed loosely Chapter 17.