ON SCALAR HYPERBOLIC CONSERVATION LAWS WITH A DISCONTINUOUS FLUX

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We study the Cauchy problem for scalar hyperbolic conservation laws with a flux that can have jump discontinuities. We introduce new concepts of entropy weak and measure-valued solution that are consistent with the standard ones if the flux is continuous. Having various definitions of solutions to the problem, we then answer the question what kind of properties the flux should possess in order to establish the existence and/or uniqueness of solution of a particular type. In any space dimension we establish the existence of measure-valued entropy solution for a flux having countable jump discontinuities. Under the additional assumption on the Hölder continuity of the flux at zero, we prove the uniqueness of entropy measure-valued solution, and as a consequence, we establish the existence and uniqueness of weak entropy solution. If we restrict ourselves to one spatial dimension, we prove the existence of weak solution to the problem where the flux has merely monotone jumps; in such a setting we do not require any continuity of the flux at zero.

Keywords: Hyperbolic scalar conservation laws; discontinuous flux; entropy solution.

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1. Introduction

The goal of this paper is to develop a theory for scalar hyperbolic conservation laws with a discontinuous flux. For simplicity, we consider the Cauchy problem. Thus, we are interested in solving the following equation

\[ u_t + \text{div} \mathbf{F}(u) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+, \]
\[ u(0, \cdot) = u_0 \quad \text{in} \quad \mathbb{R}^d, \]  

(1.1)
where $\mathbb{R}^{d+1}_+ := (0, \infty) \times \mathbb{R}^d$ ($d$ denotes the spatial dimension), $u : \mathbb{R}^{d+1}_+ \to \mathbb{R}$ is an unknown and $F : \mathbb{R} \to \mathbb{R}^d$ is a given flux of the quantity $u$. For $F$ continuous on $\mathbb{R}$ and Lipschitz continuous at 0, the existence of an entropy weak solution is well-known (see Refs. 10 and 16) provided that $u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $F$ satisfies a suitable "growth" condition if $q < \infty$. Recall that for $F$ continuous, a function $u$ is called entropy weak solution of (1.1) if $u$ satisfies (1.1) in the sense of distributions and if

$$E(u)_t + \text{div } Q(u) \leq 0 \quad (1.2)$$

holds in the sense of distribution for all entropy/entropy flux pairs $(E, Q)$, where $E$ is an arbitrary convex smooth function and $Q$ is given by relation $Q'(s) := E'(s)F'(s)$. We wish to emphasize that the continuity of $F$ is essential in Refs. 4, 6, 10, 11, 16–19, for proving the existence of $u$ solving (1.1) and (1.2).

The main aim of this paper is to find an extended framework suitable to $F$ with jump discontinuities with respect to the variable $u$. Our strategy, inspired by recent papers on implicit constitutive theory due to Rajagopal, is the following. First, we identify a given discontinuous $F$ with a continuous curve that consists of the graph of $F$ and abscissae that fill the jumps. Consequently, instead of a discontinuous $F$ of the variable $u$, we rather deal with an implicit relation

$$G(F, u) = 0. \quad (1.3)$$

Equation (1.3) represents a curve in $\mathbb{R}^{d+1}$ and we have one degree of freedom to set up the “optimal” unknown (independent variable). To make this statement more precise, let us consider the case $d = 1$, then (1.3) leads to a curve in $\mathbb{R}^2$, and $F$ is replaced by a scalar $F$. If $F$ is an (explicit) continuous function of $u$, then the choice of $u$ as a primary unknown is optimal. On the other hand, it may happen that $u$ can be written as a continuous function of $F$ and not vice versa. If this happens, then the choice of $u$ as an unknown leads to difficulties. However, if one sets $F$ as the unknown and then considers $u$ as a function of $F$, then one can incorporate the theory developed for the problems where the flux is continuous. This leads us to the observation that it might be better in such cases to reformulate (1.1) and (1.2) in terms of the unknown $F$ instead of in terms of the unknown $u$. This also clearly indicates that one has to be careful in selecting the appropriate unknown. Although this simple one-dimensional procedure does not work in higher dimensions, the idea to transform the original problem into a new problem that has better properties (continuous flux) has resulted

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a In fact, it is enough if $F$ is $\alpha$-Hölder continuous at zero with $\alpha \geq \frac{d-1}{d}$, see Theorem 4.1 below or Refs. 16 and 17.

b Implicit constitutive theory motivates our study. We would like to contribute to the development of the mathematical theory for evolutionary problems of elasticity with implicit or discontinuous (multi-valued) relations between the Cauchy stress and the deformation gradient and include for example the responses such as the one drawn at Fig. 1 in Ref. 15, at least in one spatial variable. Note that even in one spatial dimension, the dynamic problems of elasticity can be reformulated as a $2 \times 2$ system of hyperbolic conservation laws. From this perspective, the understanding of scalar hyperbolic conservation laws with a discontinuous (or multi-valued) flux represent a good starting point towards our aim.
in the method that we present in this paper. In fact, in the multi-dimensional case we construct an appropriate parametrization of a discontinuous curve in $\mathbb{R}^{d+1}$ such that the composition of $F$ with this parametrization has a natural continuous extension. Doing so, we can even consider $F$ with infinite number of jump discontinuities.

In the whole paper we thus assume that $F \in L_{\text{loc}}^\infty(\mathbb{R})^d$ is jump continuous, which means that limits $\lim_{t \to s^-} F(t)$ exist for all $s \in \mathbb{R}$. The set of jump continuous functions on a compact interval is a Banach space and each jump continuous function on a compact interval is a uniform limit of a sequence of piecewise continuous functions, i.e. having only finite number of jumps. Moreover, each jump continuous function possesses at most countably many jumps, see Chap. VI in Ref. 1. We associate with every jump continuous function a multi-valued mapping that takes, at a point of discontinuity, all values between the limit of $F$ from the left and the limit of $F$ from the right.

For jump continuous $F$ there are only a few results concerning mathematical analysis of the Cauchy problem (1.1). The most general result we are aware of analyzes the case $d = 1$ and concerns $F$ of the form $F := F_1(u) + (F_2(u) - F_1(u))H(u)$ where $F_1$ and $F_2$ are smooth and $H(u)$ denotes the Heaviside function. For such $F$, Dias et al. 5 established the existence of a weak solution that also satisfies the following generalization of (1.2)

$$E(u)_t + Q(u)_x - E'(0) w_x \leq 0,$$

where $E$ is arbitrary smooth convex function and $Q(s) := \int_0^s E'(s)[F'_1(s) + (F'_2(s) - F'_1(s))H(s)]ds$ and $w \in L^\infty(\mathbb{R}_+^{d+1})$ such that $w(x,t) \in H(u(x,t))$. This existence result has been proved under the hypothesis that the initial data are more regular, namely, $u_0 \in BV(\mathbb{R})$.

Another approach, presented by Gimse, 9 is based on the front tracking method. The author did not formulate any entropy conditions, but showed the existence of non-expansive semigroup in $L^1(\mathbb{R})$.

A problem of a different, but relevant sort, which is currently widely studied, concerns scalar conservation laws with a flux that is discontinuous with respect to the spatial variables. Among various approaches, which we do not mention here, we point out two papers. 2,12 In Ref. 2, Audusse and Perthame introduce a partial adaptation of Kružkov’s entropies, whereas Panov 12 shows that Audusse–Perthame solutions can be obtained by a method based on the change of the unknown function, similarly as in our approach.

Finally, we wish to mention that we have already incorporated the idea to transform the original problem into a new problem that has better properties in our earlier study 3 where we analyzed steady flows of incompressible fluids of power-law type characterized by the constitutive equation that can be associated with a maximal monotone graph. In Ref. 3, we relied on a very useful characterization of

$^c H$ is defined as a multi-valued function such that $H(s) := 1$ if $s < 0$, $H(s) := 0$ if $s > 0$ and $H(0) := [0,1]$. 
maximal monotone graphs in terms of 1-Lipschitz continuous mappings and strictly monotone operators presented in Francfort et al. 5

This paper aims to extend the theory presented in Refs. 5, 10 and 16 in several directions. First of all, we analyze (1.1) in arbitrary dimension. In comparison to former studies, we consider more general relationships for \( F \) and we introduce, to our opinion, a more suitable notion of weak entropy solution requiring that \( u_0 \) belongs only to \( L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) (not to BV space). Note that it follows easily from the proofs that the results can be straightforwardly extended to the case \( u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), q < \infty \) (by using the methods introduced in Ref. 16).

In order to distinguish scalar- and vector-valued functions of one or \( d, (d + 1) \) variables we will use the following notation. Small letters in italics always denote scalar functions mapping \( \mathbb{R}^{d+1}_+, \mathbb{R}^d \) respectively on \( \mathbb{R} \), i.e. \( v(x, t): \mathbb{R}^{d+1}_+ \to \mathbb{R} \) and \( v(x): \mathbb{R}^d \to \mathbb{R} \). Real functions of one real variable will always be denoted by capital letters, i.e. \( P: \mathbb{R} \to \mathbb{R} \). On the other hand, vector-valued functions of one real variable are denoted by capital bold letters, i.e. \( \mathbf{Q}: \mathbb{R} \to \mathbb{R}^d \). Similarly vector-valued functions of \( (t, x) \) or \( x \) respectively will be denoted by small bold letters, i.e. \( \mathbf{v}(x, t): \mathbb{R}^{d+1}_+ \to \mathbb{R}^d \) and \( \mathbf{v}(x): \mathbb{R}^d \to \mathbb{R}^d \).

The structure of the paper is as follows. In Sec. 2, we define notion of entropy weak solution to the problems with a continuous flux and establish its equivalent formulation. This step inspired by implicit constitutive relation (1.3) is the keystone of the whole paper. Then, in Sec. 3, we prove that in one-dimensional setting there exists a weak solution to (1.1) for a class of monotone \( F \) with jump discontinuities. Here we do not require any continuity of \( F \) at zero. In Sec. 4, we deal with notion of entropy measure-valued solution and we establish its existence and, under the assumption of \( \alpha \)-Hölder continuity of \( F \) at zero, its uniqueness. We also analyze the support of these measures. Consequently, in Sec. 5 we obtain the existence and uniqueness of entropy weak solution (again we need \( \alpha \)-Hölder continuity of \( F \) at zero).

2. Equivalent Definition of Entropy Solution

In this section we introduce a new concept of entropy solution for a smooth flux \( F \) that is different from, but equivalent to the usual one. This new concept helps us to extend the definition of entropy solution to problems where the flux has jump discontinuities. The key observation is the following.

**Lemma 2.1.** Let \( F \) be smooth. Assume that there are \( u \in L^\infty(\mathbb{R}^{d+1}_+), g \in L^\infty(\mathbb{R}^{d+1}_+) \) and smooth invertible strictly increasing functions \( U, G \) such that \( U = G^{-1} \). Moreover, let \( u = U(g) \) a.e. in \( \mathbb{R}^{d+1} \). Then \( u \) solves (1.1)_1 and (1.2) in the sense of distribution if and only if \( g \) fulfills (in the sense of distribution)

\[
\begin{align*}
U(g)_t + \text{div} F(U(g)) &= 0, \quad (2.1) \\
Q_u(g)_t + \text{div} Q_F(g) &\leq 0, \quad (2.2)
\end{align*}
\]
for arbitrary “flux-flux” pair \((Q_u, Q_F)\) given by relations

\[
Q_u' = U'\tilde{E}', \quad Q_F' = (F \circ U)'\tilde{E}',
\]

where \(\tilde{E}\) is an arbitrary convex \(^3\) smooth function (“entropy”).

**Remark 2.1.** The above lemma is formulated for smooth functions \(F, U, G\). However, it is easy to observe that it holds for \(F, U, G\) that are Lipschitz continuous as well. This is a consequence of formulas

\[
Q_u(s) = \int_0^s U'(t)E'(t)dt + c \quad \text{and} \quad Q_F(s) = \int_0^s (F \circ U)'(t)\tilde{E}'(t)dt + c.
\]

**Proof.** (Proof of Lemma 2.1) The equivalence of (1.1)\(_1\) and (2.1) simply follows from the relation \(U(g) = u\). Hence, it remains to prove equivalence of (1.2) and (2.2). For simplicity, we prove that (1.2) \(\Rightarrow\) (2.2). The opposite implication can be proved in a similar manner.

Let \(\tilde{E}\) be an arbitrary smooth convex function and \(Q_u, Q_F\) fulfill (2.3). We want to show that they satisfy (2.2). Let us define

\[
E(s) := Q_u(G(s)), \quad Q(s) := Q_F(G(s)).
\]

Since \(U(g) = u \Leftrightarrow g = G(u)\), we easily conclude that \(E(u) = Q_u(g)\) and \(Q(u) = Q_F(g)\) a.e. in \(\mathbb{R}^{d+1}_+\). Thus, the proof follows from the inequality (1.2) provided that we verify that \((E, Q)\) is an entropy/entropy flux pair. For this purpose it suffices to prove that \(E\) is convex and that \(Q' = F'E'\). Since \(\tilde{E}\) is convex and \(G\) is increasing, the convexity of \(E\) follows from the following computation

\[
E''(s) \overset{(2.4)}{=} (Q_u'(G(s))G''(s))' \overset{(2.3)}{=} (\tilde{E}'(G(s))U'(G(s))G'(s))' = \frac{\phi}{\phi' U(G(s)) = 1}
\]

\[
= \tilde{E}''(G(s))G''(s) \geq 0.
\]

Finally, the fact that \(Q' = F'E'\) follows from the following identities

\[
E'(s) = \tilde{E}'(G(s)),
\]

\[
Q'(s) \overset{(2.4)}{=} Q_F'(G(s))G'(s) \overset{(2.3)}{=} F'(U(G(s)))U'(G(s))G'(s)\tilde{E}'(G(s)) = F'(s)\tilde{E}'(G(s)).
\]

The proof of Lemma 2.1 is complete. \(\square\)

Despite its simplicity, Lemma 2.1 opens the possibility to find a continuous function (parametrization) \(U\) with possibly discontinuous inverse \(G\) so that \(F \circ U\) is

\(^3\)In case we consider \(U\) and \(G\) decreasing, we need to have concave \(\tilde{E}\).
continuous even if $F$ is discontinuous. If we succeed in finding such a $U$, we can use the “entropy” inequality (2.2) and prove the same convergence results that are usually consequences of the entropy inequality (1.2). Even more, if we find such a $U$ we can introduce a notion of entropy solution in terms of the function $g$. Consequently, we are able to establish the existence and uniqueness for $u$. Another nice feature of our approach is the fact that solution (of any kind) of the problem in consideration is obtained as an appropriate limit of solutions to standard regularized problem of (1.1)–(1.2).

3. Existence of a Weak Solution for $d = 1$

This section is devoted to the problem (1.1) with discontinuous $F$ in one spatial dimension. We will introduce a concept of weak solution to (1.1) and establish its existence, for a class of $F$ that have at most countable number of monotone jumps, without requiring any smoothness of $F$ at zero.

**Definition 3.1.** Let $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $F \in L^\infty_{loc}(\mathbb{R})$ be jump continuous.\(^6\) We say that a couple $(u, f)$ is a weak solution to (1.1) if

\[
\begin{align*}
  u &\in L^\infty(0, T; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})), \\
  f &\in L^\infty(0, T; L^\infty(\mathbb{R})), \\
  f(x, t) &\in F(u(x, t)) \quad \text{for a.a. } (t, x) \in \mathbb{R}^2, \\
\end{align*}
\]

and the identity

\[
-\int_0^\infty \int_{-\infty}^{\infty} u \varphi_t \, dx \, dt - \int_{-\infty}^{\infty} \int_0^\infty f \varphi_x \, dx \, dt = \int_\mathbb{R} u_0(x) \varphi(0, x) \, dx
\]

holds for all $\varphi \in D(\mathbb{R}^2)$.

We establish the existence of a weak solution to (1.1) for $F$ of the form

\[
F(s) = \lambda s + G(s) \quad \text{for } s \in [-M, M],
\]

where

\[
M = \|u_0\|_\infty, \lambda \in \mathbb{R} \quad \text{and } G \text{ is monotone so that } U = G^{-1} \in C(G([-M, M])),
\]

Functions fulfilling (3.4)–(3.5) are called one-side Lipschitz. The class characterized by (3.4)–(3.5) is quite general; it is easy to observe that this class includes arbitrary $F$ that are piecewise $C^1$ functions with arbitrary, but countable number of monotone jumps.\(^7\) Indeed, assuming for clarity that all jumps are increasing, one can define $-\lambda := \inf_{s \in [-M, M]} F'_-(s) - 1$ and set $G(s) := F(s) - \lambda s$. This definition immediately implies the relations and properties stated in (3.4)–(3.5).

\(^6\)To avoid misunderstanding, we recall that $f \in F(u)$ means that (i) $f = F(u)$ if $F$ is continuous at $u$, and (ii) $f \in \lim_{t \to u^-} F(t)$ if $F$ is jump discontinuous at $u$.

\(^7\)Here, monotone jumps mean that if $F$ has jumps at $a$ and at $b$, then $(F(a)_+ - F(a)_-)(F(b)_+ - F(b)_-) \geq 0$, where $F(s)_\pm := \lim_{k \to s \pm} F(k)$.
Theorem 3.1. Let \( u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \) and \( F \in L^\infty_{loc}(\mathbb{R}) \) be jump continuous and fulfills (3.4)–(3.5). Then there exists a weak solution to (1.1) within the meaning of Definition 3.1.

Proof. For simplicity we assume that the function \( G \), introduced in (3.4)–(3.5), is strictly increasing on \([-M, M]\). Let \( \rho \in D(-1, 1) \) be a mollification kernel, i.e. \( \rho(s) = \rho(-s) \) and \( \int_{-1}^1 \rho(s) \, ds = 1 \). We define \( F^n := F * \rho^n \), where \( \rho^n(s) := n \rho(ns) \). Then it is easy to deduce that \( F^n(s) = \lambda s + G^n(s) \) with \( G^n := G * \rho^n \). Since \( G \) is strictly increasing on \([-M, M]\), the same property holds for \( G^n \) and consequently we can find \( U^n \in C(\mathbb{R}) \), such that \( (U^n)^{-1} = (G^n) \) on \([-M, M]\). Note that \( F^n \to F \) in the sense that \( \lim_{n \to \infty} F^n(s) = F(s) \) for arbitrary \( s \in \mathbb{R} \). Moreover, we observe that

\[
U^n \to U \quad \text{strongly in } C[G(-M), G(M)] \quad (n \to \infty),
\]

where \( U := G^{-1} \).

Since \( F^n \) is smooth, then according to Ref. 10 there exists uniquely defined \( u^n \) such that \( u^n \in L^\infty(0, \infty; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})) \) solves

\[
u^n_t + F^n(u^n) = 0, \quad E(u^n), \quad Q^n(u^n) \leq 0,
\]

and satisfies the initial condition \( u_0 \) in the following sense \( \lim_{t \to 0} \int_{\mathbb{R}} |u^n(t, x) - u_0(x)| \, dx = 0 \). Moreover, the following uniform estimates are available

\[
\sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}} \|u^n(t)\|_1 \leq \|u_0\|_1, \quad \sup_{n \in \mathbb{N}} \sup_{t \in \mathbb{R}} \|u^n(t)\|_\infty \leq \|u_0\|_\infty.
\]

Next, we set \( f_1^n(x, t) := G^n(u^n(x, t)) \) and \( f^n(x, t) := \lambda u^n(x, t) + f_1^n(x, t) \). Letting \( n \to \infty \) and using (3.9) and (3.4)–(3.5) we find subsequences of \((u^n, f^n)\) and a couple \((u, f)\) such that (modulo subsequence)

\[
u^n \rightharpoonup u \quad \text{weakly in } L^\infty(0, \infty; L^\infty(\mathbb{R})), \quad f^n \rightharpoonup f \quad \text{weakly in } L^\infty(0, \infty; L^\infty(\mathbb{R})), \quad f_1^n \rightharpoonup f = \lambda u \quad \text{weakly in } L^\infty(0, \infty; L^\infty(\mathbb{R})), \quad u^n = U^n(f_1^n) \rightharpoonup U = u \quad \text{weakly in } L^\infty(0, \infty; L^\infty(\mathbb{R})).
\]

Thus, it is easy to deduce from (3.10)–(3.12) and (3.7) that \((u, f)\) satisfies (3.3). In order to complete the proof, it remains to show that \( f(x, t) \in F(u(x, t)) \) for a.a. \((t, x) \in \mathbb{R}^2_+\). Since \( f_1 = f - \lambda u \), this is equivalent to prove that \( f_1(x, t) \in G(u(x, t)) \) a.e. in \( \mathbb{R}^2_+ \) or equivalently that \( U(f_1(x, t)) = u(x, t) \).

Towards this goal we follow step-by-step the proof of Theorem 6, p. 57 in Ref. 7 with only one small but essential difference: instead of analyzing behavior of \( \{u^n\}_{n=1}^\infty \) as \( n \to \infty \), we focus on behavior \( \{f_1^n\}_{n=1}^\infty \). Thus, applying Lemma 2.1, we deduce that \( f_1^n \) fulfills

\[
U^n(f_1^n) + \lambda U^n(f_1^n) + f_1^n = 0, \quad Q^n_\theta(f_1^n) + Q_\theta^2(f_1^n) \leq 0,
\]

for all \( \theta \in (0, 1) \).
where $Q^n_F, Q^n_u$ are given by
\[
(Q^n_F(s))' = (F^n(U^n(s)))' \tilde{E}'(s), \quad (Q^n_u(s))' = (U^n(s))' \tilde{E}'(s).
\]
Since $(Q^n_F(s))' = (F^n(U^n(s)))' \tilde{E}'(s) = \lambda(U^n(s))' \tilde{E}'(s) + \tilde{E}'(s)$ we see that (3.15) can be equivalently rewritten as
\[
Q^n_u(f^n_1)_t + \lambda Q^n_u(f^n_1)_x + \tilde{E}(f^n_1)_x \leq 0.
\] (3.16)
Note that having (3.6), (3.10)–(3.13) we immediately obtain
\[
\tilde{E}(f^n_1) - \ast \tilde{E} \quad \text{weakly* in } L^\infty(0, \infty; L^\infty(\mathbb{R})),
\]
\[
Q^n_u(f^n_1) - \ast \tilde{Q}_u - \ast \tilde{Q}_u(f^n_1) \quad \text{weakly* in } L^\infty(0, \infty; L^\infty(\mathbb{R})),
\]
where $Q'_u = \tilde{E}U'. \text{Defining } v^n := (U^n(f^n_1), \lambda U^n(f^n_1) + f^n_1)$ and $w^n := (\lambda Q^n_u(f^n_1) + \tilde{E}(f^n_1), -Q^n_u(f^n_1))$ we see that $\text{div}_{t,x} v^n = 0$ and $\text{curl}_{t,x} w^n \leq 0$. Thus, div-curl lemma implies that
\[
U^n(f^n_1)(\lambda Q^n_u(f^n_1) + \tilde{E}(f^n_1)) - Q^n_u(f^n_1)(\lambda U^n(f^n_1) + f^n_1)
\]
\[
\quad - \lambda \langle \lambda \tilde{Q}_u + \tilde{E} \rangle - \tilde{Q}_u(\lambda U + f_1),
\]
which is equivalent to
\[
U^n(f^n_1)\tilde{E}(f^n_1) - Q^n_u(f^n_1)f^n_1 = \tilde{E}\tilde{E} - \tilde{Q}_u f_1.
\]
Using uniform convergence of $U^n$, see (3.6), we observe that the last statement is equivalent to
\[
U(f^n_1)\tilde{E}(f^n_1) - Q_u(f^n_1)f^n_1 = \tilde{E}\tilde{E} - \tilde{Q}_u f_1.
\]
Thus, denoting $\nu_{t,x}$ a Young measure corresponding to $f_1$ we observe that for a.a. $(t, x) \in \mathbb{R}_+^2$
\[
\int_{\mathbb{R}} (U(\alpha) - \tilde{U}(x, t)) \tilde{E}(\alpha) - Q_u(\alpha)(\alpha - f_1(x, t))d\nu_{t,x}(\alpha) = 0.
\]
Setting $\tilde{E}(s) := |s - f_1(x, t)|$, we obtain (note that $U$ is increasing) $Q_u(s) = |U(s) - U(f_1(x, t))|$. Thus,
\[
\int_{\mathbb{R}} (U(\alpha) - \tilde{U}(x, t))|\alpha - f_1(x, t)| - |U(\alpha) - U(f_1(x, t))|(\alpha - f_1(x, t))d\nu_{t,x}(\alpha) = 0.
\]
Since $U$ is increasing it implies that
\[
(U(f_1(x, t)) - \tilde{U}(x, t))\int_{\mathbb{R}} |\alpha - f_1(x, t)|d\nu_{t,x}(\alpha) = 0.
\]
Thus, $U(f_1(x, t)) = \tilde{U}(x, t) = u(x, t)$ for a.a. $(t, x) \in \mathbb{R}_+^2$. The proof of Theorem 3.1 is complete. \qed
Theorem 3.1 can be extended to a more general class of initial conditions, namely \( u_0 \in L^1(\mathbb{R}) \cap L^r(\mathbb{R}) \) for some \( r < \infty \). Indeed, the proof will follow the same lines, we merely need to verify that div-curl lemma is applicable. Saying differently, we need to verify that the sequences \( \| u^n \| L^1, F(u^n) \) and \( \lambda |u^n|^2 \) belong to some \( L^q_{\text{loc}}(\mathbb{R}) \) for some \( q > 1 \). However, it follows from the uniform estimates that \( \sup_t \| u^n(t) \|_r \leq c \). Consequently, prescribing suitable growth condition on \( F \), we will be able to extend Theorem 3.1 to a more general setting. Thus, we can formulate (without proof) the following theorem.

**Theorem 3.2.** Let \( u_0 \in L^1(\mathbb{R}) \cap L^r(\mathbb{R}) \). Assume that \( F \) is of the form

\[
F(s) = \lambda s + G(s)
\]

where \( G \) is strictly increasing and has a continuous inverse \( U \) on \( \mathbb{R} \). Let \( G \) satisfy

\[
|G(s)| \leq c(1 + |s|), \quad q \geq 0.
\]

Let \( r > \max\{2, q + 1\} \). Then there exist \( u \in L^\infty(0, \infty; L^1(\mathbb{R}) \cap L^r(\mathbb{R})) \) and \( f \in L^\infty(0, \infty; L^2_{\text{loc}}(\mathbb{R})) \) such that (3.3) holds and \( f(x, t) \in F(u(x, t)) \) for a.a. \( (t, x) \in \mathbb{R}^2_+ \). Moreover, if \( \lambda = 0 \) then it suffices to assume that \( r > q + 1 \).

## 4. Entropy Measure-Valued Solution in Arbitrary Dimension

In this section we introduce a new concept of entropy measure-valued solution to scalar conservation laws in several space variables for \( F \) that is jump discontinuous. This new type of measure-valued solution has the following two properties. First, it is equivalent to the standard definition for a continuous \( F \). Second, if a sequence of smooth fluxes \( \{F^n\}_{n=1}^\infty \) converges to a discontinuous \( F \) in the sense of graphs, then the (new) entropy measure-valued solution corresponding to \( F \) is attainable as the limit of uniquely defined entropy weak solutions \( \{u^n\}_{n=1}^\infty \) corresponding to \( F^n \).

In what follows, we assume that \( F \in L^\infty_{\text{loc}}(\mathbb{R}) \) is jump continuous and we denote by \( z_k \in \mathbb{R}, k \in \mathbb{N} \) the points such that \( \lim_{t \to -(z_k)_+} F(t) \neq \lim_{t \to -(z_k)_-} F(t) \). Recall that the set of such points is countable. We also assume that

- there exists nondecreasing \( U \in C(\mathbb{R}) \) such that \( \lim_{s \to \pm \infty} U(s) = \pm \infty \),
- let \( \alpha_k := \inf_{\alpha, U(\alpha) = z_k} \alpha \), \( \beta_k := \sup_{\beta, U(\beta) = z_k} \beta \), then for all \( k \in \mathbb{N} \) there holds \( \alpha_k < \beta_k < \alpha_{k+1} \),
- the function \( U \) is constant on \( [\alpha_k, \beta_k] \) and strictly increasing on \( (\beta_k, \alpha_{k+1}) \) for all \( k \in \mathbb{N} \),
- there exists \( A \in C(\mathbb{R})^d \), such that \( A(s) \in F(U(s)) \),
- the function \( A \) is linear on \( [\alpha_k, \beta_k] \) for all \( k \in \mathbb{N} \).

(4.1)

Note that if \( U \) and \( A \) satisfy (4.1) then

\[
A(s) = \begin{cases} 
F(U(s)) & \text{for } s \in (\beta_k, \alpha_{k+1}), \\
\frac{F_+(z_k) - F_-(z_k)}{\beta_k - \alpha_k} (s - \alpha_k) + F_-(z_k) & \text{for } s \in [\alpha_k, \beta_k],
\end{cases}
\]

(4.2)

where \( F_\pm(z_k) := \lim_{t \to -(z_k)_\pm} F(t) \).
Inspired by Lemma 2.1, we provide the following definition of entropy measure-valued solution to (1.1) and (1.2).

**Definition 4.1.** Let \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) and \( F \) be jump continuous and satisfy (4.1). We say that a Young measure \( \nu : \mathbb{R}^{d+1} \to \text{Prob}(\mathbb{R}) \) is entropy measure-valued solution to (1.1) and (1.2) if

\[
\sup_{t>0} \| (\langle |U|^r, \nu \rangle \|_1 \leq \| u_0 \|_r, \quad \text{for all } r \in [1, \infty),
\]

\[
\text{supp} \nu \subset \{ \lambda; |U(\lambda)| \leq \| u_0 \|_\infty \} \quad \text{for a.a. } (x, t) \in \mathbb{R}^{d+1}
\]

and

\[
- \int_{\mathbb{R}^{d+1}} \langle \langle U(\lambda) - U(\mu), \nu(\lambda) \rangle \varphi, \mu \rangle \varphi'dxdt + \langle (A(\lambda) - A(\mu))\text{sgn}(\lambda - \mu), \nu(\lambda) \rangle \cdot \nabla \varphi dxdt \leq \int_{\mathbb{R}^d} |u_0 - U(\mu)| \varphi(0) dx, \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^{d+1}), \ \varphi \geq 0, \ \text{and all } \mu \in \mathbb{R}. \quad (4.4)
\]

Note that this definition is consistent with the standard one for continuous \( F \). Indeed, if \( F \) is continuous and \( U \) is arbitrary strictly increasing continuous function, then (4.4) can be obtained formally from (2.2) by taking \( \dot{E}(\lambda) = |\lambda - \mu| \).

At this point, we formulate two main theorems of this paper and provide their proofs. The first theorem establishes the uniqueness of entropy measure-valued solution under the additional hypothesis on Hölder continuity of \( F \) at 0. It also shows “independence” of this type of solution on a particular couple \( A \) and \( U \) that comes from the Assumption (4.1). The second theorem establishes the existence of such a solution.

**Theorem 4.1.** (Uniqueness) Let \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) and \( F \) be jump continuous, satisfy (4.1) and be \( \alpha \)-Hölder continuous at 0 with \( \alpha > \frac{d-1}{d} \). Assume that \((A_1, U_1)\) and \((A_2, U_2)\) are different couples of functions satisfying (4.1). Let \( \nu^1\)-corresponding to \((A_1, U_1)\) and \( \nu^2\)-corresponding to \((A_2, U_2)\) be two measure-valued entropy solutions that satisfy

\[
\liminf_{t \to 0} \int_K \int_{\mathbb{R}^d} |U_1(\lambda) - u_0(x)| + |U_2(\mu) - u_0(x)| d\nu^1_{(x,t)}(\lambda) d\nu^2_{(x,t)}(\mu) dx = 0
\]

for arbitrary compact \( K \subset \mathbb{R}^d \). \quad (4.5)

Then for a.a. \((x, t)\) there exists \( \lambda_0 \) such that

\[
\text{supp} \nu^1_{(x,t)} \subset \{ \lambda; U_1(\lambda) = U_1(\lambda_0) \},
\]

\[
\text{supp} \nu^2_{(x,t)} \subset \{ \mu; U_2(\mu) = U_1(\lambda_0) \}. \quad (4.6)
\]

In particular, defining \( u(x, t) := \int_{\mathbb{R}} U_1(\lambda) d\nu^1_{(x,t)}(\lambda) \), we have for all \( M \in C(\mathbb{R}) \)

\[
M(u(x, t)) = \int_{\mathbb{R}} M(U_1(\lambda)) d\nu^1_{(x,t)}(\lambda) = \int_{\mathbb{R}} M(U_2(\mu)) d\nu^2_{(x,t)}(\mu) \quad \text{a.e.} \quad (4.7)
\]
Theorem 4.2. (Existence) Let \( u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) and \( F \in L^\infty(\mathbb{R}) \) be jump continuous. Then there exist \( A, U \) satisfying (4.1) and a measure-valued solution \( \nu \) to (1.1) and (1.2) in the sense of Definition 4.1. Moreover, this solution satisfies

\[
\liminf_{t \to 0} \int_K \int_{\mathbb{R}} |U(\lambda) - u_0(x)| dv_{(x,t)}(\lambda) dx = 0 \quad \text{for any compact } K \subset \mathbb{R}^d.
\]

It is worthwhile to note that no additional assumption on regularity of \( u_0 \) is required in the above theorems. Also, by the method introduced in Ref. 16 one can easily extend the results of Theorems 4.1 and 4.2 to initial data that are not bounded and prove the following result.

Theorem 4.3. Let \( u_0 \in L^1(\mathbb{R}^d) \cap L^r(\mathbb{R}^d) \) for some \( r > 1 \). Let \( F \) be jump continuous satisfying \( |F(s)| \leq (1 + |s|^q) \) for some \( 0 \leq q < r \). Then there exists measure-valued solution in the sense of Definition 4.1 where we replace (4.3) by

\[
\sup_{t > 0} \|U^s\|_1 \leq \|u_0\|_{\infty}^s \quad \text{for all } s \in [1, r].
\]

Moreover, let \( F \) be \( \alpha \)-Hölder continuous at 0 with \( \alpha > \frac{d-1}{r} \) and let \( (A_1, U_1) \) and \( (A_2, U_2) \) be two couples of functions satisfying (4.1). Let \( \nu^1 \)-corresponding to \( (A_1, U_1) \) and \( \nu^2 \)-corresponding to \( (A_2, U_2) \) be two measure-valued entropy solutions that satisfy (4.5). Then the relations (4.6)–(4.7) are valid for all \( M \in C(\mathbb{R}) \) fulfilling \( |M(s)| \leq (1 + |s|^r) \).

We omit the proof of Theorem 4.3 since it easily follows from the proofs of Theorems 4.1 and 4.2 and the method developed in Ref. 16. Hence, in the remaining part of this section, we focus on proving the results stated in Theorems 4.1 and 4.2. We start with the proof of Theorem 4.1.

Proof. The proof of Theorem 4.1 follows from the proof given in Ref. 16. The only modification is due to the fact that we are interested in the behavior of different Young measure. For this reason, we sometimes give only formal proof since all technical details can be found in Ref. 16.

To start, we denote \( \alpha_k^1, \beta_k^1, \alpha_k^2, \beta_k^2 \) the numbers from (4.1) corresponding to \( U_1 \) and \( U_2 \) and we assume that \( \nu^1 \) and \( \nu^2 \) solve

\[
\langle |U_1(\lambda) - U_2(\mu)|, \nu^1_{(x,t)}(\lambda) \otimes \nu^2_{(x,t)}(\mu) \rangle_t + \text{div} \langle Q(\lambda, \mu), \nu^1_{(x,t)}(\lambda) \otimes \nu^2_{(x,t)}(\mu) \rangle \leq 0
\]

in the sense of distribution, where \( Q(\lambda, \mu) \) is defined through

\[
Q(\lambda, \mu) := \begin{cases} 
(A_1(\lambda) - A_2(\mu)) \text{sgn} (\frac{\lambda - \alpha_k^1}{\beta_k^1 - \alpha_k^1} - \frac{\mu - \alpha_k^2}{\beta_k^2 - \alpha_k^2}) \\
(A_1(\lambda) - A_2(\mu)) \text{sgn} (U_1(\lambda) - U_2(\mu)) 
\end{cases}
\]

if there is \( k \) such that \( \lambda \in [\alpha_k^1, \beta_k^1], \mu \in [\alpha_k^2, \beta_k^2] \),

\[
(A_1(\lambda) - A_2(\mu)) \text{sgn} (U_1(\lambda) - U_2(\mu)) 
\]

otherwise.

It easily follows from (4.1) and (4.2) that \( Q \in C(\mathbb{R}^d) \).

\textsuperscript{8} Note that \( \nu^1_{(x,t)}(\lambda) \otimes \nu^2_{(x,t)}(\mu) \) denotes the product measure on \( \mathbb{R}^2 \).
We split the proof into two parts: first, we show that the statements of Theorem 4.1 follow from (4.10) that is supposed to hold a priori, and then we prove (4.10).

Towards the first aim, we multiply (4.10) by \( \psi^n \in \mathcal{D}(\mathbb{R}^d) \), integrate the result over \( \mathbb{R}^d \), use the integration by parts and finally integrate over time interval \((t_1, t_2)\), where \( t_1, t_2 \) are Lebesgue points of the first function in (4.10). By doing so, we conclude that

\[
\int_{\mathbb{R}^d} \langle |U_1(\lambda) - U_2(\mu)|, \nu_{(x,t)}^1(\lambda) \otimes \nu_{(x,t)}^2(\mu) \rangle \psi^n(x) dx 
\leq \int_{\mathbb{R}^d} \langle |U_1(\lambda) - U_2(\mu)|, \nu_{(x,t)}^1(\lambda) \otimes \nu_{(x,t)}^2(\mu) \rangle \psi^n(x) dx 
+ \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \langle \mathcal{Q}(\lambda, \mu), \nu_{(x,t)}^1(\lambda) \otimes \nu_{(x,t)}^2(\mu) \rangle \cdot \nabla \psi^n(x) dx dt 
=: I_1(t_1, n) + I_2(t_1, t_2, n). \tag{4.12}
\]

Assume, at this point, that \( \text{supp} \psi^n \subset K \), where \( K \) is a compact subset of \( \mathbb{R}^d \). Then, at least for a subsequence,

\[
I_1(t_1, n) \leq C \int_K \int_{\mathbb{R}^d} \langle |U_1(\lambda) - u_0(x)| + |U_2(\mu) - u_0(\lambda)|, \nu_{(x,t)}^1(\lambda) \otimes \nu_{(x,t)}^2(\mu) \rangle dx dt 
\xrightarrow{(4.5)} 0 \quad \text{as } t_1 \to 0.
\]

To estimate \( I_2(t_1, t_2, n) \) we first deduce from \( \alpha \)-Hölder continuity of \( F \) at 0 and the facts that \( \nu_1, \nu_2 \) are uniformly compactly supported (see Assumption (4.3)2) and \( U_1, U_2 \) are monotone that

\[
|A_1(\lambda) - A_2(\mu)| \leq C(|U_1(\lambda)|^\alpha + |U_2(\mu)|^\alpha) \quad \text{for all } \lambda, \mu \in \text{supp} \nu_1 \otimes \nu_2.
\]

Thus, we arrive at the estimate (note that \( |\mathcal{Q}(\lambda, \mu)| \leq |A_1(\lambda) - A_2(\mu)| \))

\[
I_2(t_1, t_2, n) \leq C \sup_{t>0} \int_{\mathbb{R}^d} (\langle |U_1|^{\alpha}, \nu_1 \rangle + \langle |U_2|^{\alpha}, \nu_2 \rangle) \langle \nabla \psi^n(x) \rangle dx.
\]

Applying Jensen’s \textsuperscript{b} and Hölder’s inequalities, and using the Assumption (4.3)\textsuperscript{1} with \( r := 1 \) leads to

\[
I_2(t_1, t_2, n) \leq C \langle \nabla \psi^n \rangle_{\frac{1}{\alpha}}^{\frac{1}{\alpha}}.
\]

Considering \( \psi^n \) such that \( \psi^n(x) = 1 \) in \( B(0, n) \), \( \psi^n(x) = 0 \) for \( x \in \mathbb{R}^d \setminus B(0, 2n) \) and \( |\nabla \psi^n| \leq \frac{c}{n} \), we observe that

\[
\langle \nabla \psi^n \rangle_{\frac{1}{\alpha}}^{\frac{1}{\alpha}} \leq cn^{-\frac{1}{\alpha}}n^d \to 0 \iff \alpha > \frac{d-1}{d}.
\]

\textsuperscript{b}In fact, we use a generalized version of the Jensen inequality, namely

\[
\langle V \circ U, \nu \rangle \geq V(\langle U, \nu \rangle)
\]

valid for all convex \( V \). To see that such an inequality holds, apply the standard Jensen inequality to \( \mu(E) = \nu(U(E)) \). It leads to \( \langle V \circ U, \nu \rangle = \langle V, \mu \rangle \geq V(\langle \text{Id}, \mu \rangle) = V(\langle U, \nu \rangle) \).
As a consequence, taking $\alpha > \frac{d-1}{d}$ we observe that $I_2(t_1, t_2, n) \to 0$ as $n \to \infty$, and (4.12) and monotone convergence theorem imply that

$$
\int_{\mathbb{R}^2} [U_1(\lambda) - U_2(\mu)] d\nu^1(x,t)(\lambda) d\nu^2(x,t)(\mu) = 0 \quad \text{a.e. in } \mathbb{R}^{d+1}. \tag{4.13}
$$

Our next step is to deduce the relations (4.6) from (4.13). Assume that $\lambda_0 \in \text{supp } \nu^1$ and that there is $\mu_0 \in \text{supp } \nu^2$ such that $U_1(\lambda_0) \neq U_2(\mu_0)$. Then there exist $0 \leq H_1, H_2 \in \mathcal{D}(\mathbb{R})$ such that $H_1(\lambda_0) = H_2(\mu_0) = 1$ and that

$$
\begin{align*}
\text{supp } H_1 & \subset \left\{ \lambda; \ |U_1(\lambda) - U_1(\lambda_0)| \leq \frac{1}{4} |U_2(\mu_0) - U_1(\lambda_0)| \right\}, \\
\text{supp } H_2 & \subset \left\{ \mu; \ |U_2(\mu) - U_2(\mu_0)| \leq \frac{1}{4} |U_2(\mu_0) - U_1(\lambda_0)| \right\}.
\end{align*}
$$

Hence, for all $(\lambda, \mu) \in \text{supp } H_1 \times H_2$, we have

$$
\begin{align*}
|U_1(\lambda) - U_2(\mu)| & \geq |U_1(\lambda_0) - U_2(\mu_0)| - |U_1(\lambda) - U_1(\lambda_0)| - |U_2(\mu) - U_2(\mu_2)| \\
& \geq \frac{1}{2} |U_1(\lambda_0) - U_2(\mu_0)| > 0.
\end{align*}
$$

Therefore

$$
0 < \int_{\mathbb{R}^2} H_1(\lambda) H_2(\mu) d\nu^1(\lambda) d\nu^2(\mu)
$$

eq \int_{\mathbb{R}^2} \frac{H_1(\lambda) H_2(\mu)}{|U_1(\lambda) - U_2(\mu)|} |U_1(\lambda) - U_2(\mu)| d\nu^1(\lambda) d\nu^2(\mu)
$$
\leq C \int_{\mathbb{R}^2} |U_1(\lambda) - U_2(\mu)| d\nu^1(\lambda) d\nu^2(\mu) = 0,
$$

which gives the contradiction and (4.6) is proved. The relation (4.7) then easily follows from (4.6).

In order to complete the proof, we have to verify the validity of (4.10). Recall that (4.4) written for $\nu^i$, $i = 1, 2$ reads formally as

$$
\langle |U_1(\lambda) - U_2(\mu)|, \nu^i(\lambda) \rangle_{\lambda} + \text{div} \langle (\mathcal{A}_i(\lambda) - \mathcal{A}_i(\alpha)) \text{sgn}(\lambda - \alpha), \nu^i \rangle \leq 0 \tag{4.14}
$$

and holds for arbitrary $\alpha \in \mathbb{R}$. Let $\mu \in \mathbb{R}$ be arbitrary. Consider first the possibility that there is $k$ such that $\mu \in (\beta_k^2, \alpha_{k+1}^2)$. Note that $U_2$ is strictly increasing on $(\beta_k^2, \alpha_{k+1}^2)$ and the same holds for $U_1$ on $(\beta_k^1, \alpha_{k+1}^1)$. Moreover, because $U_2(\beta_k^2, \alpha_{k+1}^2) = U_1(\beta_k^1, \alpha_{k+1}^1)$, we can find uniquely defined $\alpha \in (\beta_k^1, \alpha_{k+1}^1)$ such that $U_1(\alpha) = U_2(\mu)$. In fact, $\alpha := (U_1)^{-1}(U_2(\mu))$. Since $\mathbf{F}$ is continuous at $U_1(\alpha)$, it follows from (4.1) that $\mathcal{A}_1(\alpha) = \mathcal{A}_2(\mu)$. Moreover, since $U_1$ is strictly increasing at $\alpha$, we observe that $\text{sgn}(\lambda - \alpha) = \text{sgn}(\lambda - (U_1)^{-1}(U_2(\mu))) = \text{sgn}(U_1(\lambda) - U_2(\mu))$. Thus, for such $\mu$ we have

$$
\langle |U_1(\lambda) - U_2(\mu)|, \nu^1(\lambda) \rangle_{\lambda} + \text{div} \langle \mathbf{Q}(\lambda, \mu), \nu^1(\lambda) \rangle \leq 0. \tag{4.15}
$$
If there exists \( k \) such that \( \mu \in [\alpha_k^2, \beta_k^2] \) and if in addition \( \lambda \) does not belong to \([\alpha_k^1, \beta_k^1] \), we can prove the validity of (4.15) analogously as above. If however also \( \lambda \in [\alpha_k^1, \beta_k^1] \) we set

\[
a := (\mu - \alpha_k^2) \frac{\beta_k^1 - \alpha_k^1}{\beta_k^2 - \alpha_k^2} + \alpha_k^1.
\]

Then it is easy to show, with help of (4.2), that \( A_1(a) = A_2(\mu) \) and that \( U_1(a) = U_2(\mu) \). Moreover,

\[
\sgn(\lambda - a) = \sgn \left( \frac{\lambda - \alpha_k^1}{\beta_k^1 - \alpha_k^1} - \frac{\mu - \alpha_k^2}{\beta_k^2 - \alpha_k^2} \right).
\]

Hence, we again obtain (4.15). Thus, (4.15) is valid for all \( \mu \in \mathbb{R} \). Similarly, one can show, for all \( \lambda \in \mathbb{R} \), that

\[
\langle |U_1(\lambda) - U_2(\mu)|, \nu^2(\mu) \rangle, \text{div} \langle Q(\lambda, \mu), \nu^2(\mu) \rangle \leq 0.
\]

(4.16)

Note that

\[
\langle |U_1(\lambda) - U_2(\mu)|, \nu^1(\lambda) \rangle, \nu^2(\mu) \rangle + \langle |U_1(\lambda) - U_2(\mu)|, \nu^2(\mu) \rangle, \nu^1(\lambda) \rangle
\]

\[
= \langle |U_1(\lambda) - U_2(\mu)|, \nu^1(\lambda) \otimes \nu^2(\mu) \rangle \text{div} \langle Q(\lambda, \mu), \nu^1(\lambda) \rangle \otimes \nu^2(\mu) \rangle.
\]

(4.17)

and

\[
\int_{\mathbb{R}} (\text{div} \langle Q(\lambda, \mu), \nu^1(\lambda) \rangle) d\nu^2(\mu) + \int_{\mathbb{R}} (\text{div} \langle Q(\lambda, \mu), \nu^1(\lambda) \rangle) d\nu^1(\lambda)
\]

\[
= \text{div} \langle Q(\lambda, \mu), \nu^1(\lambda) \otimes \nu^2(\mu) \rangle.
\]

(4.18)

Consequently, integrating (4.15) over \( \mathbb{R} \) w.r.t. measure \( \nu^2(\mu) \), integrating (4.16) over \( \mathbb{R} \) w.r.t. measure \( \nu^1(\lambda) \), summing the resulting inequalities and using the relations (4.17) and (4.18), we finally get the inequality (4.10). Hence, the proof is complete. \( \Box \)

In what follows, we focus on proving Theorem 4.2.

**Proof.** In the first part of the proof (of Theorem 4.2) we assume that \( F \) has finite number of jumps, i.e. there are points \( z_k \in \mathbb{R}, k \in \{1, \ldots, S\} \) such that \( \lim_{t \to -(z_k)} F(t) \neq \lim_{t \to -(z_k)} F(t) \).

We also assume that to such an \( F \) there exist sequences of Lipschitz continuous functions \( \{F^n\}_{n=1}^\infty \) and \( \{U^n\}_{n=1}^\infty \), where \( U^n \) are strictly increasing mappings of \( \mathbb{R} \) onto \( \mathbb{R} \) that are “linear” at \( \pm \infty \) (i.e. there is a continuous inverse function to \( U^n \)), such that

\[
F^n \to F \quad \text{in the sense of graphs},
\]

(4.19)

\[
U^n \to U \quad \text{strongly in } C(\mathbb{R}),
\]

(4.20)

\[
F^n \circ U^n \to A \quad \text{strongly in } C(\mathbb{R})^d.
\]

(4.21)

It should be mentioned that the last part of the proof (relations (4.17)–(4.18)) was formal. However, one can make it rigorous by incorporating a suitable regularization scheme as that introduced for example in Ref. 16.
According to Ref. 10, for Lipschitz continuous $F^n$, we are able to find unique entropy weak solution $u$ to

$$\begin{align*}
u^n_t + \text{div } F^n(u^n) &= 0 \quad \text{in } \mathbb{R}^{d+1}_+,
E(u^n)_t + \text{div } Q^n(u^n) &\leq 0 \quad \text{in } \mathbb{R}^{d+1}_+,
\quad u^n(0, x) = u_0(x) \quad \text{in } \mathbb{R}^d,
\end{align*}$$

for all entropy—entropy flux pairs $(E, Q^n)$ corresponding to $F^n$. Moreover, solutions $u^n$ satisfy the following uniform estimate

$$\sup_n \sup_{t \geq 0} \|u^n\|_r \leq \|u_0\|_r \quad \text{for all } r \in [1, \infty].$$

(4.23)

Next, setting $G^n := (U^n)^{-1}$ and $g^n(x, t) := G^n(u^n(x, t))$, we observe, by Lemma 2.1, that $g^n$ solve

$$\begin{align*}
U^n(g^n)_t + \text{div } F^n(U^n(g^n)) &= 0 \quad \text{in } \mathbb{R}^{d+1}_+,
Q^n_u(g^n)_t + \text{div } Q^n_r(g^n) &\leq 0 \quad \text{in } \mathbb{R}^{d+1}_+,
\quad U^n(g^n(0, x)) = u_0(x) \quad \text{in } \mathbb{R}^d,
\end{align*}$$

(4.24)

where $Q^n_u$ and $Q^n_r$ form a “flux—flux” pair characterized by (2.3). Moreover, (4.23) implies that

$$\sup_n \sup_{t > 0} \|g^n\|_{\infty} \leq G(\|u_0\|_{\infty}) \leq C.$$

(4.25)

Hence, we can find a Young measure $\nu(x, t)$ corresponding to a suitable subsequence of $\{g^n\}_{n=1}^{\infty}$ (again denoted $\{g^n\}_{n=1}^{\infty}$). Note that this measure is compactly supported for a.e. $(x, t)$ in a ball $B(0, G^n(R) + G^n(0)) \subset B(0, R_0)$ where $R := \|u_0\|_{\infty}$ and $R_0$ is a fixed number. Moreover, we immediately deduce (4.3) from (4.20). Next, using (4.20) and (4.21) we also observe that

$$L(F^n(U^n(g^n)), g^n) \rightharpoonup^* \int_{\mathbb{R}} L(\lambda, \lambda) \, d\nu(\lambda) \quad \text{weakly}^* \text{ in } L^\infty_{\text{loc}}(\mathbb{R}^{d+1}_+)^d,$$

(4.26)

$$L(U^n(g^n)) \rightharpoonup^* \int_{\mathbb{R}} L(U(\lambda)) \, d\nu(\lambda) \quad \text{weakly}^* \text{ in } L^\infty_{\text{loc}}(\mathbb{R}^{d+1}_+)^d,$$

(4.27)

for any continuous $L : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $L : \mathbb{R} \to \mathbb{R}$. In particular, we have

$$u^n \rightharpoonup^* u = \int_{\mathbb{R}} U(\lambda) d\nu(\lambda) \quad \text{weakly}^* \text{ in } L^\infty(0, \infty; L^r(\mathbb{R}^d))$$

(4.28)

for all $r \in [1, \infty]$. Having these convergence results, we can finally (using the same procedure as in the proof of Theorem 3.1) set $E(\lambda) := |\lambda - \mu|$ in (2.3) and consequently find $Q^n_u := |U^n(\lambda) - U^n(\mu)|$ and $Q^n_r := (F^n(U^n(\lambda)) - F^n(U^n(\mu))) \text{sgn}(\lambda - \mu)$. Upon inserting these relations into (4.24) we obtain

$$|U^n(g^n) - U^n(\mu)|_t + \text{div}((F^n(U^n(g^n)) - F^n(U^n(\mu))) \text{sgn}(g^n - \mu)) \leq 0.$$
Thus, letting \( n \to \infty \) and using (4.26) and (4.27) we arrive at (4.4). The attainment of the initial condition, i.e. the relation (4.8) can be proved similarly as within the proof of Theorem 3.2 in Ref. 16.

Thus, to finish the first part of the proof it remains to construct \( F^n, U^n \) and \( A \) satisfying (4.19)–(4.21) and (4.1). First, we denote \( F = (F^1, \ldots, F^d) \). Then, we define for any \( \ell = 1, \ldots, d \) the points \( a^\ell_i, b^\ell_i, c^\ell_i, d^\ell_i, e^\ell_i, f^\ell_i \in \mathbb{R} \), such that \( a^\ell_i \) are such points where \( F^\ell \) has an increasing jump, i.e.

\[
a^\ell_i : \quad F^\ell_+(a^\ell_i) := c^\ell_i > d^\ell_i := F^\ell_-(a^\ell_i),
\]

where \( F^\ell_+(s) := \lim_{t \to s^+} F^\ell(t) \) and \( F^\ell_-(s) := \lim_{t \to s^-} F^\ell(t) \). Similarly, \( b^\ell_i \) are such points where \( F^\ell \) have decreasing jumps, i.e.

\[
b^\ell_i : \quad F^\ell_+(b^\ell_i) := e^\ell_i < f^\ell_i := F^\ell_-(b^\ell_i).
\]

Note that for all \( i, \ell \): \( a^\ell_i, b^\ell_i \in \{z_1, \ldots, z_S\} \), where \( z_k \) are jumps of \( F \) (see (4.1)). Next, we define (generalized Heaviside functions)

\[
H^\ell_+(s) := \begin{cases} 
0 + \sum_{i:s > a^\ell_i \geq 0} (c^\ell_i - d^\ell_i) & \text{if } s > 0, \\
0 - \sum_{i:s \leq a^\ell_i \leq 0} (c^\ell_i - d^\ell_i) & \text{if } s \leq 0,
\end{cases}
\]

\[
H^\ell_-(s) := \begin{cases} 
0 + \sum_{i:s > b^\ell_i \geq 0} (e^\ell_i - f^\ell_i) & \text{if } s > 0, \\
0 - \sum_{i:s \leq b^\ell_i \leq 0} (e^\ell_i - f^\ell_i) & \text{if } s \leq 0.
\end{cases}
\]

Finally, we set

\[
H(s) := \sum_{\ell=1}^d H^\ell_+(s) - H^\ell_-(s),
\]

\[
G(s) := s + H(s).
\]

Moreover, to “extract” discontinuities from \( F \), we set \( F^\ell_c := F^\ell - H^\ell_+ + H^\ell_- \) and define \( F_c := (F^1_c, \ldots, F^d_c) \). Hence, \( F_c \) is continuous. Without loss of generality let us assume that \( F \) is continuous at 0 and consequently the same holds for \( H^\ell_+, H^\ell_- \), \( H \) and therefore \( H(0) = 0 \). Since \( G \) is strictly increasing, we can define its inverse \( U \) as

\[
U(s) := \begin{cases} 
s + C_k & \text{for } s \in (G_+(z_{k-1}), G_-(z_k)), \\
G_-(z_k) + C_k & \text{for } s \in [G_-(z_k), G_+(z_k)],
\end{cases}
\]

\[
C_{k+1} := G_-(z_k) - G_+(z_k) + C_k,
\]

where \( C_1 \) is chosen such that \( U(0) = 0 \). Note that in (4.36) we used notation \( z_0 := -\infty \) and \( z_{S+1} := \infty \). Also, note that in fact \( \alpha_k = G_-(z_k), \beta_k = G_+(z_k) \), where \( \alpha_k, \beta_k \) appeared in (4.1). Finally, we are able to construct desired regularizations that will
satisfy (4.19)–(4.21). First, we introduce a sequence of functions \( U^n \) that will be strictly increasing, Lipschitz continuous and will converge to \( U \) uniformly on \( \mathbb{R} \), i.e. will satisfy (4.20). We define \( U^n \) in the following way:

\[
U^n(s) := \begin{cases} 
  s + C_k, & \text{for } s \in \left( G_+(z_{k-1}) + \frac{1}{n}, G_-(z_k) - \frac{1}{n} \right), \\
  \frac{2(s - G_-(z_k) + \frac{1}{n})}{n(G_+(z_k) - G_-(z_k)) + 2} + G_-(z_k) + C_k - \frac{1}{n}, & \text{for } s \in \left( G_-(z_k) - \frac{1}{n}, G_+(z_k) + \frac{1}{n} \right].
\end{cases}
\]

(4.38)

Further, we define regularization of \( H^\ell_\pm \), for sufficiently large \( n \), as follows:

\[
(H^\ell_+)^n(s) := \begin{cases} 
  \frac{n}{2}(s - a_i^\ell + \frac{1}{n})(c_i^\ell - d_i^\ell) + H^\ell_+(a_i^\ell - \frac{1}{n}), & \text{for } s \in \left( a_i^\ell - \frac{1}{n}, a_i^\ell + \frac{1}{n} \right], \\
  H^\ell_+(s) & \text{for } s \in \left( b_i^\ell - \frac{1}{n}, b_i^\ell + \frac{1}{n} \right],
\end{cases}
\]

(4.39)

\[
(H^\ell_-)^n(s) := \begin{cases} 
  \frac{n}{2}(s - b_i^\ell + \frac{1}{n})(e_i^\ell - f_i^\ell) + H^\ell_-(b_i^\ell - \frac{1}{n}), & \text{for } s \in \left( b_i^\ell - \frac{1}{n}, b_i^\ell + \frac{1}{n} \right].
\end{cases}
\]

(4.40)

Next, let \( F^n_c \) be arbitrary mollification of \( F_c \) that converges to \( F_c \) uniformly in \( C(\mathbb{R})^d \). Then it is clear that \( F^n := F^n_c + ((H^\ell_+)^n - (H^\ell_-)^n, \ldots, (H^\ell_+)^n - (H^\ell_-)^n) \) converges to \( F \) in the sense of graphs, i.e. (4.19) is satisfied. Moreover, we immediately obtain that \( F^n_c \circ U^n \to F_c \circ U \) in \( C(\mathbb{R})^d \) (note that this function is constant on intervals \([\alpha_k, \beta_k] \)). Thus, to show (4.21), it remains to discuss the behavior of \( (H^\ell_+)^n \circ U^n \). In order to avoid technical difficulties we assume that all \( z_j > 0 \), therefore we have that \( (H^\ell_+)^n(U^n(y)) = 0 \) for all \( y \leq 0 \) and we also have that \( C_1 = 0 \) (this constant appears in (4.36)). For simplicity, we will only discuss the behavior of \( (H^\ell_+)^n \circ U^n \). For \( H^\ell_- \) the proof is the same. Thus, using (4.38) we have

\[
(H^\ell_+)^n(U^n(y)) = \begin{cases} 
  (H^\ell_+)^n(y + C_k), & \text{for } y \in \left( G_+(z_{k-1}) + \frac{1}{n}, G_-(z_k) - \frac{1}{n} \right], \\
  \frac{2(y - G_-(z_k) + \frac{1}{n})}{n(G_+(z_k) - G_-(z_k)) + 2} + G_-(z_k) + C_k - \frac{1}{n}, & \text{for } y \in \left( G_-(z_k) - \frac{1}{n}, G_+(z_k) + \frac{1}{n} \right].
\end{cases}
\]

(4.41)

\[\text{This is the point where we need to know that } F \text{ has finite number of jump discontinuities.}\]
Note that from the definition of $G_{\pm}(z_k)$ and $C_k$, it is evident that $G_{-}(z_k) + C_k$ and $G_{+}(z_k) + C_{k+1}$ are the points of $H$ where singularities appear. Note that if any of these singular points of $H$ is not a singular point of $H^\ell_+$ (for some $\ell$), then $(H^\ell_+)^n(U^n(y))$ remains constant in the neighborhood of this point for all $n$. Hence, assume that $H^\ell_+$ has singularity at $G_{-}(z_k) + C_k =: a^j_j$ for some $j$. Then

$$(H^\ell_+)^n(U^n(y)) = \begin{cases} 
0 + \frac{1}{n} \sum_{i=1}^{j-1} (c_i - d_i^j), & \text{for } y \in \left( G_+(z_{k-1}) + \frac{1}{n}, G_-(a^j_j) - \frac{1}{n} \right), \\
n \left( \frac{2}{n} \frac{y - G_-(a^j_j) + \frac{1}{n}}{G_+(a^j_j) - G_-(a^j_j)} \right) (c_j - d_j^j) + \frac{1}{n} \sum_{i=1}^{j-1} (c_i - d_i^j), & \text{for } y \in \left( G_-(a^j_j) - \frac{1}{n}, G_+(a^j_j) + \frac{1}{n} \right), \\
0 + \frac{1}{n} \sum_{i=1}^{j} (c_i - d_i^j), & \text{for } y \in \left( G_+(a^j_j) + \frac{1}{n}, G_-(z_{k+1}) - \frac{1}{n} \right), 
\end{cases}$$

and we see that $(H^\ell_+)^n(U^n(y))$ converges in $C(\mathbb{R})$ to an affine function that is linear on all intervals $[\alpha_k, \beta_k]$. The first part of the proof regarding $F$ with finite number of jump discontinuities is complete.

In order to finish the proof for $F$ with infinite number of jump discontinuities, we start to use the symbol $F^S$ to denote a function having $S$ jumps. Clearly, each jump continuous $F$ can be approximated by the sequence of piecewise continuous functions $(F^S)^{\infty}_{S=1}$ such that $F^S$ converges uniformly to $F$ as $S \to \infty$ on the compact interval $[-(\|u_0\|_\infty + 1), \|u_0\|_\infty + 1]$. The strategy of the proof is to consider the flux $F^S$ and corresponding $U^S$ and $A^S$, to find the corresponding entropy measure-valued solution $\nu^S$ (such solution exists since $F^S$ has a finite number of jumps) and finally let $S \to \infty$ and study the properties of limit objects.

We construct the homeomorphism $H_S : \mathbb{R} \to \mathbb{R}$ such that $H_S(1) = 1$ and

$$H_S(x) = \begin{cases} 
\frac{\beta_k - \alpha_k}{2^{\sigma(k)}} & \text{for } x \in [\alpha_k, \beta_k], \\
1 & \text{for } x \in (\beta_k, \alpha_{k+1}).
\end{cases}$$

(4.42)

Here $\sigma : \{1, \ldots, S\} \to \{1, \ldots, S\}$ is a permutation such that $\sigma(i) < \sigma(j)$ if $|F^S_+(z_i) - F^S_+(z_j)| > |F^S_+(z_j) - F^S_+(z_j)|$ or if $|F^S_+(z_i) - F^S_-(z_i)| = |F^S_+(z_j) - F^S_-(z_j)|$ and $z_i < z_j$. Then we define $U^S_S(\cdot) := U^S(H^{-1}_S(\cdot))$ and also corresponding $A^S_S(\cdot) := A^S(H^{-1}_S(\cdot))$.

Such construction guarantees that the transport of measure for the uniform limit (that surely exists) $U^S_{H} \to U$ on $\mathbb{R}$ will be well defined, i.e. we need that $[-\|u_0\|_\infty, \|u_0\|_\infty] \subset U(\mathbb{R})$. It is not difficult to observe that there exists $\delta > 0$ such that $F^S \to F$ uniformly on the set $[-(\|u_0\|_\infty + \delta), \|u_0\|_\infty + \delta]$ and therefore $A^S_{H} \to A$ uniformly on $U^{-1}([-\|u_0\|_\infty + \delta, \|u_0\|_\infty + \frac{\delta}{2})]$. For all $B$, Borel measurable sets, we define $\mu^S(B) := \nu^S(H_S(B))$. Since $\mu^S \to \mu$ and the corresponding functions integrated in (4.4) and

...
converge uniformly, we can easily pass to the limit in these formulas and complete the proof.

In this section we introduced a notion of entropy measure-valued solution to (1.1) and (1.2). We showed that it can be constructed as a limit (in fact strong limit due to Theorem 4.1) of standard entropy weak solutions to regularized problems. Moreover, we established uniqueness in our class of entropy measure-valued solution. Consequently, the entropy measure-valued solution, as constructed above, can be accepted as suitable (physical) concept of solution provided we are able to show its following stability property: if $F^n$ is an arbitrary sequence of smooth functions that converges to $F$ in the sense of graphs, then the corresponding sequence $u^n$ of entropy weak solution generates (as $n \to \infty$) an entropy measure-valued solution within the meaning of Definition 3.3. Thus, due to uniqueness (assuming Hölder continuity of $F$ at 0), we observe that this concept of entropy measure-valued solution does not depend on the choice of our sequence $F^n$. The required stability property is established in the following theorem.

**Theorem 4.4.** Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $F$ be $\alpha$-Hölder continuous at 0 with $\alpha > \frac{d-1}{d}$. Let $\{F^n\}_{n=1}^\infty$ be sequence of smooth functions that converges to some $F$ in the sense of graphs. Then the whole sequence $\{u^n\}_{n=1}^\infty$ of uniquely defined weak entropy solutions to (1.1) and (1.2) with the flux $F^n$ converges to some $u$ strongly in all $L^p_{\text{loc}}(\mathbb{R}^{d+1})$. Moreover, there exists a function $U$ satisfying (4.1) and there exists a Young measure $\nu$-entropy measure-valued solution in accordance with Definition 4.1 such that $u = \int_{\mathbb{R}} U(\lambda) d\nu(\lambda)$.

**Proof.** First, we can find sequences of functions $(H^\pm_n)$ corresponding to those introduced in (4.39) and (4.40) such that

- $(H^\pm_n) \to H^\pm$ in the sense of graphs, where $H^\pm$ are defined in (4.32) and (4.33),
- $F^n \to F_c$ strongly in $C(\mathbb{R})^d$, where $F_c$ is defined after (4.35).

Next, similarly as in the proof of Theorem 4.2, we set $G(s) = s + \sum \ell (H^\pm(s) - H^\pm(\ell))$, $U := (G)^{-1}$ and introduce its mollification $U^n$. Then following step-by-step the second part of the proof of Theorem 4.2, one can deduce that $F^n \circ U^n \to A$ strongly in $C(\mathbb{R})^d$ where $A$ satisfies the Assumption (4.1). Thus, we can define $g^n(x,t) = G(u^n(x,t))$ and find a Young measure $\nu$ corresponding to $g$, which is a weak limit of $g^n$. Moreover, from the proof of Theorem 4.2 it is clear that $\nu$ is entropy measure-valued solution in accordance with Definition 4.1. The uniqueness result stated in Theorem 4.1 implies that the whole sequence $u^n$ must converge to $u := \int_{\mathbb{R}} U(\lambda) d\nu(\lambda)$. The remaining arguments follow from Theorems 4.1 and 4.2. □

5. Entropy Weak Solution

We introduce a notion of entropy weak solution for discontinuous $F$ that is, according to Lemma 2.1, equivalent to the notion of entropy weak solution for continuous flux $F$. 

(4.8)
Moreover, assuming Hölder continuity of $F$ at 0 we establish the existence and uniqueness of such a solution. It is also worth noticing that due to Theorem 4.4, the solution does not depend on the way of how $F$ is regularized; we show that the solution can be obtained as a strong limit of weak entropy solutions corresponding to smooth approximations $F^n$ of $F$.

**Definition 5.1.** Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $F \in L^\infty_{\text{loc}}(\mathbb{R})$ be jump continuous. We say that $u \in L^\infty(0, \infty; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ is an entropy weak solution to (1.1) and (1.2) if there exist $U \in C(\mathbb{R})$ and $A \in C(\mathbb{R})$ satisfying (4.1) and there is $g \in L^\infty(\mathbb{R}^{d+1})$ such that

$$U(g(x, t)) = u(x, t), \quad A(g(x, t)) \in F(u(x, t)) \quad \text{a.e. in } \mathbb{R}^{d+1},$$

$$u_t + \text{div} A(g) = 0 \quad \text{in the sense of distribution},$$

$$\liminf_{t \to 0} \int_{K} |u(x, t) - u_0(x)| \, dx = 0 \quad \text{for any compact } K \subset \mathbb{R}^d,$$

and for all smooth convex $\tilde{E}$, that are for all $k \in \mathbb{N}$ linear on $(\alpha_k, \beta_k)$, there holds

$$Q_u(g)_x + \text{div} Q_A \leq 0 \quad \text{in the sense of distribution},$$

with $Q_u$ and $Q_A$ given by

$$Q_u(s) = U'(s)\tilde{E}'(s), \quad Q_A(s) = A'(s)\tilde{E}'(s),$$

where $\alpha_k, \beta_k$ are defined in (4.1).

The following theorem establishes the existence and uniqueness of weak entropy solution.

**Theorem 5.1.** Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $F \in L^\infty_{\text{loc}}(\mathbb{R})$ be jump continuous and $\alpha$-Hölder continuous at 0 with $\alpha > \frac{d-1}{d}$. Then there exists a unique weak entropy solution $u$ to (1.1) and (1.2).

Similarly, as in the preceding section, we formulate the following generalization of Theorem 5.1.

**Theorem 5.2.** Let $u_0 \in L^1(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ with some $r > 1$. Let $F$ be jump continuous, $\alpha$-Hölder continuous at 0 and satisfy $|F(s)| \leq c(1 + |s|^q)$ for some $q < r$. Then there exists unique $u \in L^\infty(0, \infty; L^1(\mathbb{R}^d) \cap L^r(\mathbb{R}^d))$ and there exist $g \in L^\infty(0, \infty; L^r_{\text{loc}}(\mathbb{R}^d))$ and $U \in C(\mathbb{R})$ satisfying (4.1) and (5.1)–(5.5), whereas we in addition assume that $\tilde{E}$ that appears in (5.5) fulfills $|\tilde{E}(s)| \leq c(1 + |s|^m)$ with $m < r + 1 - q$.

**Proof.** A part of the proof of Theorem 5.1 concerning the existence of $u$ easily follows from Theorems 4.1 and 4.2. Indeed, if we denote $\nu$ an entropy measure-valued solution within the meaning of Definition 4.1 that was constructed in Theorem 4.2
and we denote \( g(x, t) := \int_{\mathbb{R}} \lambda d\nu(x, t)(\lambda) \), then according to Theorem 4.1 we have that

\[
\nu(x, t) = \delta_{g(x, t)} \quad \text{if } g(x, t) \in (\beta_k, \alpha_{k+1}) \text{ for some } k, \quad (5.6)
\]

\[
\text{supp } \nu(x, t) \subset [\alpha_k, \beta_k] \quad \text{if } g(x, t) \in [\alpha_k, \beta_k] \text{ for some } k. \quad (5.7)
\]

Next, we define \( u(x, t) = U(g(x, t)) \) \((5.6), (5.7) \equiv \int_{\mathbb{R}} U(\lambda)d\nu(x, t)(\lambda) \). Since \( A \) is linear on \([\alpha_k, \beta_k]\) and (5.6) and (5.7) hold, it is easy to observe that \( A(g(x, t)) = \int_{\mathbb{R}} A(\lambda)d\nu(x, t)(\lambda) \). This and (4.1) imply that \( A(g(x, t)) \in F(u(x, t)) \). Thus, the properties (5.1)–(5.3) are proved. To show (5.4) we observe that

\[
\langle U(\lambda) - U(\mu), \nu(\lambda) \rangle = \langle A(g(x, t)) - A(\mu), \nu(\lambda) \rangle = \langle (A(\lambda) - A(\mu))\text{sgn}(\lambda - \mu), \nu(\lambda) \rangle.
\]

Moreover, if \( \mu \in (\beta_k, \alpha_{k+1}) \) for some \( k \), then \( (A(\lambda) - A(\mu))\text{sgn}(\lambda - \mu) \) is linear on all \([\alpha_k, \beta_k]\), \( k \in \mathbb{N} \). Therefore, for such \( \mu \), we have

\[
\langle (A(\lambda) - A(\mu))\text{sgn}(\lambda - \mu), \nu(\lambda) \rangle = (A(g(x, t)) - A(\mu))\text{sgn}(g(x, t) - \mu). \quad (5.9)
\]

Using (4.4), (5.8) and (5.9) we obtain (5.4) for all \( \tilde{E}(s) = |s - \mu| \) where \( \mu \) is arbitrary number belonging to some \((\beta_k, \alpha_{k+1})\). By using the standard limiting procedure one can then obtain the validity of (5.4) for all convex entropies that are linear on each interval \((\alpha_k, \beta_k)\).

To prove uniqueness, it is enough to show that for a given entropy weak solution \( u \) there exists a Young measure \( \nu \) that is entropy measure-valued solution and that satisfies \( u(x, t) = \int_{\mathbb{R}} U(\lambda)d\nu(x, t)(\lambda) \). If this holds, Theorem 4.1 then implies uniqueness of such \( u \). Hence, we define for a.a. \((x, t)\)

\[
\nu(x, t) := \begin{cases}
\delta_{g(x, t)} & \text{if } g(x, t) \in \mathbb{R} \setminus \bigcup_{k \in \mathbb{N}} (\alpha_k, \beta_k), \\
\beta_k - g(x, t) & \text{if } g(x, t) \in (\alpha_k, \beta_k),
\end{cases} \quad (5.10)
\]

Note that \( \nu(x, t) \) is a probability measure. Moreover, using (5.10), we see that for arbitrary continuous function \( B \) that is linear on each interval \((\alpha_k, \beta_k)\) we have for a.a. \((x, t)\)

\[
\int_{\mathbb{R}} B(\lambda)d\nu(x, t)(\lambda) = B(g(x, t)). \quad (5.11)
\]

Thus setting \( \tilde{E}(s) := |s - \mu| \) for arbitrary \( \mu \not\in (\alpha_k, \beta_k) \), we see that such \( \tilde{E} \) and the corresponding \( Q_u \) and \( Q_A \) defined in (5.5) are linear on all intervals \((\alpha_k, \beta_k)\). Therefore,

\[
\langle Q_u(\lambda), \nu(x, t)(\lambda) \rangle = \langle \int U(\lambda) - U(\mu), \nu(x, t)(\lambda) \rangle = |U(g(x, t)) - U(\mu)|
\]

\[
= Q_u(g(x, t)),
\]

\[
\langle Q_A(\lambda), \nu(x, t)(\lambda) \rangle = \langle (A(\lambda) - A(\mu))\text{sgn}(\lambda - \mu), \nu(x, t)(\lambda) \rangle
\]

\[
= (A(g(x, t)) - A(\mu))\text{sgn}(g(x, t) - \mu) = Q_A(g(x, t)).
\]

(5.12)
Although (5.4) is formulated only for smooth \( E \), a standard mollification procedure justifies the validity of the inequality (5.4) also for non-smooth \( E \).
Moreover, using the definition of $\nu$ (5.10), we deduce
\[
\gamma((A(\lambda) - A(\alpha_k))\text{sgn}(\lambda - \alpha_k), \nu(x,t)(\lambda)) \\
+ (1 - \gamma)((A(\lambda) - A(\beta_k))\text{sgn}(\lambda - \beta_k), \nu(x,t)(\lambda)) \\
= \frac{\gamma(g(x,t) - \beta_k)}{\beta_k - \alpha_k}(A(\beta_k) - A(\alpha_k)) + \frac{(1 - \gamma)(g(x,t) - \alpha_k)}{\beta_k - \alpha_k}(A(\beta_k) - A(\alpha_k)) \\
= \frac{\gamma(2g(x,t) - \beta_k + \beta_k - g(x,t))}{\beta_k - \alpha_k}(A(\beta_k) - A(\alpha_k))
\] (5.18)
and we observe that the R.H.S. of (5.17) is equal to that of (5.18), and as a consequence (5.16) follows. The proof of Theorem 5.1 is complete. \qed

In the final lemma we will address the relation between the solution introduced in Definition 5.1 and the solution satisfying (1.4) (the relation (1.4) corresponds to relation (2.5) in Ref. 5).

**Lemma 5.1.** Let $d = 1$ and $u$ be the entropy weak solution to (1.1) and (1.2) in the sense of Definition 5.1. Assume that $F$ takes the form $F(u) = F_1(u) + (F_2(u) - F_1(u))H(u)$ where $F_1$ and $F_2$ are smooth functions normalized by the conditions $F_1(0) = 1$ and $F_2(0) = 0$, and $H(u)$ denotes the Heaviside function. Then $u$ satisfies (1.4) in the sense of distribution for arbitrary smooth function $E$, $Q(s) = \int_0^s E'(s)[F_1'(s) + (F_2'(s) - F_1'(s))H(s)]ds$ and $w \in L^\infty(\mathbb{R}^{d+1})$ such that $w(x,t) \in \tilde{H}(u(x,t))$, where $\tilde{H}(y) = H(y)$ if $y \neq 0$ and $\tilde{H}(0) = [0,1]$. Note that $w$ is independent of the choice of $E$.

**Proof.** Let $\alpha, \beta$ correspond to the notation introduced in (4.1) with a single jump at $u = 0$. We rewrite (5.4) as follows:
\[
\int_{\mathbb{R}^2_+} Q_u(g)\varphi_t \, dx \, dt + \int_{\mathbb{R}^2_+} Q_A(g)\varphi_x \, dx \, dt \geq 0
\] (5.19)
for all $\varphi \in D(\mathbb{R}^2)$. On the sets where $U$ is equal to zero, (5.5) provides that $Q_u(g) = \text{const}$. Without loss of generality we can assume $Q_u(g) = 0$. Consequently, the relation (2.4) implies that the first integral in (5.19) reduces to
\[
\int_{\mathbb{R}^2_+} E(u)\varphi_t \, dx \, dt.
\]
Next, using (5.5) and recalling that $\tilde{E}$ is linear on the set $(\alpha, \beta)$, we observe that for $\alpha \leq \xi \leq \beta$
\[
Q_A(\xi) = \int_\alpha^\xi Q_A'(s) \, ds = \tilde{E}'(\xi) \int_\alpha^\xi A'(s) \, ds = \tilde{E}'(\alpha)(A(\xi) - A(\alpha)).
\] (5.20)
Since $A$ is affine and (5.1) holds, we see that $1 = A(\alpha) \geq A(\xi) \geq A(\beta) = 0$ for $\alpha \leq \xi \leq \beta$. Consequently, using $\tilde{E}(\alpha) = E(0)$, we conclude
\[
Q(u) = (Q_A - \tilde{E}'(0)\tilde{H})(G(u))
\]
(although $Q_A(G(u))$ or $\hat{H}(G(u))$ alone are not defined at $u = 0$), where

$$\hat{H}(s) = \begin{cases} 
0 & \text{for } s < \alpha, \\
\frac{\alpha - s}{\alpha - \beta} & \text{for } \alpha \leq s \leq \beta, \\
1 & \text{for } \beta < s.
\end{cases}$$

Indeed, for $u \neq 0$ it holds $Q'(u) = E'(u)F'_{\lambda}(u)$ for $u < 0$ and $Q'(u) = E'(u)F'_{\lambda}(u)$ for $u > 0$. On the other hand, we have the relation

$$\frac{d}{du} Q_A(G(u)) = Q'_A(G(u))G'(u) = A'(G(u))\hat{E}'(G(u))G'(u) = F'(u)E'(u).$$

Since $\hat{H}(u(t, x)) \geq \hat{H}(g(x, t)) = w(x, t)$ we obtain

$$\int_{\mathbb{R}^2_+} E(u)\varphi_t + (Q(u) - E'(0)w)\varphi_x dxdt \geq 0,$$

which implies (1.4). The proof is finished. \hfill $\square$

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