Graph coloring

Lecture notes, vol. 11, Vizing's Theorem. Chromatic Number of $\mathbb{R}^k$.

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Theorem 1. *(Vizing) For every graph G:*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Proof. (Sketch) Let $\Delta := \Delta(G)$. We have to show that there is an edge coloring with $\Delta + 1$ colors.

If $G = \overline{K_n}$ we are done.

Otherwise choose an edge $e = uv \in E(G)$. Fix $C = \{1, \ldots, \Delta + 1\}$. By induction $(\Delta(G - e) \leq \Delta)$ color the edges of $G - e$. Let $f \colon E(G) \setminus \{e\} \to C$ denote the coloring. We need to define a color of e, possibly changing some existing colors. We say a color c:

1. *appears at a vertex* x if $f(xy) = c$ for some $xy \in E(G)$.
2. *is missing at* x, otherwise.

Since $|C| = \Delta + 1$, each vertex has at least one missing color. Define $v_0 := v$ and set c_0 to be any color missing at v_0. If c_0 is missing at u, then set $f(uv_0) := c_0$ and we are done. Otherwise if c_0 appears at u, then set v_1 to be a vertex such that $f(uv_1) = c_0$ and c_1 any color missing at v_1. If c_1 is missing at u shift the colors from uv_1 to uv_0. If c_1 appears at u then choose a vertex v_2 such that $f(uv_2) = c_1$ and any color c_2 missing at v_2. Recursively we define v_i as any vertex with $f(uv_i) = c_{i-1}$ and c_i as any missing color at v_i. This process stops when either

1. c_i is also missing at u, then shift colors from uv_i to uv_0.
2. $c_i = c_j$ for $0 \leq j < i$.

Suppose $c_i = c_j$ for $0 \leq j < i$. Let c be some color missing at u. If c is also missing at v_i, set $f(uv_i) = c$ and shift colors uv_i to uv_0, otherwise c appears at v_i. Consider the graph $f^{-1}(c) \cup f^{-1}(c_i)$. It contains a path starting at v_i. Where does it end? If the path ends at v_j, shift colors from uv_j down to uv_0 set $f(uv_j) = c$ and flip the colors on the path $v_i \to v_j$. Otherwise it ends at v_{j+1} or somewhere else, and these two cases are left as an exercise. □

Remark 2. We only relied on the existence of missing colors at every vertex. We can use this observation, for example:

Proposition 3. *If G has only one vertex of maximal degree, then $\chi'(G) = \Delta(G)$.*

Proof. Let u have $\deg(u) = \Delta = \Delta(G)$. Pick an edge $e = uv \in E(G)$. Now $\Delta(G - e) \leq \Delta - 1$. Color the edges of $G - e$ with Δ colors (Vizing). Again every vertex has a missing color. The recoloring part of the proof gives now an edge coloring of G with Δ colors. □

Chromatic number of the Euclidean spaces

In this part we will have infinite graphs.

Definition 4. *$\chi(\mathbb{R}^d)$ is the minimal number of colors required to color all points in $\mathbb{R}^d$ so that if $d(x, y) = 1$ then x, y have different colors for all $x, y \in \mathbb{R}^d$, where*

$$d(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$$

Definition 5. *For $X \subset \mathbb{R}^d$ define a graph U_X (U for "unit") with vertex set X and edges*

$$x_1 x_2 \in E(U_X) \text{ iff } d(x_1, x_2) = 1.$$

Observation 6. $\chi(\mathbb{R}^d) = \chi(U_{\mathbb{R}^d})$.

Example 7. $U_{\mathbb{R}}$: $xy \in E(U_{\mathbb{R}})$ iff $|x - y| = 1$. $U_{\mathbb{R}}$ is a union of infinitely many (uncountably many) bi-infinite paths. $\chi(U_{\mathbb{R}}) = 2 = \chi(\mathbb{R})$.

Remark 8. All invariants $(\omega, \chi, \alpha, \Delta, \dots)$ we defined still make sense for infinite graphs, except that they might be equal to ∞. $\omega(G) \le \chi(G)$ and $H \subset G \Rightarrow \chi(H) \le \chi(G)$ etc. still hold.

Theorem 9. *Suppose G is a graph (which may be infinite). If every* finite *subgraph of G can be colored with k colors, then G can be colored with k colors.*

Proof. Let $G = (V, E)$ be a graph and let X be the set of all functions $f \colon V \to \{1, \dots, k\}$, i.e. $X = \prod_{v \in V} \{1, \dots, k\} = \{1, \dots, k\}^V$. View $\{1, \dots, k\}$ as a discrete topological space and equip X with the product topology. $\{1, \dots, k\}$ is finite, so it is compact. By Tychonoff's theorem X is compact. For any $F \subset E$ let $X_F \subset X$ be defined as those $f \colon V \to \{1, \dots, k\}$ which are proper colorings of (V, F).

1. $X_{\{e\}}$ is closed in X since

$$X_{\{e\}} = \bigcup_{i \neq j} \{f \in X : f(u) = i, f(v) = j, e = uv\}$$

   is a finite union of closed sets.

2. $X_{F_1} \cap X_{F_2} = X_{F_1 \cup F_2}$.

3. For any $F \subset E$, X_F is closed since $X_F = \bigcap_{e \in F} X_{\{e\}}$, is an intersection of closed sets, hence closed.

Now: Take the family $\mathcal{F} = \{X_F\}_{\substack{F \subset E \\ F \text{ finite}}}$. All sets in $\mathcal{F}$ are closed, and all intersections of finitely many from $\mathcal{F}$ are non-empty (second claim: $X_{F_1} \cap \cdots \cap X_{F_n} = X_{F_1 \cup \cdots \cup F_n} \neq \emptyset$ because $(V, F_1 \cup \cdots \cup F_n)$ is finite, hence k-colorable) Then the intersection of all sets in $\mathcal{F}$ is non-empty (by compactnes of X). $f \in \bigcap_{\substack{F \subset E \\ |F| < \infty}} X_F$ is a proper coloring on *every* edge of G. $\square$

What about $\chi(\mathbb{R}^2)$

Lemma 10. *(easy upper-bound)* $\chi(\mathbb{R}^2) \le 9$.

Proof. Take the 3×3-square where the length of the diagonals in each little square is 0.99. Color every such square with 9 colors (choose any neighboring color on the commom edges). Use this square to tile the plane. Take two points x, y of the same color. Then

1. x, y are in the same small square and so $d(x, y) \le 0.99$, or

2. x, y are in two different big squares and $d(x, y) \ge 2 \cdot 0.99 \cdot 1/\sqrt{2} > 1$

so $d(x, y) \neq 1$. $\square$

References

[1] West, *Introduction to graph theory*