Problem 1. (2pt)
a) Prove or disprove: if the only complex roots of $P_G(t)$ are 0 and 1 then $G$ is a forest.

b) How many non-isomorphic graphs have chromatic polynomial $t^2(t-1)^8$?

c) Find all non-isomorphic graphs with chromatic polynomial $t(t-1)^3(t-2)$.

Problem 2. (2pt) A vertex coloring of $G$ will be called brilliant if (1) every two adjacent vertices have different colors and (2) every two vertices which have a common neighbour also have different colors. Let $\chi_b(G)$ be the minimal number of colors required for a brilliant coloring of a simple graph $G$, and let $P_b(G, t)$ be the number of brilliant colorings of $G$ with colors $\{1, \ldots, t\}$.

Find all graphs $G$ with $\chi_b(G) \leq 2$ and show that $P_b(G, t)$ is a polynomial in $t$ for every graph $G$.

Problem 3. (2pt) Prove that $\chi_l(G) + \chi_l(G) \leq |V(G)| + 1$ for any graph $G$, where $\chi_l$ is the list chromatic number.

Problem 4. (2pt) (This is an experimental problem; I am not expecting any proofs.) Let $g(n)$ be the expected number of colors used by the greedy algorithm to color a random graph from $G(n, \frac{1}{2})$.

- Compute and plot an experimental approximation of $g(n)$ for a sequence of reasonably large values of $n$, for example $n = 100, 200, \ldots, 2000$.

- Speculate about the asymptotic behaviour of $g(n)$ as $n \to \infty$. In particular, what do you think about $\lim_{n \to \infty} \frac{g(n)}{n/\log_2 n}$?

- Find information about the expected value of $\chi(G)$ for $G \in G(n, \frac{1}{2})$. How well does the greedy algorithm perform?

Problem 5. (2pt) Choose and solve one of these problems.

(5.1) Let $G$ be a nonempty graph. Simplify the expression

$$\sum_I P(G - I, -1)$$

where the sum runs over all independent sets $I$ in $G$ (including the empty one) and, as always, $G - X$ denotes the subgraph of $G$ induced by the vertex set $V(G) - X$.

Hint: More generally, consider $\sum_I P(G - I, t)$.

(5.2) A vertex $v$ of a directed graph is called a source if all the edges incident to $v$ are pointing out of $v$. Suppose $G$ is a nonempty graph with $n$ vertices. Prove that the number of acyclic orientations of $G$ having exactly one source equals $n \cdot (-1)^{n-1} \cdot [t]P_G(t)$.

Hint: Find a deletion-contraction rule for $a(G, v_0)$ := the number of acyclic orientations of $G$ in which some prescribed vertex $v_0$ is the unique source.

Deadline: Friday week 11, 18/03/2016, 10:16am.