Lecture Notes in Functional Analysis

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LECTURE 1
Banach spaces

1.1. Introducton to Banach Spaces

Definition 1.1. Let X be a $\mathbb{K}$–vector space. A functional $p : X \to [0, +\infty)$ is called a seminorm, if

(a) $p(\lambda x) = |\lambda| p(x), \quad \forall \lambda \in \mathbb{K}, x \in X,$
(b) $p(x + y) \le p(x) + p(y), \quad \forall x, y \in X.$

Definition 1.2. Let p be a seminorm such that $p(x) = 0 \Rightarrow x = 0$. Then, p is a norm (denoted by $\|\cdot\|$).

Definition 1.3. A pair $(X, \|\cdot\|)$ is called a normed linear space.

Lemma 1.4. *Each normed space* $(X, \|\cdot\|)$ *is a metric space* (X, d) *with a metric given by* $d(x, y) = \|x - y\|$.

Definition 1.5. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a normed space $(X, \|\cdot\|)$ is called a Cauchy sequence, if

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{N} \quad \forall n, m \ge N(\varepsilon) \;\Rightarrow\; \|x_n - x_m\| \le \varepsilon.$$

Definition 1.6. A sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x (which is denoted by $\lim_{n \to +\infty} x_n = x$), if

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{N} \quad \forall n \ge N(\varepsilon) \;\Rightarrow\; \|x_n - x\| < \varepsilon.$$

Definition 1.7. If every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ converges in X, then $(X, \|\cdot\|)$ is called a complete space.

Definition 1.8. A normed linear space $(X, \|\cdot\|)$ which is complete is called a Banach space.

Lemma 1.9. *Let $(X, \|\cdot\|)$ be a Banach space and U be a closed linear subspace of X. Then, $(U, \|\cdot\|)$ is a Banach space as well.*

1.2. Examples of Banach spaces

Example 1. Let $\mathbf{B}(T)$ be a space consisting of all bounded maps $x : T \longrightarrow \mathbb{K}$. For each $x \in \mathbf{B}(T)$ we set

$$\|x\|_\infty = \sup_{t\in T} |x(t)|.$$

Then, $(\mathbf{B}(T), \|\cdot\|_\infty)$ is a Banach space. To prove the assertion we need to show that

(a) $\|\cdot\|_\infty$ is a norm,
(b) each Cauchy sequence converges to an element from $\mathbf{B}(T)$.

Concerning claim (a), let $\lambda \in \mathbb{K}$ and $x \in B(T)$. Then,

$$\|\lambda x\|_\infty = \sup_{t\in T} |\lambda \cdot x(t)| = |\lambda| \cdot \sup_{t\in T} |x(t)| = \lambda \|x\|_\infty. \tag{1.10}$$

Let $t_o \in T$ and $x, y \in \mathbf{B}(T)$. Then,

$$|x(t_o) + y(t_o)| \leq |x(t_o)| + |y(t_o)| \leq \sup_{t\in T} |x(t)| + \sup_{t\in T} |y(t)| = \|x\|_\infty + \|y\|_\infty.$$

The right hand side of the inequality above is independent on t. Therefore, taking supremum of both sides of the inequality over $t \in T$ yields

$$\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty. \tag{1.11}$$

Finally, let x be such that $\|x\|_\infty = 0$, which is equivalent to $\sup_{t\in T} |x(t)| = 0$. This implies that $|x(t)| = 0$ for each t. Thus, $x = 0$.

Now, we shall prove the statement (b). Let $\{x_n\}$ be a Cauchy sequence. Then, for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n, m \geq N$ it holds that $\|x_n - x_m\|_\infty < \varepsilon$. In particular,

$$|x_n(t) - x_m(t)| < \varepsilon, \quad \forall t \in T.$$

Thus, for any $t \in T$ the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ converges to some $x(t)$, due to the completeness of $\mathbb{K}$ (real and complex numbers are complete spaces). Define a candidate for a limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$, that is, $x : T \longrightarrow \mathbb{K}$ as

$$x(t) = \lim_{n\to+\infty} x_n(t).$$

If follows from the statement above that there exists $N_o = N_o(\varepsilon, t)$ such that

$$|x_n(t) - x(t)| < \varepsilon, \quad \forall n \geq N_o. \tag{1.12}$$

Without loss of generality we can assume that $N_o(\varepsilon, t) \geq N(\varepsilon)$ for each $t \in T$. Then, for $n \geq N$ it holds that

$$\begin{aligned} |x_n(t) - x(t)| &\leq \left|x_n(t) - x_{N_o(\varepsilon,t)}(t)\right| + \left|x_{N_o(\varepsilon,t)}(t) - x(t)\right| \\ &\leq \left\|x_n - x_{N_o(\varepsilon,t)}\right\|_\infty + \varepsilon < 2\varepsilon, \end{aligned}$$

where we used the fact that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and (1.12). Moreover, for each $t \in T$ and $N = N(\varepsilon)$ we have

$$|x(t)| \le |x_N(t)| + |x_N(t) - x(t)| \le \|x_N\|_\infty + 2\varepsilon,$$

which implies $\|x\|_\infty \le \|x_N\|_\infty + 2\varepsilon$ and so forth $x \in \mathbf{B}(T)$.

Example 2. Let T be a metric space and $\mathbf{C}_b(T)$ be a space of bounded continuous functions on T. Then, $(\mathbf{C}_b, \|\cdot\|_\infty)$ is a Banach space.

To prove the assertion, it is sufficient to show that $\mathbf{C}_b(T)$ is a closed subspace of $\mathbf{B}(T)$ (due to the Lemma 1.9), that is, to show that every sequence in $\mathbf{C}_b(T)$ which converges in $\mathbf{B}(T)$ converges to a point from $\mathbf{C}_b(T)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of bounded, continuous functions convergent to $x \in \mathbf{B}(T)$. We need to show that x is a continuous function. For any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$, such that $\|x_N - x\|_\infty < \varepsilon/3$, since the sequence is convergent. Now, let $t_o \in T$. By the continuity of x_N, there exists $\delta = \delta(\varepsilon, t_o) > 0$ such that

$$d(t, t_o) < \delta \quad \Longrightarrow \quad |x_N(t) - x_N(t_o)| < \varepsilon/3.$$

Therefore, for all t such that $d(t, t_o) < \delta$ it holds that

$$\begin{aligned} |x(t) - x(t_o)| &\le |x(t) - x_N(t)| + |x_N(t) - x_N(t_o)| + |x_N(t_o) - x(t_o)| \\ &\le 2\|x - x_N\|_\infty + |x_N(t) - x_N(t_o)| \le 2 \cdot \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

which ends the proof.

Example 3. A space of continuous functions vanishing at infinity

$$\mathbf{C}_o(\mathbb{R}^n) = \left\{ f \in \mathbf{C}_b(\mathbb{R}^n) \; : \; \lim_{|t|\to+\infty} |f(t)| = 0 \right\}$$

with a $\|\cdot\|_\infty$ norm is a Banach space.

Example 4. The following spaces

$$\begin{aligned} c_o &= \left\{ \{t_n\}_{n\in\mathbb{N}} \; : \; t_n \in \mathbb{K}, \; \lim_{n\to+\infty} t_n = 0 \right\}, \\ c &= \left\{ \{t_n\}_{n\in\mathbb{N}} \; : \; t_n \in \mathbb{K}, \; \lim_{n\to+\infty} t_n \text{ exists} \right\} \end{aligned}$$

with a $\|\cdot\|_\infty$ norm are Banach spaces.

Remark 1. $\mathbf{B}(\mathbb{N})$ is often denoted by l^∞.

1.3. l^p spaces

Definition 1.13. Let $1 \le p < +\infty$. We define

$$l^p = \left\{ x \in l^\infty \; : \; \sum_{n=1}^{+\infty} |x_n|^p < +\infty \right\} \quad \text{and} \quad \|x\|_p = \sqrt[p]{\sum_{n=1}^{+\infty} |x_n|^p}.$$

Lemma 1.14. *(Hölder inequality for sequences). Let $1 \le p, q \le +\infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $x \in l^p$ and $y \in l^q$ it holds that*

(a) $x \cdot y \in l^1$,
(b) $\|x \cdot y\|_1 \leq \|x\|_p \cdot \|y\|_q$.

Lemma 1.15. *(Minkowski inequality for sequences). Let* $1 \leq p \leq +\infty$ *and* $x, y \in l^p$. *Then*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Example 5. Spaces $\left(l^p, \|\cdot\|_p\right)$ are Banach spaces for $1 \leq p \leq +\infty$.

Space $(l^\infty, \|\cdot\|_\infty)$ coincides with $(\mathbf{B}(\mathbb{N}), \|\cdot\|_\infty)$, therefore we assume that $p < +\infty$. Similarly as in the Example 1, to prove the assertion we need to show that

(a) $\|\cdot\|_p$ is a norm,
(b) each Cauchy sequence converges to an element from l^p.

Claim (a) is straightforward, when one uses the Minkowski inequality for the proof of the triangle inequality. For the proof of completness, consider a Cauchy sequence $\{x^n\}_{n\in\mathbb{N}}$. Each element $x^n \in l^p$ is a sequence given by $x^n = (x_1^n, x_2^n, \dots)$. Note that $l^p \subset l^\infty$, what holds due to the following estimate

$$\|x\|_p = \sqrt[p]{\sum_{k=1}^{+\infty} |x_k|^p} \geq \sqrt[p]{\sup_{k\in\mathbb{N}} |x_k|^p} = \sup_{k\in\mathbb{N}} |x_k| = \|x\|_\infty,$$

for $x = (x_1, x_2, \dots)$. Thus, if we consider the sequence $\{x^n\}_{n\in\mathbb{N}}$ as a sequence of elements of l^∞ space, we conclude that there exists exactly one $x \in l^\infty$ such that $\lim_{n\to+\infty} \|x^n - x\|_\infty = 0$ (this follows from the completeness of $(l^\infty, \|\cdot\|_\infty)$). In particular,

$$\lim_{n\to+\infty} x_k^n = x_k, \quad \text{for each } k \in \mathbb{N}. \tag{1.16}$$

We shall show that $x = \{x_k\}_{k\in\mathbb{N}}$ is an element of l^p space and that $\{x^n\}_{n\in\mathbb{N}}$ converges to x in l^p. For any $\varepsilon > 0$ there exists $N = N(\varepsilon)$, such that for all $n, m \geq N$ it holds that

$$\|x^n - x^m\|_p < \varepsilon.$$

In particular, for every $K \in \mathbb{N}$

$$\sqrt[p]{\sum_{k=1}^{K} \left|x_k^n - x_k^m\right|^p} \leq \|x^n - x^m\|_p < \varepsilon.$$

Using (1.16) and passing to the limit with x_k^m we obtain

$$\sqrt[p]{\sum_{k=1}^{K} \left|x_k^n - x_k\right|^p} < \varepsilon.$$

Since the estimate is valid for all K and the right hand side of the inequality is independent on K, it holds also that

$$\sqrt[p]{\sum_{k=1}^{+\infty} \left|x_k^n - x_k\right|^p} < \varepsilon.$$

Therefore $\|x^n - x\|_p < \varepsilon$, which proves that x is a limit of the sequence in l^p. Moreover,

$$\|x\|_p \le \|x - x^N\|_p + \|x^N\|_p \le \varepsilon + \|x^N\|_p < +\infty.$$

1.4. Minkowski functional

Definition 1.17. Set A is called an absorbing set if for each $x \in X$ there exists $t \in \mathbb{K}$, such that $t \cdot x \in A$.

Definition 1.18. Set A is called a balanced set if $x \in A \Rightarrow -x \in A$.

Definition 1.19. Let A be a convex, absorbing and balanced set. A functional $\mu_A : X \longrightarrow [0, +\infty)$ defined by

$$\mu_A(x) = \inf \left\{ t \in (0, +\infty) \ : \ \frac{x}{t} \in A \right\} \tag{1.20}$$

is called Minkowski functional.

Lemma 1.21. *Minkowski functional generates a seminorm on X. If additionally A is bounded in each direction, that is, for each $x \in X$ a set $(A \cap lin\{x\})$ is a bounded set, then it is a norm.*

Proof. We shall concentrate on the essential part of the proof, that is, showing that μ_A fulfills the triangle inequality. Fix $\varepsilon > 0$ and let $t = \mu_A(x) + \varepsilon$, $s = \mu_A(y) + \varepsilon$. Then, $t^{-1}x, s^{-1}y \in A$, what follows from the definition of the Minkowski functional. Set A is convex, therefore

$$\frac{t}{t+s} \cdot \frac{x}{t} + \frac{s}{t+s} \cdot \frac{y}{s} = \frac{x+y}{t+s} \in A,$$

which implies that

$$\mu_A(x+y) = \inf \left\{ z \in (0, +\infty) \ : \ \frac{x+y}{z} \in A \right\} \le t + s = \mu_A(x) + \mu_A(y) + 2\varepsilon.$$

Due to a freedom in the choice of ε the assertion is proved. □

1.4.1. Examples of normed spaces with a norm introduced by the Minkowski functional

Let $F = F(\Omega)$ be a space of real valued, Lebesgue measurable functions on Ω.

Example 6. *Orlicz spaces $L_M(\Omega)$.*
Let M be a non-negative convex function on $[0, +\infty)$, such that

$$M(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} M'(t) = +\infty.$$

Define a set

$$A = \left\{ f \in F \ : \ \int_\Omega M(|f(x)|) \, \mathrm{d}x \le 1 \right\}. \tag{1.22}$$

Orlicz space $L_M(\Omega)$ is the smallest linear space containing A. It can be checked that Minkowski functional μ_A defines a norm on $L_M(\Omega)$.

Example 7. *Lebesgue spaces*

$$\mathbf{L^p}(\Omega) = \left\{ f \in F \, : \, \int_\Omega |f(x)|^p \mathrm{d}x < +\infty \right\}.$$

The most important class of Orlicz spaces arises when we set $M(x) = x^p$, where $1 < p < +\infty$. In this case we obtain Lebesgue spaces $\mathbf{L^p}(\Omega)$. Analogously as in the example above,

$$A = \left\{ f \in F \, : \, \int_\Omega |f(x)|^p \mathrm{d}x \le 1 \right\}.$$

It turns out that Minkowski functional μ_A is given by the following formula

$$\mu_A(f) = \sqrt[p]{\int_\Omega |f(x)|^p \mathrm{d}x}.$$

Note that A is a convex, absorbing and balanced set, therefore μ_A is a seminorm on $\mathbf{L^p}(\Omega)$. Moreover, if $\mu_A(f) = 0$, then $\int_\Omega |f(x)|^p \mathrm{d}x = 0$, which implies $f = 0$ a.e. Thus, μ_A defines a norm on $\mathbf{L^p}(\Omega)$.

Example 8. *Generalized Lebesgue spaces*

$$\mathbf{L^{p(\cdot)}}(\Omega) = \left\{ f \in F \, : \, \int_\Omega |f(x)|^{p(x)} \mathrm{d}x < +\infty \right\}.$$

The next important class of Orlicz spaces is created when one sets $M(x) = x^{p(x)}$, where $p(x)$ fulfills $1 < p_1 \le p(x) \le p_2 < +\infty$ for some p_1, p_2. In this case we obtain generalized Lebesgue spaces $\mathbf{L^{p(\cdot)}}(\Omega)$. Similarly as before,

$$A = \left\{ f \in F \, : \, \int_\Omega |f(x)|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

1.5. $\mathbf{L^p}$ spaces.

Definition 1.23. Let $1 \le p < +\infty$. The space $\mathbf{L^p}(\Omega)$ consists of equivalence classes of Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega |f(x)| \mathrm{d}x < +\infty,$$

where two measurable functions are equivalent if they are equal a.e. The $\mathbf{L^p}$ norm of $f \in \mathbf{L^p}(\Omega)$ is defined by

$$\| \cdot \|_{\mathbf{L^p}} = \sqrt[p]{\int_\Omega |f(x)| \mathrm{d}x}.$$

For $p = +\infty$ the definition is slightly different. We say that a function f is essentially bounded, if

$$\operatorname{essup}|f| = \inf_{N : |N| = 0} \sup_{(\Omega \setminus N)} |f(x)| < +\infty.$$

The space $\mathbf{L^\infty}(\Omega)$ consists of equivalence classes (two functions are equivalent if they are equal a.e.) of measurable, essentially bounded functions $f : \Omega \to \mathbb{R}$ with a norm

$$\| \cdot \|_{\mathbf{L^\infty}} = \operatorname{essup}|f|.$$

Remark 2. The reason to regard functions that are equal a.e. as equivalent is so that $\|f\|_{\mathbf{L^p}} = 0$ implies that $f = 0$ and thus $\|\cdot\|_{\mathbf{L^p}}$ is a norm. For example, the characteristic function of the rational numbers $\mathbb{Q}$ is equivalent to 0 in $\mathbf{L^p}(\mathbb{R})$, for $1 \le p \le +\infty$.

Lemma 1.24. *(Hölder inequality for integrals). Let $1 \le p \le +\infty$ and $\frac{1}{p} + \frac{1}{g} = 1$. Let $f \in \mathbf{L^p}(\Omega)$, $g \in \mathbf{L^q}(\Omega)$. Then, $f \cdot g \in \mathbf{L^1}(\Omega)$ and*

$$\|f \cdot g\|_{\mathbf{L^1}} \le \|f\|_{\mathbf{L^p}} \cdot \|g\|_{\mathbf{L^q}}.$$

Theorem 1.25. *Orlicz spaces $L_M(\Omega)$, Lebesgue spaces $\mathbf{L^p}(\Omega)$ and generalized Lebesgue spaces $\mathbf{L^{p(\cdot)}}(\Omega)$ are Banach spaces.*

We prove the theorem only for the Lebesgue spaces. In the proof we shall use the following lemma.

Lemma 1.26. *For any normed space $(X, \|\cdot\|)$ the following conditions are equivalent*

(a) $(X, \|\cdot\|)$ is a complete space.
(b) If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X, such that $\sum_{n=1}^{+\infty} \|x_n\| < +\infty$, then there exists $x \in X$ such that

$$\lim_{N\to+\infty} \|\sum_{n=1}^{N} x_n - x\| = 0.$$

The condition (b) simply states that any absolutely convergent series is convergent.

Proof of Lemma 1.26.
(a) $\Rightarrow$ (b). The implication follows from the fact that $S_N = \sum_{n=1}^{N} x_n$ is a Cauchy sequence.
(b) $\Rightarrow$ (a). Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence. For each $k \in \mathbb{N}$ there exists N_k, such that

$$\|x_m - x_n\| < 2^{-k}, \quad \forall n, m \ge N_k.$$

We choose a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\|x_{n_{k+1}} - x_{n_k}\| < 2^{-k}, \quad \forall k \in \mathbb{N}$$

and denote $y_1 = x_{n_1}$, $y_k = (x_{n_{k+1}} - x_{n_k})$ for $k > 1$. Therefore $\sum_{i=1}^{+\infty} \|y_i\| < +\infty$. From assumptions it follows that there exists $y \in X$, such that

$$\lim_{N\to+\infty} \|\sum_{n=1}^{N} y_n - y\| = \lim_{N\to+\infty} \|x_{n_{N+1}} - y\| = 0.$$

Therefore, the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges in X. A Cauchy sequence, which has a convergent subsequence, converges as well, which ends the proof. □

Proof of Theorem 1.25 for the Lebesgue spaces, $p < +\infty$.

Checking that $\|\cdot\|_{\mathbf{L^p}}$ is a norm, when one uses Minkowski inequality, is straightforward. Note that we have proved the assertion in an alternative way for $1 < p < +\infty$ in the Example 7. For the proof of completeness we shall use claim (b) from Lemma 1.9. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbf{L^p}(\Omega)$ such that $M = \sum_{n=1}^{+\infty} \|f_n\|_{\mathbf{L^p}} < +\infty$. We need to construct a function $f \in \mathbf{L^p}(\Omega)$, such that $\lim_{N\to+\infty} \| \sum_{n=1}^{N} f_n - f\|_{\mathbf{L^p}} = 0$. Define $\hat{g}_n, \hat{g} : \Omega \to \mathbb{R}$ as following

$$\hat{g}_n(x) = \sum_{i=1}^{n} |f_i(x)| \quad \text{and} \quad \hat{g}(x) = \sum_{n=1}^{+\infty} |f_n(x)|.$$

From Minkowski inequality we obtain

$$\|\hat{g}_n\|_{\mathbf{L^p}} = \| \sum_{i=1}^{n} |f_i| \|_{\mathbf{L^p}} \leq \sum_{i=1}^{n} \|f_i\|_{\mathbf{L^p}} \leq \sum_{n=1}^{+\infty} \|f_n\|_{\mathbf{L^p}} = M < +\infty,$$

By construction, $\hat{g}_n$ converges monotonically to $\hat{g}$. Therefore, from the monotone convergence theorem and the inequality above it follows that

$$\int_\Omega (\hat{g}(x))^p \,\mathrm{d}x = \int_\Omega \lim_{n\to+\infty} (\hat{g}_n(x))^p \,\mathrm{d}x = \lim_{n\to+\infty} \int_\Omega (\hat{g}_n(x))^p \,\mathrm{d}x < M^p.$$

which implies that $\hat{g} \in \mathbf{L^p}(\Omega)$ and in particular $\hat{g}$ is finite a.e. From the latter fact we conclude that

$$f(x) := \sum_{n=1}^{+\infty} f_n(x)$$

is finite a.e. and $f \in \mathbf{L^p}(\Omega)$ with $\|f\|_{\mathbf{L^p}} \leq \|\hat{g}\|_{\mathbf{L^p}}$. Note that

$$0 \leq \left| f(x) - \sum_{i=1}^{n} f_i(x) \right|^p = \left| \sum_{i=n+1}^{+\infty} f_i(x) \right|^p \leq \left(\sum_{i=n+1}^{\infty} |f_i(x)| \right)^p \leq (\hat{g}(x))^p < M^p.$$

Thus, by the Lebesgue dominated convergence theorem

$$\lim_{n\to+\infty} \int_\Omega \left| f(x) - \sum_{i=1}^{n} f_i(x) \right|^p \mathrm{d}x = 0,$$

which ends the proof due to the Lemma 1.26. □

LECTURE 2
Linear Operators

Definition 2.1. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces. A space consisting of linear, bounded operators $A : E \to F$ is denoted as $\alpha(E, F)$. A bounded operator is understood here as the operator, which maps bounded sets onto bounded sets.

Remark 3. $\alpha(E, F)$ is a normed space with a norm given by

$$\|A\|_{\alpha(E,F)} = \sup_{x \in E, x \neq 0} \frac{\|A(x)\|_F}{\|x\|_E}.$$

Remark 4. For any $A \in \alpha(E, F)$ the following equalities hold

$$\|A\|_{\alpha(E,F)} = \sup_{x\,:\,\|x\|_E \leq 1} \|A(x)\|_F = \sup_{x\,:\,\|x\|_E = 1} \|A(x)\|_F. \tag{2.2}$$

Theorem 2.3. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces and $T : E \to F$ be a linear operator. Then, the following conditions are equivalent:*

(a) T is continuous,
(b) T is continuous at 0,
(c) there exists M such that for each $x \in E$ it holds that $\|Tx\|_F \leq M\|x\|_E$,
(d) T is uniformly continuous.

Proof of Theorem 2.3.
$(c) \Rightarrow (d)$. Condition (c) implies that T is Lipschitz continuous and thus, also uniformly continuous.
$(d) \Rightarrow (a) \Rightarrow (b)$. The proof is trivial.
$(b) \Rightarrow (c)$. We shall prove this implication by contradiction. Assume that T is continuous at 0 and (c) does not hold, which implies that there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that

$$\|Tx_n\|_F > n\|x_n\|_E.$$

Define $y_n = \dfrac{x_n}{n\|x_n\|_E}$. Then, $\|y_n\|_E = 1/n$ and $\lim_{n\to+\infty}\|y_n - 0\|_E = 0$. Moreover,

$$\|Ty_n\|_F = \frac{\|Tx_n\|_F}{n\|x_n\|_E} > 1.$$

Therefore, $\lim_{n\to+\infty}\|y_n - 0\|_E = 0$ and $\|Ty_n - T(0)\|_F > 1$, which is a contradiction due to the fact that T is continuous. □

Lemma 2.4. *In an infinite dimensional Banach space there exist unbounded operators, which are defined everywhere.*

Lemma 2.5. *Let $D \subset E$ be a dense subset of a normed space $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be a Banach space. Then, for each $T \in \alpha(D,F)$ there exists a unique $\hat{T}$ such that*

$$\hat{T} : E \longrightarrow F, \quad \hat{T}|_D = T, \quad \text{and} \quad \|\hat{T}\|_{\alpha(E,F)} = \|T\|_{\alpha(D,F)}.$$

Proof of Lemma 2.5. For each $x \in E$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$, such that $\lim_{n\to+\infty}\|x_n - x\|_E = 0$. Define an operator

$$\hat{T}x = \lim_{n\to+\infty} Tx_n.$$

The operator $\hat{T}$ is well defined, since $\{Tx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and F is a Banach space, thus the limit exists. Note that $\hat{T}$ is unique (it does not depend on the choice of the sequence $\{x_n\}_{n\in\mathbb{N}}$). Indeed, if there exist sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{x'_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} x_n = \lim_{k\to+\infty} x'_k = x,$$

then $\lim_{n,k\to+\infty}\|x_n - x'_k\|_E = 0$ and from the boundedness of T,

$$\lim_{n,k\to+\infty}\|Tx_n - Tx'_k\|_F \le M \lim_{n,k\to+\infty}\|x_n - x'_k\|_E = 0.$$

Moreover, $\hat{T}$ is bounded since

$$\|\hat{T}x\|_F = \lim_{n\to+\infty}\|Tx_n\|_F \le \lim_{n\to+\infty} M\|x_n\|_E \le \sup_{n\in\mathbb{N}} M\|x_n\|_E < C.$$

It also holds that

$$\|\hat{T}\|_{\alpha(E,F)} = \sup_{x\in E,\, x\neq 0}\frac{\|\hat{T}(x)\|_F}{\|x\|_E} = \sup_{x\in D, x\neq 0}\frac{\|\hat{T}(x)\|_F}{\|x\|_D} = \sup_{x\in D, x\neq 0}\frac{\|T(x)\|_F}{\|x\|_D} = \|T\|_{\alpha(D,F)}.$$

□

Theorem 2.6. *Let* $(F, \|\cdot\|_F)$ *be a Banach space. Then,* $(\alpha(E,F), \|\cdot\|_{\alpha(E,F)})$ *is a Banach space as well.*

Proof of Theorem 2.6. We need to prove that a normed space $(\alpha(E,F), \|\cdot\|_{\alpha(E,F)})$ is complete. Let $\{A_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\alpha(E,F)$, that is,

$$\forall\varepsilon > 0 \quad \exists N = N(\varepsilon) \quad \forall n,m \geq N \quad \|A_n - A_m\|_{\alpha(E,F)} = \sup_{x\in E,\, x\neq 0} \frac{\|A_n x - A_m x\|_F}{\|x\|_E} < \varepsilon, \tag{2.7}$$

which implies that $\{A_n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in F for each fixed $x \in E$ as well. Since F is a Banach space, this sequence is convergent. Thus, an operator $A : E \to F$ given by

$$Ax = \lim_{n\to+\infty} A_n x$$

is well defined. It is also linear and bounded. The latter claim holds due to the following

$$\frac{\|Ax\|_F}{\|x\|_E} = \lim_{n\to+\infty} \frac{\|A_n x\|_F}{\|x\|_E} \leq \sup_{n\in\mathbb{N}} \frac{\|A_n x\|_F}{\|x\|_E} < +\infty.$$

Now, we want to prove that $\lim_{n\to+\infty} \|A_n - A\|_{\alpha(E,F)} = 0$. Let x be such that $\|x\|_E = 1$. Condition (2.7) implies then

$$\forall\varepsilon > 0 \quad \exists N = N(\varepsilon) \quad \forall n,m > N \quad \|A_n x - A_m x\|_F < \varepsilon.$$

If we let $m \to +\infty$, then $\|A_n x - Ax\|_F < \varepsilon$. This holds for each x such that $\|x\|_E = 1$, therefore

$$\sup_{x\,:\,\|x\|_E = 1} \|A_n x - Ax\|_F < \varepsilon,$$

which ends the proof due to the equality (2.2). □

Lemma 2.8. *Let* X, Y, Z *be normed spaces and* $S \in \alpha(X,Y)$*,* $T \in \alpha(Y,Z)$*. Then,*

$$\|T \circ S\|_{\alpha(X,Z)} \leq \|T\|_{\alpha(Y,Z)} \cdot \|S\|_{\alpha(X,Y)}.$$

Proof of Lemma 2.8. Let $x \in X$ and $x \neq 0$. Then,

$$\|(T\circ S)(x)\|_Z = \|T(Sx)\|_Z \leq \|T\|_{\alpha(Y,Z)} \cdot \|Sx\|_Y \leq \|T\|_{\alpha(Y,Z)} \cdot \|S\|_{\alpha(X,Y)} \cdot \|x\|_X.$$

Thus,

$$\frac{\|(T\circ S)(x)\|_Z}{\|x\|_X} \leq \|T\|_{\alpha(Y,Z)} \cdot \|S\|_{\alpha(X,Y)}.$$

Taking supremum over $x \in X$ ends the proof. □

Example 9. Examples of linear operators $T : X \to Y$:

(a) *Identity* : $X = Y$, $T = \mathrm{Id}$. The norm $\|T\|_{\alpha(X,Y)}$ of T is equal to 1.

(b) *A linear map between finite dimensional spaces* : $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ and $A \in \mathbb{R}^n \times \mathbb{R}^m$ is a matrix $A = \{a_{ij}\}$, $1 \le i \le n$, $1 \le j \le m$,

$$T(x) = A \cdot x.$$

(c) *Differentiation* : $X = \mathbf{C}^1([0,1])$, $Y = \mathbf{C}([0,1])$,

$$T(f) = f'.$$

(d) Let $g \in \mathbf{L^q}([0,1],\mu)$ and p, q be such that $\frac{1}{p} + \frac{1}{q} = 1$. $X = \mathbf{L^p}([0,1],\mu)$, $Y = \mathbf{L^1}([0,1],\mu)$,

$$T_g(f) = \int_0^1 fg \, \mathrm{d}\mu.$$

(e) *Fredholm integral operator* : $X = Y = \mathbf{C}([0,1])$,

$$(T(f))(y) = \int_0^1 k(x,y) f(x) \mathrm{d}x,$$

where $k \in \mathbf{C}([0,1] \times [0,1])$.

(f) An operator defined analogously as in the example (e), but with $X = Y = \mathbf{L^2}([0,1],\mu)$ and $k \in \mathbf{L^2}([0,1] \times [0,1],\mu)$.

Remark 5. (to example (e)) Note that continuity of $T(f)$ follows directly from the uniform continuity of k. Indeed, from a definition of uniform continuity

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \left(|y - y'| < \delta \ \Rightarrow \ \sup_{x \in [0,1]} |k(x,y) - k(x,y')| < \varepsilon \right).$$

Therefore, it also holds that

$$|(T(f))(y) - (T(f))(y')| \le \int_0^1 |k(x,y) - k(x,y')| \cdot |f(x)| \, \mathrm{d}x \le \varepsilon \, \|f\|_\infty.$$

Note that

$$\|Tf\|_\infty = \sup_{y \in [0,1]} \left| \int_0^1 k(x,y) f(x) \mathrm{d}x \right| \le \|f\|_\infty \cdot \sup_{y \in [0,1]} \|k(\cdot,y)\|_\infty \le \|f\|_\infty \|k\|_\infty.$$

Definition 2.9. An operator $T \in \alpha(X,Y)$ is called an isomorphism, if there exists an inverse operator $T^{-1} \in \alpha(Y,X)$. Moreover, if

$$\|T\|_{\alpha(X,Y)} = \|T^{-1}\|_{\alpha(X,Y)} = 1,$$

then T is called an isometry and X, Y are said to be isometrically isomorphic.

Two isometrically isomorphic normed spaces share the same structure, so they are usually identified with each other.

Theorem 2.10. *Let $(X, \|\cdot\|_X)$ be a normed space and $T \in \alpha(X,X) =: \alpha(X)$. Then, the following implication holds*

$$\sum_{n=0}^{+\infty} T^n \text{ converges in } \alpha(X) \quad \Rightarrow \quad (\mathrm{Id} - T)^{-1} \text{ exists and } (\mathrm{Id} - T)^{-1} = \sum_{n=0}^{+\infty} T^n,$$

where $T^n = \underbrace{T \circ \cdots \circ T}_{n \text{ times}}$.

Proof of Theorem 2.10. Define $S_m = \sum_{n=0}^{m} T^n$. Then, $(\mathrm{Id} - T)S_m = S_m(\mathrm{Id} - T)$ and

$$(\mathrm{Id} - T)S_m = \mathrm{Id} \circ S_m - T \circ S_m = \sum_{n=0}^{m} T^n - \sum_{n=1}^{m+1} T^n = \mathrm{Id} - T^{m+1}.$$

Since $\sum_{n=0}^{+\infty} T^n$ converges in $\alpha(X)$, we have that $\lim_{n\to+\infty} T^n x = 0$ for each $x \in X$. Therefore, passing to the limit in the equalities above yields

$$\mathrm{Id}(x) = \lim_{m\to+\infty} (\mathrm{Id} - T^{m+1})x = \lim_{m\to+\infty} (\mathrm{Id} - T)S_m x = (\mathrm{Id} - T) \lim_{m\to+\infty} S_m x = (\mathrm{Id} - T)(\sum_{n=0}^{+\infty} T^n)x.$$

Similar argument proves that $\mathrm{Id}(x) = (\sum_{n=0}^{+\infty} T^n)(\mathrm{Id} - T)x$. Therefore,

$$(\mathrm{Id} - T)^{-1}x = \sum_{n=0}^{+\infty} T^n x, \quad \forall x \in X.$$

□

Remark 6. If $(X, \|\cdot\|)_X$ is a Banach space it is sufficient to assume that $\|T\|_{\alpha(X)} < 1$. In this case

$$\|(\mathrm{Id} - T)^{-1}\|_{\alpha(X)} \le \frac{1}{1 - \|T\|_{\alpha(X)}}. \tag{2.11}$$

Indeed, note that

$$\sum_{n=0}^{+\infty} \|T^n\|_{\alpha(X)} \le \sum_{n=0}^{+\infty} \|T\|^n_{\alpha(X)} < +\infty,$$

since $\|T\|_{\alpha(X)} < 1$. Since X is a Banach space, then the convergence of the series $\sum_{n=0}^{+\infty} T^n$ follows from Lemma 1.26. The inequality (2.11) follows from a formula on a sum of a geometric series.

Exercise 1. Find a solution $x \in \mathbf{C}([0,1])$ to the following equation

$$x(s) - \int_0^1 k(s,t)x(t)\,\mathrm{d}t = y(s),$$

where $k \in \mathbf{C}([0,1] \times [0,1])$ and $y \in \mathbf{C}([0,1])$ are given.

LECTURE 3
Dual Spaces

Definition 3.1. Let $(X, \|\cdot\|_X)$ be a normed space. Space $\alpha(X, \mathbb{K})$ consisting of linear, bounded functionals on X is called a dual space to X. It is denoted by X' or X^*.

Remark 7. Dual space X' is a Banach space even if X is not a Banach space.

Theorem 3.2. *Let $1 \le p, q < +\infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, an operator*

$$T : l^q \to (l^p)', \quad (Tx)(y) = \sum_{n=1}^{+\infty} x_n y_n,$$

where $x = (x_1, x_2, \dots) \in l^q, y = (y_1, y_2, \dots) \in l^p$, is an isometric isomorphism.

Proof of Theorem 3.2. Linearity of T is straightforward. T is bounded, which follows from Hölder inequality

$$|(Tx)(y)| \le \|x\|_q \cdot \|y\|_p \quad \Rightarrow \quad \|Tx\|_{(l^p)'} \le \|x\|_q. \tag{3.3}$$

To show that T is an isomorphism we need to prove that it is injective and surjective. Indeed,

$$Tx = 0 \quad \Rightarrow \quad x_n = (Tx)(e_n) = 0, \quad \forall n \in \mathbb{N},$$

where $e_n = (0, \dots, 0, 1, 0, \dots)$, that is, the n-th coordinate is the only non-zero element of e_n. Thus, $\ker(T) = \{0\}$, which proves that T is an injection. For the proof of surjectivity we have to show that

$$\forall y' \in (l^p)' \;\; \exists x \in l^q \;\; \text{such that} \;\; T(x) = y', \;\; \text{i.e.,} \;\; T(x)(y) = y'(y) \;\; \forall y \in l^p.$$

It is sufficient to prove that the equality holds for $y = e_n$, $n \in \mathbb{N}$. Indeed, assume that $T(x)(e_n) = y'(e_n)$ for each $n \in \mathbb{N}$. Since $T(x)$ and y' are linear, then the equality holds for all $y \in \{\text{lin}\{e_n\} : n \in \mathbb{N}\}$. However, the operators are also continuous and therefore the equality holds for

$$\overline{\{\text{lin}\{e_n\} : n \in \mathbb{N}\}}^{\|\cdot\|_p} = l^p$$

as well. Now, fix $y' \in (l^p)'$ and define $s_n = y'(e_n)$, $x = \{s_n\}_{n\in\mathbb{N}}$. We prove that $x \in l^q$. To this end, let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence given by

$$t_n = \begin{cases} \frac{|s_n|^q}{s_n} & \text{for } s_n \neq 0, \\ 0 & \text{for } s_n = 0. \end{cases}$$

Therefore, for each $N \in \mathbb{N}$ it holds that

$$\sum_{n=1}^{N} |t_n|^p = \sum_{n=1}^{N} |s_n|^{p(q-1)} = \sum_{n=1}^{N} |s_n|^q$$

and

$$\begin{aligned} \sum_{n=1}^{N} |s_n|^q &= \sum_{n=1}^{N} t_n s_n = \sum_{n=1}^{N} t_n y'(e_n) = y'\left(\sum_{n=1}^{N} t_n e_n\right) \\ &\leq \|y'\|_{(l^p)'} \cdot \left(\sum_{n=1}^{N} |t_n|^p\right)^{\frac{1}{p}} = \|y'\|_{(l^p)'} \cdot \left(\sum_{n=1}^{N} |s_n|^q\right)^{\frac{1}{p}}. \end{aligned}$$

Therefore,

$$\left(\sum_{n=1}^{N} |s_n|^q\right)^{\frac{1}{q}} \leq \|y'\|_{(l^p)'} \quad \Rightarrow \quad \|x\|_q \leq \|y'\|_{(l^p)'} \quad \Rightarrow \quad x \in l^q,$$

since the first inequality holds for all $N \in \mathbb{N}$. It is straightforward now, that

$$T(x)(e_n) = s_n = y'(e_n).$$

The fact that $\|x\|_q \leq \|y'\|_{(l^p)'}$ together with (3.3) imply that T is an isometry. □

Theorem 3.4. *Let $1 \leq p < +\infty$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$ and (Ω, Σ, μ) be a measure space, where μ is a σ-finite measure. Then,*

$$T : \mathbf{L^q}(\Omega, \mu) \to (\mathbf{L^p}(\Omega, \mu))', \quad (Tg)(f) = \int_{\Omega} fg \, \mathrm{d}\mu$$

is an isometric isomorphism.

Proof of Theorem 3.4. T is clearly a linear operator. It follows from Hölder inequality that

$$|T(g)(f)| \leq \|fg\|_{\mathbf{L^1}} \leq \|f\|_{\mathbf{L^p}} \cdot \|g\|_{\mathbf{L^q}} \quad \Rightarrow \quad \|Tg\|_{(\mathbf{L^p})'} \leq \|g\|_{\mathbf{L^q}}.$$

Thus, the norm of the operator T is at most 1. In particular, the function f defined as below

$$f = \frac{\bar{g}}{|g|}\left(\frac{|g|}{\|g\|_{\mathbf{L^q}}}\right)^{\frac{q}{p}}, \quad \text{for } 1 < p < +\infty,$$

(where $\bar{g}$ is a complex conjugate of g) is such that $\|f\|_{\mathbf{L^p}} = 1$ and

$$(Tg)(f) = \int fg \, \mathrm{d}\mu = \|g\|_{\mathbf{L^q}},$$

which implies that $\|Tg\|_{(\mathbf{L^p})'} \geq \|g\|_{\mathbf{L^q}}$ and thus T is an isometry. For $p = 1$ we set $f = \frac{\bar{g}}{|g|}$ and use similar arguments.

Injectivity of T is straightforward. Indeed, assume that $T(g) = 0$, which implies that

$$\int_\Omega fg \, \mathrm{d}\mu = 0, \quad \forall f \in \mathbf{L^p}(\Omega, \mu).$$

Thus, $g = 0$ $\mu - a.e.$ and $g = 0$ in $\mathbf{L^q}(\Omega,\mu)$. Now, we need to show that the operator is surjective, that is,

$$\forall y' \in (\mathbf{L^p})' \;\; \exists g \in \mathbf{L^q} \;\text{ such that }\; T(g) = y', \;\text{ i.e., }\; (Tg)(f) = \int_\Omega fg \, \mathrm{d}\mu = y'(f), \quad \forall f \in \mathbf{L^p}.$$

Assume that $\mu(\Omega) < +\infty$ and define

$$\nu : \Sigma \to \mathbb{K}, \quad \nu(E) = y'(\chi_E).$$

Since μ is finite, $\chi_E \in \mathbf{L^p}(\Omega,\mu)$ and thus ν is well defined. It is clear that ν is additive and in the case $p < +\infty$ it is also σ-additive (why?). Thus, ν is a signed (or complex) measure. From the construction it follows that ν is absolutely continuous with respect to μ. Indeed,

$$\mu(E) = 0 \quad \Rightarrow \quad \chi_E = 0 \;\; \mu - a.e. \quad \Rightarrow \quad \chi_E \in \mathbf{L^p}(\Omega,\mu).$$

Since $\chi_E = 0$ in $\mathbf{L^p}(\Omega,\mu)$, it holds that

$$\nu(E) = y'(\chi_E) = y'(0) = 0.$$

By Radon-Nikodym theorem, there exists $g \in \mathbf{L^1}(\Omega,\mu)$ such that

$$y'(\chi_E) = \nu(E) = \int_E g \, \mathrm{d}\mu = \int_\Omega \chi_E g \, \mathrm{d}\mu, \quad \forall E \in \Sigma.$$

Characteristic functions are dense in $(\mathbf{L}^\infty, \|\cdot\|_{\mathbf{L}^\infty})$ (and in $(\mathbf{L^p}, \|\cdot\|_{\mathbf{L^p}})$), which implies that

$$y'(f) = \int_\Omega fg \, \mathrm{d}\mu \quad \forall f \in \mathbf{L}^\infty(\Omega,\mu). \tag{3.5}$$

Now, it remains to prove that $g \in \mathbf{L^q}(\Omega,\mu)$. Cases for $q < +\infty$ and $q = +\infty$ are considered separately. Assume that $q < +\infty$ and define a μ-measurable function

$$f(x) = \begin{cases} \frac{|g(x)|^q}{g(x)} & \text{if } g(x) \neq 0, \\ 0 & \text{if } g(x) = 0. \end{cases}$$

Then, $|g(x)|^q = (fg)(x) = |f(x)|^p$. Define a μ-measurable set

$$E_n = \{x \;:\; |g(x)| \le n\}.$$

Then, $\chi_{E_n} f \in \mathbf{L}^\infty(\Omega,\mu)$ and thus we can plug it into (3.5).

$$\begin{aligned} \int_\Omega (\chi_{E_n} f) g \, \mathrm{d}\mu &= y'(\chi_{E_n} f) \le \|y'\|_{(\mathbf{L^p})'} \cdot \|\chi_{E_n} f\|_{\mathbf{L^p}} \\ &= \|y'\|_{(\mathbf{L^p})'} \left(\int_{E_n} |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} = \|y'\|_{(\mathbf{L^p})'} \left(\int_{E_n} |g|^q \, \mathrm{d}\mu \right)^{\frac{1}{p}}, \end{aligned}$$

On the other hand

$$\int_\Omega (\chi_{E_n} f) g \, \mathrm{d}\mu = \int_{E_n} |g|^q \, \mathrm{d}\mu.$$

Combining the both expressions we obtain

$$\left(\int_{E_n} |g|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}} \leq \|y'\|_{(\mathbf{L^p})'}.$$

The inequality above holds also when the integral is taken over the whole Ω. Indeed,

$$\sup_{n\in\mathbb{N}} \int_{E_n} |g|^q \, \mathrm{d}\mu = \lim_{n\to\mathbb{N}} \int_{E_n} |g|^q \, \mathrm{d}\mu = \int_{\Omega} |g|^q \, \mathrm{d}\mu,$$

due to the monotone convergence theorem. Thus,

$$\left(\int_{\Omega} |g|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}} = \|g\|_{\mathbf{L^q}} \leq \|y'\|_{(\mathbf{L^p})'}.$$

Now, consider the case $q = +\infty$. Define

$$E = \left\{x \,:\, g(x) > \|y'\|_{(\mathbf{L^1})'}\right\} \quad \text{and} \quad f(x) = \chi_E(x)\,\frac{|g(x)|}{g(x)}.$$

It is clear that $f \in \mathbf{L}^\infty(\Omega,\mu)$ and $\|f\|_{\mathbf{L^1}} = \mu(E)$. If $\mu(E) > 0$, then

$$\mu(E)\|y'\|_{(\mathbf{L^1})'} < \int_E |g| \, \mathrm{d}\mu = \int_\Omega fg \, \mathrm{d}\mu = y'(f) \leq \|y'\|_{(\mathbf{L^1})'} \cdot \|f\|_{\mathbf{L^1}}$$

and $\mu(E) < \|f\|_{\mathbf{L^1}}$, which is a contradiction. □

Definition 3.6. Let Σ be a σ-algebra of subsets of Ω and $\mathbb{K} = \{\mathbb{N}, \mathbb{C}\}$. $\nu : \Sigma \to \mathbb{K}$ is called a signed (or complex) finite measure, if

(a) $\nu(\varnothing) = 0$,
(b) $\nu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \nu(A_i)$, if $A_i \cap A_j = \varnothing$ for $i \neq j$.

Theorem 3.7. *Let Σ be a σ-algebra of subsets of Ω, μ be a finite measure and ν be a signed (or complex) measure on Σ. Moreover, let ν be absolutely continuous with respect to μ, that is,*

$$\mu(E) = 0 \quad \Rightarrow \quad \nu(E) = 0.$$

Then, there exists $g \in \mathbf{L^1}(\Omega,\mu)$ such that

$$\nu(E) = \int_E g \, \mathrm{d}\mu, \quad \forall E \in \Sigma.$$

3.1. Extensions of functionals

Definition 3.8. Let X be a vector space. Functional $p : X \to \mathbb{R}$ is called sublinear, if

(a) $p(\lambda x) = \lambda p(x), \quad \forall \lambda \geq 0, x \in X,$
(b) $p(x+y) \leq p(x) + p(y), \quad \forall x, y \in X.$

Remark 8. Any seminorm is a sublinear functional.

Theorem 3.9. *(Hahn-Banach) Let X be a vector space and U be its subspace. If*

$$p : X \to \mathbb{R} \quad \text{is a sublinear functional,}$$
$$l : U \to \mathbb{R} \quad \text{is a linear functional such that } l(x) \le p(x),\ \forall x \in U,$$

then there exists an extension $L : X \to \mathbb{R}$, such that

$$L|_U = l \quad \text{and} \quad L(x) \le p(x),\ \forall x \in X.$$

In the proof we shall use the following lemma.

Lemma 3.10. *(Zorn lemma) Let $(A, \le)$ be a non-empty partially ordered set, such that every non-empty totally ordered subset has an upper bound in A. Then, the set A contains at least one maximal element.*

A totally ordered set is the set with partial order under which every pair of elements is comparable.

Proof of Theorem 3.9. Assume that $\dim(X/U) = 1$. With this assumption our task is to extend the functional l to $X = U \oplus \mathbb{R}x_o$, where $x_o \in X \setminus U$. Note, that in this case any $x \in X$ can be written as $x = u + \lambda x_o$, where $u \in U$ and $\lambda \in \mathbb{R}$. Define

$$L_r(x) = l(u) + \lambda r. \tag{3.11}$$

We claim that L_r is a linear extension of l, such that $L_r(x) \le p(x)$. If $\lambda = 0$, we have that $L_r(x) = l(u) + \lambda r = l(u) \le p(u)$. Hence, we can assume that $\lambda \ne 0$. Let $\lambda > 0$. Then, inequality $L_r(x) \le p(x)$ is equivalent to the following

$$\begin{aligned} \lambda r &\le p(u + \lambda x_o) - l(u) \\ r &\le p\left(\frac{u}{\lambda} + x_o\right) - l\left(\frac{u}{\lambda}\right) \\ r &\le \inf_{v \in U} (p(v + x_o) - l(v)). \end{aligned}$$

Similarly, for $\lambda < 0$ we obtain

$$\begin{aligned} \lambda r &\le p(u + \lambda x_o) - l(u) \\ -r &\le p\left(\frac{u}{-\lambda} - x_o\right) - l\left(\frac{u}{-\lambda}\right) \\ r &\ge \sup_{w \in U} (l(w) - p(w - x_o)). \end{aligned}$$

Therefore, there exists $r \in \mathbb{R}$ such that $L_r \le p$, if

$$l(w) - p(w - x_o) \le p(v + x_o) - l(v),$$

which is equivalent to

$$l(v) + l(w) \le p(v + x_o) + p(w - x_o), \quad \forall v, w \in U.$$

But $l(v) + l(w) = l(v + w) \le p(v + w) \le p(v + x_o) + p(w - x_o)$, since $v + w \in U$ and p is sublinear.

Now, consider the collection

$$A = \{(V, L) \,:\, V \subset X \text{ is a linear subspace, } L : V \to \mathbb{R} \text{ is a linear extension of } l \text{ such that } L(x) \le p(x),\ \ \forall x \in V\}.$$

We define a partial order on A by setting

$$(V_1, L_1) \le (V_2, L_2) \quad \Leftrightarrow \quad V_1 \subset V_2 \text{ and } V_2|_{Z_1} = V_1.$$

Now, we check that every chain (i.e., totally ordered set)

$$\{(V_i, L_i) \,:\, i \in I\}$$

in A has an upper bound. Let $\bar{V} = \bigcup_{i \in I} V_i$. Thus, $\bar{V}$ is a subset of X and $\bar{V}$ contains each V_i. Let define $\bar{L} : \bar{V} \to \mathbb{R}$ in the following way. If $x \in \bar{V}$, then there exists $i \in I$ such that $x \in V_i$ and we set $\bar{L}(x) = V_i(x)$. The definition does not depend on the choice of i because of the chain definition. Moreover, it implies that $\bar{L}$ is linear and $(V_i, L_i) \le (\bar{V}, \bar{L})$ for every $i \in I$. In other words this is an upper bound for the chain. Zorn's lemma guarantees the existence of a maximal element $(V, L) \in A$. If $V \ne X$, then it follows from the first part of the proof, that we can extend the functional and thus, it is a contradiction due to the fact that (V, L) is maximal. □

Theorem 3.12. *Let X be a $\mathbb{C}$-vector space. Then,*

(a) If $l : X \to \mathbb{R}$ is a $\mathbb{R}$-linear functional, that is,

$$l(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 l(x_1) + \lambda_2 (x_2), \ \ \forall \lambda_1, \lambda_2 \in \mathbb{R}, x \in X,$$

then,

$$h(x) = l(x) - i l(ix)$$

is a $\mathbb{C}$-linear functional such that $\operatorname{Re} h = l$.

(b) If $h : X \to \mathbb{C}$ is a $\mathbb{C}$-linear functional, then $l = \operatorname{Re} h$ is a $\mathbb{R}$-linear functional.

(c) If p is a seminorm and $h : X \to \mathbb{C}$ is a $\mathbb{C}$-linear functional, then for all $x \in X$ the following equivalence holds

$$|h(x)| \le p(x) \quad \Leftrightarrow \quad |\operatorname{Re} h(x)| \le p(x).$$

(d) If $(X, \|\cdot\|_X)$ is a normed space and $h : X \to \mathbb{C}$ is a $\mathbb{C}$-linear continuous functional, then

$$\|h\|_{\alpha(X,\mathbb{C})} = \|\operatorname{Re} h\|_{\alpha(X,\mathbb{R})}.$$

Theorem 3.13. *(Hahn-Banach theorem for $\mathbb{C}$) Let X be a $\mathbb{C}$-vector space and U be its subspace. If*

$$p : X \to \mathbb{R} \quad \text{is a sublinear functional,}$$
$$l : U \to \mathbb{C} \quad \text{is a linear functional such that } \operatorname{Re} l(x) \le p(x), \ \forall x \in U,$$

then there exists an extension $L : X \to \mathbb{C}$, such that

$$L|_U = l \quad \text{and} \quad \operatorname{Re} L(x) \le p(x), \ \forall x \in X.$$

Theorem 3.14. *Let $(X, \|\cdot\|_X)$ be a normed space and U be its subspace. Then, for every continuous linear functional $u' : U \to \mathbb{K}$ there exists a continuous linear funtional $x' : X \to \mathbb{K}$, such that*

$$x'|_U = u' \quad \text{and} \quad \|x'\|_{\alpha(X,\mathbb{K})} = \|u'\|_{\alpha(U,\mathbb{K})}.$$

Proof of Theorem 3.14. (only for $\mathbb{K} = \mathbb{R}$) Define $p : X \to \mathbb{R}$ as following

$$p(x) = \|u'\|_{\alpha(U,\mathbb{K})} \cdot \|x\|_X.$$

It is clear that p is a sublinear functional and $u'(x) \le \|u'\|_{\alpha(U,\mathbb{K})} \cdot \|x\|_X$, for all $x \in U$. Thus, by Hahn-Banach theorem there exists $x' : X \to \mathbb{R}$, such that

$$x'|_U = u' \quad \text{and} \quad x'(x) \le p(x), \ \ \forall x \in X.$$

Note that $x'(-x) \le p(-x) = p(x)$, which implies that

$$|x'(x)| \le \|u'\|_{\alpha(U,\mathbb{K})} \cdot \|x\|_X, \quad \forall x \in X,$$

which implies that x' is continuous and $\|x'\|_{\alpha(X,\mathbb{K})} \le \|u'\|_{\alpha(U,\mathbb{K})}$. For the proof of the inverse inequality notice that

$$\|u'\|_{\alpha(U,\mathbb{K})} = \sup_{u \in B_u} |u'(u)| = \sup_{u \in B_u} |x'(u)| \le \sup_{x \in B_x} |x'(x)| = \|x'\|_{\alpha(X,\mathbb{K})},$$

where

$$B_u = \{u \in U \ : \ \|u\|_X \le 1\} \quad \text{and} \quad B_x = \{x \in X \ : \ \|x\|_X \le 1\}.$$

□

Remark 9. Let X be a normed space. Then, for every $x_o \in X, x_o \ne 0$ there exists $x' \in X'$ such that

$$\|x'\|_{\alpha(X,\mathbb{K})} = 1 \ \ \text{and} \ \ x'(x_o) = \|x_o\|_X.$$

Indeed, let $M = \mathbb{K}x_o$. Define $x' : M \to \mathbb{R}$ by

$$x'(\lambda x_o) = \lambda \|x_o\|_X, \quad \forall \lambda \in \mathbb{K}.$$

One can easily check that

$$\|x'\|_{\alpha(M,\mathbb{R})} = 1 \ \ \text{and} \ \ x'(x_o) = \|x_o\|_X.$$

The existence of the extension $x' : X \to \mathbb{K}$ follows from the Hahn-Banach theorem.

Remark 10. Let X be a normed space. Then, for every $x_1, x_2 \in X$ such that $x_1 \ne x_2$ there exists $x' \in X'$ such that $x'(x_1) \ne x'(x_2)$. This is the immediate conequence of the previous remark.

Remark 11. For any normed space $(X, \|\cdot\|_X)$ it holds that

$$\|x\|_X = \sup_{x' \in B_{x'}} |x'(x)|, \ \ \forall x \in X.$$

Indeed, since $x' \in B_{x'} = \{x' \in X' \ : \ \|x'\| \le 1\}$ we have that

$$|x'(x)| \le \|x'\| \cdot \|x\|_X \le \|x\|_X, \quad \forall x' \in B_{x'}.$$

On the other hand, it follows from the previous remark that there exists x' such that $\|x'\| = 1$ (and thus $x' \in B_{x'}$) and

$$\|x\|_X = x'(x) \le |x'(x)| \le \sup_{x' \in B_{x'}} |x'(x)|.$$

Lemma 3.15. *Let $(X, \|\cdot\|_X)$ be a normed space and $V \subset X$ be its convex and open subset such that $0 \notin V$. Then, there exists $x' \in X'$ such that* Re $x'(x) < 0$*, for each $x \in V$.*

Proof of Lemma 3.15. (only for $\mathbb{K} = \mathbb{R}$) Define sets $A + B$ and $A - B$ as following

$$A \pm B = \{a \pm b \; : \; a \in A, b \in B\}.$$

Moreover, for $x_o \in V$ define

$$y_o = -x_o \quad \text{and} \quad U = V - \{x_o\}.$$

Set U is convex, open and $y_o \notin U$, $0 \in U$. For the set U define a Minkowski functional p_U. Note that p_U is sublinear and $p_U(y_o) \ge 1$. Now, consider a linear functional

$$y' : \text{lin}\{y_o\} \to \mathbb{R}, \quad y'(t \cdot y_o) = t \cdot p_U(y_o).$$

It holds that $y'(y) \le p_U(y)$, for each $y \in Y$. Indeed,

$$\begin{aligned} y'(t \cdot y_o) \le 0 \le p_U(t \cdot y_o), &\quad \text{for} \quad t \le 0, \\ y'(t \cdot y_o) = p_U(t \cdot y_o), &\quad \text{for} \quad t > 0. \end{aligned}$$

Let x' be an extension of y' such that $x' \le p_U$. There exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subset U$ such that

$$|x'(x)| = \max\{x'(x), x'(-x)\} \le \max\{p_U(x), p_U(-x)\} \le \frac{1}{\varepsilon}\|x\|_X.$$

It is straightforward that $x'(y_o) = p_U(x_o) \ge 1$ and for each $x = (u - y_o) \in V$, where $u \in U$ it holds that

$$x'(x) = x'(u) - x'(y_o) \le p_U(u) - 1 < 0.$$

□

Theorem 3.16. *(Hahn-Banach theorem) Let X be a normed space, $V_1, V_2 \subset X$ be convex and V_1 be open. If $V_1 \cap V_2 = \varnothing$, then there exists $x' \in X'$ such that*

$$\text{Re}\, x'(v_1) < \text{Re}\, x'(v_2) \quad \forall v_1 \in V_1, v_2 \in V_2.$$

Proof of Theorem 3.16. Let define $V = V_1 - V_2 = (V_1 + (-V_2))$, which is in fact given by

$$V = \bigcup_{x \in V_2} (V_1 - \{x\}).$$

Note that V is open and since $V_1 \cap V_2 = \varnothing$ it holds that $0 \notin V$. Therefore, by Lemma (3.15) there exists a functional x' such that

$$\text{Re}\, x'(v_1 - v_2) < 0$$

for all $v_1 \in V_1$ and $v_2 \in V_2$ and thus, Re $x'(v_1) <$ Re $x'(v_2)$. □

Theorem 3.17. *(Hahn-Banach theorem) Let X be a normed space, $V \subset X$ be a closed and convex set and $x \notin V$. Then, there exists a functional $x \in X'$ such that*

$$\operatorname{Re} x'(x) < \inf\{\operatorname{Re} x'(v) \,:\, v \in V\},$$

that is, there exists $\varepsilon > 0$ such that

$$\operatorname{Re} x'(x) < \operatorname{Re} x'(x) - \varepsilon \leq \operatorname{Re} x'(v).$$

Proof of Theorem 3.17. Since V is closed, there exists an open ball with radius r, such that

$$B(x,r) \cap V \leq \varnothing.$$

From Theorem 3.16 if follows that

$$\operatorname{Re}\ x'(x+u) < \operatorname{Re}\ x'(v), \quad \forall u \in B(0,r),\ v \in V.$$

Therefore,

$$\operatorname{Re} x'(x) + \operatorname{Re} x'(u) < \operatorname{Re} x'(v)$$

and

$$\begin{aligned} \operatorname{Re} x'(x) + \|\operatorname{Re} x'\|_{X'} \cdot r &\leq \operatorname{Re} x'(v) \\ \operatorname{Re} x'(x) + \|\operatorname{Re} x'\|_{X'} \cdot r &\leq \inf\{\operatorname{Re} x'(v) \,:\, v \in V\}. \end{aligned}$$

□

LECTURE 4
Weak convergence and reflexivity

Let $(X, \|\cdot\|_X)$ be a normed space. X'' is defined as a space of linear functionals

$$i(x) : X' \to \mathbb{K}$$

given by

$$\big(i(x)\big)(x') = x'(x). \tag{4.1}$$

Remark 12. Operator i is a linear isometry (in general not surjective). Indeed,

$$\left|\big(i(x)\big)(x')\right| = |x'(x)| \leq \|x'\| \cdot \|x\|_X \quad \Rightarrow \quad \|i(x)\| \leq \|x\|_X.$$

According to Remark 9, for every $x \in X$, $x \neq 0$, there exists a functional x' such that $x'(x) = \|x\|_X$ and $\|x'\| = 1$. Thus,

$$\|x\|_X = |x'(x)| = |i(x)(x')| \leq \|i(x)\|.$$

Definition 4.2. A Banach space X is called reflexive if i defined as in (4.1) is surjective. We write then $X \cong X''$.

Example 10.

(a) Finite dimensional spaces are reflexive,
(b) Banach spaces l^p and $\mathbf{L^p}$ for $1 < p < +\infty$ are reflexive,
(c) Banach spaces $\mathbf{L^1}$, $\mathbf{L^\infty}$, $(\mathbf{C}(K), \|\cdot\|_\infty)$ are not reflexive.

Definition 4.3. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a space X. We say that $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to some x and denote it as $x_n \rightharpoonup x$, if

$$\lim_{n\to+\infty} x'(x_n) = x(x) \text{ in } \mathbb{K}, \quad \forall x' \in X'.$$

Definition 4.4. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in a space $Y \cong X'$. We say that $\{y_n\}_{n\in\mathbb{N}}$ converges weakly* to some x and denote it as $x_n \overset{*}{\rightharpoonup} x$, if

$$\lim_{n\to+\infty} y_n(x) = y(x) \text{ in } \mathbb{K}, \quad \forall x \in X.$$

Theorem 4.5. *(Sequential version of Banach-Alaoglu theorem) Let* $(X, \|\cdot\|_X)$ *be a separable normed space and* $Y \cong X'$*. If* $\{y_n\}_{n\in\mathbb{N}}$ *is a sequence such that* $\sup_{n\in\mathbb{N}} \|y_n\|_Y \le 1$*, then there exists a subsequence* $\{y_{n_k}\}_{k\in\mathbb{N}}$ *and an element* $y \in Y$ *such that*

$$y_{n_k} \overset{*}{\rightharpoonup} y.$$

Proof of Theorem 4.5. If X is separable, then there exists a set $L = \{x_j\}_{l\in\mathbb{N}} \subset X$ which is linearly independent and linearly dense in X. Since $|y_n(x_1)|$ is uniformly bounded with respect to n, then there exists a subsequence n_k^1 such that $y_{n_k^1}(x_1)$ converges to some element $y^1 \in Y$. Similarly, $|y_{n_k^1}(x_2)|$ is bounded and thus, there exists a subsequence n_k^2 such that $y_{n_k^2}(x_2)$ converges to some $y^2 \in Y$. Iterating this procedure we obtain sequences

$$y_{n_k^m} \subset y_{n_k^{m-1}} \subset \cdots \subset y_{n_k^1} \subset y_n.$$

By a diagonal argument (setting $y_{n_k} = y_{n_k^k}$) we chose a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} y_{n_k}(x_l) = y^l, \quad \forall l \in \mathbb{N}.$$

To complete the proof we need to show that y_{n_k} converges for all $x \in X$. To this end, fix $\varepsilon > 0$. Since L is linearly dense, there exist $N = N(\varepsilon)$ and a sequence $\{(\lambda_{l_i}, x_{l_i})\}_{i\in\mathbb{N}}$, $\lambda_{l_i} \in \mathbb{K}$, $x_{l_i} \in L$, such that

$$\left\| x - \sum_{i=1}^{N(\varepsilon)} \lambda_{l_i} x_{l_i} \right\|_X < \varepsilon.$$

Define a linear functional $y : \mathrm{lin}\{x_l \,:\, x_l \in L\} \to \mathbb{K}$ as

$$y(\sum_{i=1}^{N(\varepsilon)} \lambda_{l_i} x_{l_i}) = \sum_{i=1}^{N(\varepsilon)} \lambda_{l_i} y^{l_i}.$$

Since the assumptions of Hahn-Banach theorem are fulfilled, we can extend y on X preserving $\|y\|_Y \le 1$. Note that for each $\varepsilon > 0$ it holds that

$$\lim_{k\to+\infty} |y_{n_k}(x) - y(x)| \le 2\varepsilon + \lim_{k\to+\infty} \left| y_{n_k}(\sum_{i=1}^{N(\varepsilon)} \lambda_{l_i} x_{l_i}) - y(\sum_{i=1}^{N(\varepsilon)} \lambda_{l_i} x_{l_i}) \right| = 2\varepsilon.$$

□

Remark 13. If X is reflexive, then weak and weak* convergence are equivalent. In such a case, Theorem 4.5 can be formulated in terms of the weak convergence as well.

Remark 14. For a general predual space X there is a topological version of Theorem 4.5, which states that the unit ball in X' is compact in the weak* topology.

Definition 4.6.

(a) A subset M of a metric space X is nowhere dense, if $\mathrm{int}(\bar{M}) = \varnothing$,
(b) A set M is of the 1-st category, if $M = \bigcup_{n\in\mathbb{N}} M_n$, where M_n are nowhere dense sets.

Theorem 4.7. *(Baire category) A non-empty, complete metric space is not of the* 1*-st category.*

Theorem 4.8. *Let X be a Banach space and Y be a normed space. Let $T_i \in \alpha(X,Y)$, where $i \in I$. If for each $x \in X$*

$$\sup_{i\in I} \|T_i x\|_Y < +\infty,$$

then

$$\sup_{i\in I} \|T_i\|_{\alpha(X,Y)} < +\infty.$$

Proof of Theorem 4.8. Let $n \in \mathbb{N}$ and define

$$E_n = \left\{x \in X \,:\, \sup_{i\in I} \|T_i x\|_Y \le n\right\} = \bigcap_{i\in I} \{x \in X \,:\, \|T_i x\| \le n\}.$$

From the assumptions it follows that sets E_n are closed and $X = \bigcup_{n\in\mathbb{N}} E_n$. Therefore, by Theorem 4.7 at least one of the sets E_n has a non-empty interior. Let $n_o \in \mathbb{N}$ be such that $\mathrm{int}E_{n_o} \neq \varnothing$. Then, for each $x_o \in \mathrm{int}E_{n_o}$ there exists $\varepsilon > 0$ such that $B(x_o,\varepsilon) \subset E_{n_o}$. Let $x \in X$ be such that $\|x\|_X \le 1$. Then,

$$\varepsilon x + x_o \in B(x_o,\varepsilon) \subset E_{n_o}.$$

If so, then

$$\|T_i(\varepsilon x + x_o)\|_Y \le n_o, \quad \forall i \in I.$$

Therefore, for any $x \in X$ such that $\|x\|_X \le 1$ it holds that

$$\|T_i x\|_Y = \frac{1}{\varepsilon}\|T_i(\varepsilon x)\|_Y = \frac{1}{\varepsilon}\|T_i(\varepsilon x + x_o) - T_i x_o\|_Y \le \frac{1}{\varepsilon}\left(n_o + \|T_i x_o\|_Y\right).$$

Since the right hand side of the inequality above does not depend on the choice of x, it also holds that

$$\|T_i\|_{\alpha(X,Y)} \le \frac{1}{\varepsilon}\left(n_o + \|T_i x_o\|_Y\right).$$

Taking supremum over $i \in I$ yields

$$\sup_{i\in I} \|T_i\|_{\alpha(X,Y)} \le \frac{1}{\varepsilon}\left(n_o + \sup_{i\in I} \|T_i x_o\|_Y\right) < +\infty.$$

□

LECTURE 5
Open mapping and closed graph theorems

Definition 5.1. We say that $T : X \to Y$ is open, if for any open set $U \subset X$ a set $T(U) \subset Y$ is open.

Lemma 5.2. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces and $T : X \to Y$ be a linear operator. Then, the following conditions are equivalent.*

(a) T is open,
(b) $\forall r > 0\ \exists \varepsilon > 0$, such that $B_Y(0, \varepsilon) \subset T(B_X(0, r))$,
(c) $\exists \varepsilon > 0$, such that $B_Y(0, \varepsilon) \subset T(B_X(0, 1))$.

Proof of Lemma 5.2. $i) \Rightarrow ii)$ and $ii) \Leftrightarrow iii)$ are trivial. We shall prove $ii) \Rightarrow i)$. Let $U \subset X$ be an open set and $x \in U$. It means that

$$\exists r > 0, \quad \text{such that} \quad x + B_X(0, r) \subset U$$

Moreover, $Tx \in T(U)$ and due to the linearity of T

$$Tx + T(B_X(0, r)) = T(x + B_X(0, 1)) \subset T(U).$$

From the assumption we have that there exists $\varepsilon > 0$ such that $B_Y(0, \varepsilon) \subset T(B_X(0, r))$. Therefore it holds that $Tx + B_Y(0, \varepsilon) \subset T(U)$ and thus $T(U)$ is open. □

Theorem 5.3. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $T \in \alpha(X, Y)$ be a surjective operator. Then, T is open.*

To prove the theorem we need the following lemma.

Lemma 5.4. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $T \in \alpha(X, Y)$ be a surjective operator. Then, there exists $\varepsilon > 0$ such that*

$$B_Y(0, \varepsilon) \subset \overline{T(B_X(0, 1))}.$$

Proof of Lemma 5.4. Since T is surjective it holds that $Y = \bigcup_{n \in \mathbb{N}} T(B_X(0, n))$. From linearity of T it follows that

$$Y = \bigcup_{n \in \mathbb{N}} n \cdot T(B_X(0, 1)).$$

Theorefore, int$\left(\overline{T(B_X(0,1))}\right) \neq \varnothing$ due to Theorem 4.7 (Baire theorem). Thus, there exists $y_o \in Y$ and $\varepsilon > 0$ such that

$$B_Y(y_o,\varepsilon) \subset \overline{T(B_X(0,1))}.$$

Since T is surjective, there exists x_o such that $Tx_o = y_o$ and

$$\begin{aligned} B_Y(0,\varepsilon) &= B_Y(y_o,\varepsilon) - y_o \subset \overline{T(B_X(0,1))} - Tx_o \subset \overline{T(B_X(0,1))} - \overline{T(B_X(0,\|x_o\|_X))} \\ &= \overline{T(B_X(0,1)) + T(B_X(0,\|x_o\|_X))} = \overline{T(B_X(0,1+\|x_o\|_X))}. \end{aligned}$$

Thus, form linearity of T it follows that $B_Y\left(0,\frac{\varepsilon}{(1+\|x_o\|_X)}\right) \subset \overline{T(B_X(0,1))}$. □

Proof of Theorem 5.3. Due to Lemma 5.4 there exists $\varepsilon_o > 0$ such that

$$B_Y(0,\varepsilon_o) \subset \overline{T(B_X(0,1))}.$$

Let $y \in B_Y(0,\varepsilon_o)$. Then, there exists $\varepsilon > 0$ such that

$$\|y\|_Y < \varepsilon < \varepsilon_o.$$

Define $\bar{y} = (\varepsilon_o/\varepsilon)y$. Then, $\|\bar{y}\|_Y < \varepsilon_o$ and $\bar{y} \in \overline{T(B_X(0,1))}$. There also exists y_o such that

$$y_o = Tx_o \in T(B_X(0,1)), \quad \|\bar{y} - y_o\| < \alpha\varepsilon_o, \quad \text{where} \quad 0 < \alpha < 1$$

and α fulfills

$$\frac{\varepsilon}{\varepsilon_o} \cdot \frac{1}{1-\alpha} < 1.$$

Note that

$$\frac{\bar{y} - y_o}{\alpha} \in B_Y(0,\varepsilon_o).$$

By the same argument we can find $y_1 = Tx_1 \in T(B_X(0,1))$ such that

$$\left\|\frac{\bar{y} - y_o}{\alpha} - y_1\right\|_Y < \alpha\varepsilon_o.$$

Therefore,

$$\|y - (y_o + \alpha y_1)\|_Y < \alpha^2\varepsilon_o.$$

Iterating the procedure we obtain

$$\|\bar{y} - \sum_{i=0}^{n} \alpha^i y_i\|_Y \le \alpha^{n+1}\varepsilon_o$$

which is equivalent to

$$\|\bar{y} - T(\sum_{i=0}^{n} \alpha_i^i)\|_Y < \alpha^{n+1}\varepsilon_o.$$

Note that $\{x_i\}_{i\in\mathbb{N}} \subset B_X(0,1)$ and $0 < \alpha < 1$, hence $\sum_{i=0}^{n} \alpha^i x_i$ converges. Since X is a Banach space we set

$$\bar{x} = \sum_{i=0}^{+\infty} \alpha^i x_i \in X$$

and by the continuity of T we obtain $\bar{y} = T\bar{x}$. Denote $x = (\varepsilon/\varepsilon_o)\bar{x}$. Then, $y = Tx$ and

$$\|x\|_X = \frac{\varepsilon}{\varepsilon_o}\|\bar{x}\|_X \le \frac{\varepsilon}{\varepsilon_o}\sum_{i=0}^{+\infty}\alpha^i\|x_i\|_X < \frac{\varepsilon}{\varepsilon_o}\sum_{i=0}^{+\infty}\alpha^i = \frac{\varepsilon}{\varepsilon_o}\cdot\frac{1}{1-\alpha} < 1.$$

□

Remark 15. Let X, Y be Banach spaces and $T \in \alpha(X,Y)$ be bijective. Then, T^{-1} is continuous.

Remark 16. If $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces and norms $\|\cdot\|_1$, $\|\cdot\|_2$ are such that

$$\|x\|_1 \le M\|x\|_2, \quad \forall x \in X,$$

then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Definition 5.5. Let X, Y be normed spaces, D be a subspace of X and $T: D \to Y$ be a linear functional. Then, T is closed if for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ such that $x_n \to x$ and $\{Tx_n\}_{n\in\mathbb{N}} \subset Y$, $Tx_n \to y$, it holds that $x \in D$ and $Tx = y$.

(If T is continuous, then it is also closed?)

Definition 5.6. For a linear functional $T: D \to Y$ we define its graph as following

$$\text{graph}(T) = \{(x, Tx) \;:\; x \in D\} \subset X \times Y.$$

Lemma 5.7. *Let $X, Y, D, T, \text{graph(T)}$ be defined as above. Then,*

(a) $\text{graph}(T)$ *is a linear subspace of* $X \times Y$*,*
(b) T *is closed if and only if* $\text{graph}(T)$ *is closed in* $X \oplus Y$*.*

Lemma 5.8. *Let X, Y be Banach spaces and $D \subset X$ be a linear subspace. If $T: D \to Y$ is closed, then*

(a) D with a norm $\|x\| = \|x\|_X + \|Tx\|_Y$ is a Banach space,
(b) T is continuous as a mapping from $(D, \|\cdot\|)$ to Y.

Proof of Lemma 5.8. a) Since T is closed, we know that for every sequences

$$\{x_n\}_{n\in\mathbb{N}} \subset D \quad \text{such that} \quad x_n \to x \in X$$

and

$$\{Tx_n\}_{n\in\mathbb{N}} \subset Y \quad \text{such that} \quad Tx_n \to y$$

it holds that $x \in D$ and $Tx = y$. $\{x_n\}_{n\in\mathbb{N}} \subset D$ is a Cauchy sequence with respect to the $\|\cdot\|$ norm, since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and $\{Tx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Moreover, its limit in X is an element from the subspace D. Thus,

$$\|x_n - x\| = \|x_n - x\|_X + \|T(x_n - x)\|_Y \to 0,$$

which implies that $(D, \|\cdot\|)$ is complete.
b) T is bounded, which holds due to the following inequality

$$\|Tx\|_Y \le \|x\|_X + \|Tx\|_Y = \|x\|.$$

Thus, T is also continuous. □

Theorem 5.9. *Let X, Y be Banach spaces and $T : X \to Y$ be a linear and closed. Then, T is continuous.*

Proof of Theorem 5.9. T is continuous with respect to the $\|\cdot\|$ norm defined in the previous lemma. Note that if we set $D = X$ in the previous lemma, then X is a Banach space with norms $\|\cdot\|_X$ and $\|\cdot\|$. Moreover, $\|x\|_X \leq \|x\|$ for all $x \in X$. Then, according to Remark 16 the norm $\|\cdot\|_X$ is equivalent to $\|\cdot\|$. Therefore, T is continuous with respect to the norm $\|\cdot\|_X$ as well. □

5.1. Application of Banach -Steinhaus Theorem

Lemma 5.10. *For a subset M of a normed space $(X, \|\cdot\|_X)$, the following conditions are equivalent:*

(a) M is bounded,
(b) for every $x' \in X'$, $x'(M) \subset \mathbb{K}$ is bounded.

Proof of Lemma 5.10. The implication $i) \Rightarrow ii)$ is trivial. To show $ii) \Rightarrow i)$ consider

$$i : X \to X'',$$

such that $i(x)$ is given by (4.1) for all $x \in X$. Note that

$$\sup_{x \in M} |x'(x)| = \sup_{x \in M} |i(x)(x')| < +\infty, \quad \forall x \in X.$$

By Banach-Steinhaus theorem we have

$$\sup_{x \in M} \|x\| = \sup_{x \in M} \|i(x)\| < +\infty.$$

□

Remark 17. Any weakly convergent sequence is bounded.

Lemma 5.11. *Let X be a Banach space, Y be a normed space and $T_n \in \alpha(X, Y)$ for all $n \in \mathbb{N}$. If for all $x \in X$ there exists $Tx = \lim_{n \to +\infty} T_n x$, then $T \in \alpha(X, Y)$.*

LECTURE 6
Adjoint Operator

Definition 6.1. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces and $T \in \alpha(X, Y)$. Then, $T' : Y' \to X'$ is defined by

$$(T'y')(x) = y'(Tx).$$

One can check that $T' \in \alpha(Y', X')$.

Theorem 6.2. *Let X, Y be Banach spaces and $T \in \alpha(X, Y)$. Then, the following conditions are equivalent*

(a) $\mathrm{Im}(T)$ *is closed,*
(b) $\mathrm{Im}(T) = \ker(T')_\perp$,
(c) $\mathrm{Im}(T')$ *is closed,*
(d) $\mathrm{Im}(T') = \ker(T)^\perp$,

where

$$U^\perp := \{x' \in X' \,:\, x'(x) = 0, \;\; \forall x \in U\}, \quad V_\perp := \{x \in X \,:\, x'(x) = 0, \;\; \forall x' \in V\}.$$

Lemma 6.3. *Let X, Y be Banach spaces and $T \in \alpha(X, Y)$ have a closed image. Then, there exists $K \geq 0$ such that*

$$\forall y \in \mathrm{Im}(Y) \;\; \exists x \in X \quad \text{s.t.} \quad Tx = y \;\text{ and }\; \|x\|_X \leq K \|y\|_Y.$$

Proof of Lemma 6.3. Define a canonical factorization $\hat{T}$ (what is precisely a canonical factorization?)

$$\hat{T} : X/\ker(T) \to \mathrm{Im}(T).$$

Note that $X/\ker(T)$ and $\mathrm{Im}(T)$ are Banach spaces and $\hat{T}$ is bijective. Then, according to Remark 15 there exists

$$\hat{T}^{-1} \in \alpha(\mathrm{Im}(T), X/\ker(T))$$

and $K \geq 0$ such that $\|\hat{T}^{-1}\| \leq K$. □

Lemma 6.4. *Let $(X, \|\cdot\|_X)$,$(Y, \|\cdot\|_Y)$ be Banach spaces and $T \in \alpha(X,Y)$. If there exists a constant C such that for all $y' \in Y'$*

$$C\|y'\|_{\alpha(Y,\mathbb{K})} \leq \|T'y'\|_{\alpha(X',\mathbb{K})},$$

then T is surjective and open.

Proof of Lemma 6.3. In order to prove that T is open we need to show that $B_Y(0,C) \subset T(B_X(0,1))$ (according to Lemma 5.2, claim (a)). In fact it is enough to prove

$$B_Y(0,C) \subset \overline{T(B_X(0,1))} =: D.$$

Let $y_o \in B_Y(0,C)$ and assume that $y_o \notin D$ (note that D is a convex set). According to Hahn-Banach theorem, there exists $y' \in Y'$, $\alpha \in \mathbb{R}$ such that for each $y \in D$

$$\text{Re}\, y'(y) \leq \alpha < \text{Re}\, y'(y_o) \leq |y'(y_o)|.$$

Note that $0 \in D$, which implies $\text{Re}\, y'(0) = 0 \leq \alpha$. We can assume that $\alpha > 0$. Define $\tilde{y}' = y'/\alpha$. Then

$$\text{Re}\, \tilde{y}'(y) = \text{Re}\, (y'(y)/\alpha) \leq 1 < \text{Re}\, (y'(y_o)/\alpha) = \text{Re}\, \tilde{y}'(y_o), \quad \forall y \in D.$$

Thus,

$$|\tilde{y}'(y)| \leq 1 < |\tilde{y}'(y_o)|, \quad \forall y \in D$$

(note that $y \in D \Rightarrow \lambda y \in D$, $\forall \lambda$ such that $|\lambda| > 1$). Moreover, for all x such that $\|x\|_X < 1$, $Tx = y \in D$ and from the definition of T'

$$|T'\tilde{y}'(x)| = |\tilde{y}'(Tx)| \leq 1,$$

which implies that $\|T'\tilde{y}'\|_{\alpha(X,\mathbb{K})} \leq 1$, but

$$|\tilde{y}'(y_o)| \leq \|\tilde{y}'\|_{\alpha(Y,\mathbb{K})} \cdot \|y_o\|_Y < C\|\tilde{y}'\|_{\alpha(Y,\mathbb{K})}$$

and $1 < C\|\tilde{y}'\|_{\alpha(Y,\mathbb{K})}$, which leads to the contradiction. □

Lemma 6.5.

$$\overline{\text{Im}(T)} = (\ker(T'))_\perp.$$

Proof of Lemma 6.5.
" $\subset$ ". Let $y \in \overline{\text{Im}(T)}$, which means that there exists $x \in X$ such that $y = Tx$. If $y' \in \ker(T')$, then $T'y'$ is a zero functional on X' and thus

$$0 = (T'y')(x) = y'(Tx) = y'(y).$$

Therefore, for each $y \in \overline{\text{Im}(T)}$ it holds that $y'(y) = 0$, if $y' \in \ker(T')$. Since

$$(\ker(T'))_\perp = \{y \in Y \;:\; y'(y) = 0, \;\; \forall y' \in \ker(T')\},$$

we conclude that $\text{Im}(T) \subset (\ker(T'))_\perp$.
" $\supset$ ". Let $U = \overline{\text{Im}(T)}$. Thus, U is a closed subspace of Y. We will show that

$$y \notin U \quad \Rightarrow \quad y \notin (\ker(T'))_\perp.$$

By Hahn-Banach theorem there exists $y' \in Y$ such that

$$y'|_U = 0 \quad \text{and} \quad y'(y) \neq 0.$$

In particular $0 = y'(Tx) = (T'y')(x)$ for all $x \in X$, that is, $T'y'$ is a zero functional on X and thus $y' \in \ker(T')$. This proves that $y \notin (\ker(T'))_\perp$, since the opposire claim would imply that $y'(y) = 0$. □

Proof of Theorem 6.2.
$(b) \Rightarrow (a)$ is straightforward.
$(b) \Leftarrow (a)$ follows from Lemma 6.5.
Now, we shall prove $(a) \Rightarrow (d)$. Clearly $\mathrm{Im}(T') \subset (\ker(T))^\perp$ (for this we do not need (a)) and $T'y'(x) = y'(Tx) = 0$ for $x \in \ker(T)$. Let $x' \in (\ker(T))^\perp$ and define a linear functional

$$z' : \mathrm{Im}(T) \to \mathbb{K}, \;\; z'(y) = x'(x), \;\; \text{for} \;\; y = Tx.$$

Note that z is continuous. Indeed, using one of the previous lemmas and the open mapping theorem we have that

$$\forall y \in \mathrm{Im}(T) \;\; \exists x \in X \quad \text{s. t.} \quad y = Tx, \;\; \text{and}$$

$$|z'(y)| = |x'(x)| \leq \|x'\|_{\alpha(X,\mathbb{K})} \cdot \|x\|_X \leq \|x'\|_{\alpha(X,\mathbb{K})} \cdot K\|y\|_Y.$$

By Hahn-Banach theorem we can define $y' \in Y'$, which is an extension of z'. Then,

$$x'(x) = z'(Tx) = y'(Tx) = (T'y')(x), \;\; \forall x \in X,$$

that is, $x' = T'y'$. This finishes the proof, since we assumed that $x' \in (\ker(T))^\perp$.
$(d) \Rightarrow (c)$ is straightforward.
$(d) \Rightarrow (a)$. Let $Z = \overline{\mathrm{Im}(T)} \subset Y$ and define $S \in \alpha(X, Z)$ by setting $Sx = Tx$. For $y' \in Y$ and $x \in X$ we have

$$(T'y')(x) = y'(Tx) = y'|_Z(Sx) = (S'(y'|_Z))(x), \;\; \forall x \in X.$$

Then, $T'y' = S'(y'|_Z)$. Therefore $\mathrm{Im}(T') \subset \mathrm{Im}(S')$. To prove the opposite inclusion assume that $S'z' \in \mathrm{Im}(S')$ for some $z' \in Z'$ and consider any extension y' of z' (by Hahn-Banach theorem such extension exists). Then,

$$S'z' = T'y' \quad \Rightarrow \quad \mathrm{Im}(T') = \mathrm{Im}(S').$$

Then, by the assumption we have $\mathrm{Im}(S') = \overline{\mathrm{Im}(S')}$. Moreover, $\mathrm{Im}(S)$ is dense in Z and thus S' is injective (?). S' is then a continuous bijection between Z' and $\mathrm{Im}(S')$. In fact it is an in isomorphism, in particular

$$C\|z'\|_{\alpha(Z,\mathbb{K})} \leq \|S'z'\|_{\alpha(X,\mathbb{K})}, \;\; \forall z' \in Z'.$$

Then, by one of the Lemma above, S is surjective (and open). Thus, $\mathrm{Im}(S) = Z = \mathrm{Im}(T)$ (?) and $\mathrm{Im}(T) = \overline{\mathrm{Im}(T)}$. □

Lemma 6.6. *Let P be a continuous projection on a normed space X (P is a projection if $P^2 = P$). Then,*

(a) $P = 0$ *or* $\|P\| \geq 1$,
(b) $\ker(P)$ *and* $\operatorname{Im}(P)$ *are closed,*
(c) $X = \ker(P) \oplus \operatorname{Im}(P)$.

Proof of Lemma 7.11.
(a) $\|P\| = \|P^2\| \leq \|P\|^2$, which implies $P = 0$ or $\|P\| \geq 1$.
(b) $\ker(P) = P^{-1}(\{0\})$ is closed (since P is continuous), $\mathrm{Id} - P$ is a continuous projection and $\operatorname{Im}(P) = \ker(\mathrm{Id} - P)$ is closed.
(c) $\forall x \in X$ it holds that $x = (\mathrm{Id} - P)x + Px$. □

LECTURE 7
Hilbert Spaces

Definition 7.1. Let X be a $\mathbb{K}$- linear space. A mapping

$$< \cdot, \cdot > \; X \times X \to \mathbb{K}$$

is called a scalar product if

(a) $< x_1 + x_2, y > = < x_1, y > + < x_2, y >$, $\forall x_1, x_2, y \in X$,
(b) $< \lambda x, y > = \lambda < x, y >$, $\forall x, y \in X, \lambda \in \mathbb{K}$,
(c) $< x, y > = \overline{< y, x >}$, $\forall x, y \in X$,
(d) $< x, x > \geq \; 0$, $\forall x \in X$,
(e) $< x, x > = \; 0 \quad \Leftrightarrow \quad x = 0$.

Lemma 7.2. *(Cauchy - Schwarz inequality). Let X be a $\mathbb{K}$-linear space with a scalar product. Then,*

$$|< x, y >|^2 \; \leq < x, x > \cdot < y, y >, \quad \forall x, y \in X.$$

The equality holds if and only if $x = \lambda y$ for $\lambda \in \mathbb{K}$.

Lemma 7.3. *Define a mapping $\| \cdot \| : X \times X \to \mathbb{R}$ by*

$$\|x\| = \sqrt{< x, x >}.$$

Then, $\| \cdot \|$ is a norm. In particular, it holds that

$$|< x, y >| \leq \|x\| \cdot \|y\|, \quad \forall x, y, \in X.$$

Definition 7.4. A normed space $(X, \| \cdot \|_X)$ is called a prehilbert space, if there exists a scalar product $< \cdot, \cdot >$ such that

$$\|x\|_X = \sqrt{< x, x >}, \quad \forall x \in X.$$

Complete prehilbert space is called a Hilbert space.

Lemma 7.5. *Let $(X, \| \cdot \|)$ be a prehilbert space and U be its dense linear subspace. If for all $u \in U$ it holds that $< x, u >= 0$, then $x = 0$.*

Proof of Lemma 7.5. Define a set

$$Y = \{y \in X \ :< x, y >= 0\}.$$

Y is closed since the map $y \to < x, y >$ is continuous. It also contains a dense subspace U. This implies that $Y = X$. In particular, $x \in Y$ and thus $\|x\|^2 = < x, x > = 0$. □

If $(X, \|\cdot\|_X)$ is a normed space, we can introduce a scalar product by the norm $\|\cdot\|_X$. More precisely, we set

$$< x, y > = \frac{1}{4}\left(\|x+y\|_X^2 - \|x-y\|_X^2\right), \text{ for } \mathbb{K} = \mathbb{R}$$

and

$$< x, y > = \frac{1}{4}\left(\|x+y\|_X^2 - \|x-y\|_X^2 + i\|x+iy\|_X^2 - i\|x-iy\|_X^2\right), \text{ for } \mathbb{K} = \mathbb{C}.$$

Lemma 7.6. *Scalar product* $< \cdot, \cdot > : X \times X \to \mathbb{K}$ *is a continuous mapping.*

Proof of Lemma 7.6. Let $x_1, x_2, y_1, y_2 \in X$. Then,

$$\begin{aligned} |< x_1, y_1 > - < x_2, y_2 >| &= |< x_1 - x_2, y_1 > + < x_2, y_1 - y_2 >| \\ &\leq \|x_1 - x_2\| \cdot \|y_1\| + \|y_1 - y_2\| \cdot \|x_2\|. \end{aligned}$$

□

Theorem 7.7. *(Parallelogram equality) A normed space* $(X, \|\cdot\|_X)$ *is a prehilbert space if and only if, for all* $x, y \in X$ *the following inequality holds*

$$\|x+y\|_X^2 + \|x-y\|_X^2 = 2\|x\|_X^2 + 2\|y\|_X^2.$$

Proof of Theorem 7.7. (for $\mathbb{K} = \mathbb{R}$). We introduce a scalar product as following

$$< x, y > = \frac{1}{4}\left(\|x+y\|_X^2 - \|x-y\|_X^2\right).$$

Clearly, it holds that $\|x\|_X = < x, x >^2$. We need to prove the properties of a scalar product form the Definition 7.1.

(a) Let $x_1, x_2 \in X$ and define

$$\begin{aligned} \alpha &:= \|x_1 + x_2 + y\|_X^2 = 2\|x_1 + y\|_X^2 + 2\|x_2\|_X^2 - \|x_1 - x_2 + y\|_X^2, \\ \beta &:= \|x_1 + x_2 + y\|_X^2 = 2\|x_2 + y\|_X^2 + 2\|x_1\|_X^2 - \|-x_1 + x_2 + y\|_X^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_1 + x_2 + y\|^2 &= \frac{\alpha + \beta}{2} \\ &= \|x_1 + y\|^2 + \|x_1\|^2 + \|x_2 + y\|^2 + \|x_2\|^2 - \frac{1}{2}\left(\|x_1 - x_2 + y\|^2 + \|-x_1 + x_2 + y\|^2\right). \end{aligned}$$

Similarly one gets

$$\|x_1 + x_2 - y\|^2 = \|x_1 - y\|^2 + \|x_1\|^2 + \|x_2 - y\|^2 + \|x_2\|^2 - \frac{1}{2}\left(\|x_1 - x_2 - y\|^2 + \|-x_1 + x_2 - y\|^2\right).$$

Finally,

$$\begin{aligned} < x_1 + x_2, y > &= \frac{1}{4}\left(\|x_1 + x_2 + y\|_X^2 - \|x_1 + x_2 - y\|_X^2\right) \\ &= \frac{1}{4}\left(\|x_1 + y\|_X^2 + \|x_2 + y\|_X^2 - \|x_1 - y\|_X^2 - \|x_2 - y\|_X^2\right) \\ &= < x_1, y > + < x_2, y > . \end{aligned}$$

Claim (b) follows from (a) for $\lambda \in \mathbb{N}$. Form the construction of $< \cdot, \cdot >$ we also have the property (b) fulfilled for $\lambda = 0$, $\lambda = -1$ and thus for all $\lambda \in \mathbb{Z}$. Let $\lambda = \frac{m}{n} \in \mathbb{Q}$. Then,

$$n < \lambda x, y > = n < m\frac{x}{n}, y > = m < x, y > = n\lambda < x, y > .$$

By the continuiuty of $\|\cdot\| : X \to \mathbb{K}$ we can extend this result for $\lambda \in \mathbb{R}$.

Claims (c), (d) and (e) are straightforward. □

Remark 18. The following claims hold:

(a) A normed space is prehilbert if and only if all its 2-dimensional subspaces are prehilbert.
(b) Any subspace of a prehilbert space is prehilbert.
(c) Closure of a prehilbert space is a Hilbert space.

Example 11. Examples of Hilbert spaces.

(a) $\mathbb{R}^n$ and $\mathbb{C}^n$ with $< s_i, t_i > = \sum_{i=1}^{n} s_i \bar{t}_i$,
(b) l^2 with $< s_i, t_i > = \sum_{i=1}^{+\infty} s_i \bar{t}_i$,
(c) $\mathbf{L^2}(\Omega, \mu)$ with $< f, g > = \int_\Omega f \cdot \bar{g} \, \mathrm{d}\mu$.

Definition 7.8. Let X be a prehilbert space. We say that x and y are orthogonal and write $x \perp y$, if $< x, y >= 0$. Sets A and B are orthogonal, if $< x, y > = 0$ for all $x \in A$ and $y \in B$. The set

$$A^\perp = \{y \in X \,:\, x \perp y, \ \ \forall x \in A\}$$

is called the orthogonal complement of A.

Remark 19. For a Hilbert space X the following claims hold:

(a) $x \perp y \quad \Rightarrow \quad \|x\|^2 + \|y\|^2 = \|x + y\|^2$,
(b) $A^\perp$ is a closed subspace of X,
(c) $A \subset (A^\perp)^\perp$,
(d) $A^\perp = (\overline{\mathrm{lin}\{A\}})^\perp$.

Theorem 7.9. *Let $(H, \|\cdot\|)$ be a Hilbert space and K be its convex and closed subset. Let $x_o \in H$. Then, there exists a unique $x \in K$ such that*

$$\|x - x_o\| = \inf_{y \in K} \|y - x_o\|.$$

This statement still holds, if H is a uniformly convex space.

Proof of Theorem 7.9. It is trivial if $x_o \in K$ (we set $x = x_o$). Let $x_o \notin K$. Without loss of generality we can assume that $x_o = 0$. In this case we define

$$d = \inf_{y \in K} \|y\|.$$

There exists $\{y_n\}_{n\in\mathbb{N}} \subset K$ and $\lim_{n\to+\infty} \|y_n\| = d$. We will prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. This is sufficient to show, since $(K, \|\cdot\|)$ is a Banach space. By Parallelogram equality we have

$$\|(y_n + y_m)/2\| + \|(y_n - y_m)/2\| = \frac{1}{2}\left(\|y_n\|^2 + \|y_m\|^2\right), \quad \forall m, n \in \mathbb{N}.$$

By convexity of K we have that $\frac{1}{2}(y_n + y_m) \in K$ and thus $\|\frac{1}{2}(y_n + y_m)\|^2 \geq d^2$. Therefore,

$$d^2 + \|(y_n - y_m)/2\| \leq \frac{1}{2}\left(\|y_n\|^2 + \|y_m\|^2\right), \quad \forall m, n \in \mathbb{N}.$$

Since the right hand side of the inequality converges to d^2, we have that $\|y_n - y_m\|$ converges to zero as n, m tend to infinity. By the comleteness of K, there exists $x \in K$ such that

$$x = \lim_{n\to+\infty} y_n \quad \text{and} \quad \|x\| = d.$$

Now, we shall show that the element is unique. Let $x, \bar{x} \in K$ be such that

$$\|x\| = \|\bar{x}\| = d.$$

Then, by Parallelogram equality

$$\|(x - \bar{x})/2\|^2 \;=\; d^2 - \|(x + \bar{x})/2\|^2.$$

Note that $(x + \bar{x})/2 \in K$, $\|(x + \bar{x})/2\|^2 \geq d^2$, which implies $\|x - \bar{x}\| = 0$ and thus $x = \bar{x}$. □

Lemma 7.10. *Let K be a closed convex subset of a Hilbert space $(H, \|\cdot\|)$ and $x_o \in H$. The, the following conditions are equivalent*

(a) $\|x_o - x\| = \inf_{y\in K} \|x_o - y\|$,

(b) $\mathrm{Re}\,\langle x_o - x, y - x\rangle \leq 0, \quad \forall y \in K.$

Proof of Lemma 7.10.

$(b) \Rightarrow (a)$

$$\begin{aligned} \|x_o - y\|^2 &= \|(x_o - x) + (x - y)\|^2 \\ &= \|x_o - x\|^2 + 2\mathrm{Re}\,\langle x_o - x, x - y\rangle + \|x - y\|^2 \geq \|x_o - x\|^2. \end{aligned}$$

$(a) \Rightarrow (b)$. Let $t \in [0,1]$, $y \in K$ and define $y_t = (1-t)x + ty \in K$. Then,

$$\begin{aligned} \|x_o - x\|^2 &\leq \|x_o - y_t\|^2 = \langle x_o - x + t(x - y), x_o - x + t(x - y)\rangle \\ &= \|x_o - x\|^2 + 2\mathrm{Re}\,\langle x_o - x, t(x - y)\rangle + t^2\|x - y\|^2. \end{aligned}$$

Finally,

$$\mathrm{Re}\,\langle x_o - x, t(y - x)\rangle \leq \frac{t}{2}\|x - y\|^2, \quad \forall t \in [0,1].$$

□

Theorem 7.11. *Let $U \neq \{0\}$ be a closed subspace of a Hilbert space $(H, \|\cdot\|)$. Then, there exists a linear projection*

$$P_U : H \to U, \quad \text{s. t.} \quad \|P_u\| = 1, \ \ker(P_U) = U^\perp.$$

Moreover, $\mathrm{Id} - P_U$ *is a projection such that*

$$(\mathrm{Id} - \mathrm{P_U}) : H \to U^\perp, \quad \text{s. t.} \quad \|\mathrm{Id} - \mathrm{P_U}\| = 1, \ (\text{if } U \neq H).$$

It also holds that $H = U \oplus U^\perp$.

Proof of Theorem 7.11. Define

$$P_U : H \to U, \quad P_U(x_o) = x,$$

where $x_o \in H$ and $x \in U$ is such that $\|x_o - x\|^2 = \inf_{y \in U} \|x_o - y\|^2$. It is straightforward that P_U is a projection. By the previous lemma

$$\mathrm{Re}\left\langle x_o - P_U(x_o), y - P_U(x_o)\right\rangle \leq 0, \quad \forall y \in U.$$

Note that $y - P_U(x_o) \in U$ and thus

$$\mathrm{Re}\left\langle x_o - P_U(x_o), y\right\rangle \leq 0, \quad \forall y \in U.$$

Using $-y$ and iy in the expression above we obtain

$$\left\langle x_o - P_U(x_o), y\right\rangle = 0, \ \forall y \in D \quad \text{and} \quad x_o - P_U(x_o) = x_o - x \in U^\perp.$$

Since $U^\perp$ is a linear subspace, it holds that for all $x_1, x_2 \in H$, $\lambda_1, \lambda_2 \in \mathbb{K}$

$$(\lambda_1 x_1 - \lambda_1 P_U(x_1)) + (\lambda_2 x_2 - \lambda_2 P_U(x_2)) \in U^\perp.$$

Therefore, $\lambda_1 x_1 + \lambda_2 x_2 = z \in H$ and $P_U(z)$ is such that $z - P_U(z) \in U^\perp$. Thus,

$$P_U(\lambda_1 x_1 + \lambda_2 x_2) = P_U(z) = \lambda_1 P_U(x_1) + \lambda_2 P_U(x_2),$$

which proves that P_U is linear. From the construction of P_U it follows that $\mathrm{Im}(P_U) = U$ and we have that $\ker(P_U) = U^\perp$. Therefore,

$$P_U(x_o) = 0 \quad \Leftrightarrow \quad x_o \in U^\perp.$$

We claim that $\mathrm{Id} - P_U$ is a projection with $\mathrm{Im}(\mathrm{Id} - P_U) = U^\perp$ and $\ker(\mathrm{Id} - P_U) = U$. By Pythagoras theorem

$$\|x_o\|^2 = \|P_U(x_o) + x_o - P_U(x_o)\|^2 = \|P_U(x_o)\|^2 + \|(\mathrm{Id} - P_U)(x_o)\|^2$$

and $P_U(x_o) \in U$, $x_o - P_U(x_o) \in U^\perp$. Finally, we get $H = U \oplus U^\perp$. Moreover, $\|P_U\| \leq 1$ and $\|\mathrm{Id} - P_U\| \leq 1$ (in fact equal to 1, if $U \neq \{0\}$ and $U \neq H$). □

Remark 20. Let H be a Hilbert space and U be its linear subspace. Then,

$$\overline{U} = (U^\perp)^\perp.$$

Indeed, for a closed set V it holds that $\mathrm{Id} - P_V = P_{V^\perp}$. Consider $V = \overline{U}$. It is clear that

$$U^\perp = V^\perp \quad \text{and} \quad \mathrm{Id} - P_{V^\perp} = P_{V^{\perp\perp}}.$$

Thus, $P_V = P_{V^{\perp\perp}}$ and thus $\overline{U} = U^{\perp\perp}$.

Theorem 7.12. *(Riesz theorem) Let H be a Hilbert space. Then, mapping*

$$\Phi : H \to H', \quad H(y) = \langle \cdot, y \rangle$$

is bijective, isometric and conjugate linear (i.e., $\Phi(\lambda y) = \bar{\lambda}\Phi(y)$). It means that

$$\forall x' \in H' \ \exists! y \in H, \quad \text{s. t.} \quad x'(x) = \langle x, y \rangle, \ \forall x \in H \ \text{ and } \ \|x'\| = \|y\|.$$

Proof of Theorem 7.12. Clearly Φ is conjugate linear. It follows from the Cauchy-Schwarz inequality that

$$\|\Phi(y)\| \leq \|y\|$$

and for $x = y/\|y\|$ we obtain

$$\Phi(y)(x) = \frac{\langle y, y \rangle}{\|y\|} = \|y\|,$$

which implies that Φ is an isometric and thus, it is injective. Let $x' \in H'$ (without loss of generality we can assume that $\|x'\| = 1$) and $U = \ker(x')$. By the previous theorem $H = U \oplus U^\perp$, where $\dim(U^\perp) = 1$. Thus, there exists $y \in H$ such that $x'(y) = 1$ and $U^\perp = \mathrm{lin}(y)$. For $x = u + \lambda y \in (U \oplus U^\perp)$

$$x'(x) = \lambda x'(y) = \lambda \quad \text{and} \quad \langle x, y \rangle = \lambda \|y\|^2.$$

Therefore,

$$\Phi(y/\|y\|^2) = x' \quad \text{and } \ \Phi \text{ is surjective.}$$

Finally, $\|y\| = 1$ and $\|x'\| = 1$. □

Remark 21. Each Hilbert space is reflexive.

7.1. Orthonormal basis

Definition 7.13. Let H be a Hilbert space. A subset $S \subset H$ is an orthonormal system if and only if for all $e, f \in S$ it holds that

$$\|e\| = \|f\| = 1 \quad \text{and} \quad \langle e, f \rangle = 0.$$

A subset $S \subset H$ is called an orthonormal basis if

$$S \subset T, \ T \ - \text{orthonormal system} \quad \Rightarrow \quad T = S.$$

Example 12. Examples of orthonormal systems:

(a) $H = \mathbf{L^2}([0,2\pi],\mathbb{R})$,

$$S = \left\{\frac{1}{2\pi}\chi_{[0,2\pi]}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\cos(n),\ n \in \mathbb{N}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\sin(n),\ n \in \mathbb{N}\right\}.$$

(b) $H = \mathbf{L^2}([0,2\pi],\mathbb{C})$,

$$S = \left\{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{in},\ n \in \mathbb{Z}\right\}.$$

Theorem 7.14. *(Gram - Schmidt theorem) Let H be a Hilbert space and $\{x_n : n \in \mathbb{N}\}$ be a linearly independent set in H. Then, there exists an orthogonal system S such that $\overline{\mathrm{lin} S} = \mathrm{lin}\{x_n : n \in \mathbb{N}\}$.*

Example 13. Let $H = \mathbf{L^2}([-1,1],\mathbb{R})$ and $x_n(t) = t^n$, for $n \in \mathbb{N}$. Then,

$$S = \left\{\sqrt{n+\frac{1}{2}} \cdot P_n(t),\ n \in \mathbb{N}\right\}, \quad \text{where} \quad P_n(t) = \frac{1}{2^n n!}\left(\frac{d}{dt}\right)^n (t^2-1)^n.$$

Theorem 7.15. *(Bessel inequality) Let H be a Hilbert space. If $S = \{e_n : n \in \mathbb{N}\} \subset H$ is an orthonormal system and $x \in H$, then*

$$\sum_{n=1}^{+\infty} \left|\langle x, e_n\rangle\right|^2 \le \|x\|^2.$$

Proof of Theorem 7.15. Let $N \in \mathbb{N}$. Define

$$x_N = x - \sum_{n=1}^{N} \langle x, e_n\rangle e_n$$

such that $x_N \perp e_k$, for $k = 1,\dots,N$. By Pythagoras theorem

$$\|x\|^2 = \|x_N\|^2 + \Big\|\sum_{n=1}^{N}\langle x, e_n\rangle e_n\Big\|^2 = \|x_N\|^2 + \sum_{n=1}^{N}\left|\langle x, e_n\rangle\right|^2 \ge \sum_{n=1}^{N}\left|\langle x, e_n\rangle\right|^2.$$

This estimate holds for all $N \in \mathbb{N}$, therefore taking the limit $N \to +\infty$ ends the proof. □

Lemma 7.16. *Let H be a Hilbert space, $S \subset H$ be an orthonormal system and $x \in H$. Then, the set*

$$S_x = \left\{e \in S : \langle x, r\rangle \neq 0\right\}$$

is countable.

Proof of Lemma 7.16. From Bessel inequality it holds that

$$S_{x,n} = \left\{e \in S : \langle x, r\rangle \ge 1/n\right\}$$

is finite and thus $S_x = \bigcup_{n\in\mathbb{N}} S_{x,n}$ is at most countable. □

LECTURE 8
Fourier Transform

Definition 8.1. Let $f \in \mathbf{L^1}(\mathbb{R}^n)$. Define

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-ix\xi}\mathrm{d}x, \quad \forall \xi \in \mathbb{R}^n.$$

$\mathcal{F}$ is called a Fourier transform.

Theorem 8.2. $\mathcal{F} : (\mathbf{L^1}(\mathbb{R}^n), \|\cdot\|_{\mathbf{L^1}}) \to (\mathbf{C}_o(\mathbb{R}^n), \|\cdot\|_\infty)$ *is a continuous linear mapping such that*

$$\|\mathcal{F}\| \leq (2\pi)^{-n/2}.$$

Here, $\mathbf{C}_o(\mathbb{R}^n)$ *denotes a space of continuous functions vanishing at infinity.*

Proof of Theorem 8.2. A linearity of $\mathcal{F}$ is obvious. Note that $|\mathrm{e}^{-ix\xi}| = 1$. Therefore,

$$\|\mathcal{F}f\|_\infty \leq (2\pi)^{-n/2}\|f\|_{\mathbf{L^1}} \quad \Rightarrow \quad \|\mathcal{F}\| \leq (2\pi)^{-n/2}.$$

Now, we want to prove

(a) $\lim_{\xi^k \to \xi} |(\mathcal{F}f)(\xi^k) - (\mathcal{F}f)(\xi)| = 0$,

(b) $\lim_{|\xi^k| \to +\infty} |(\mathcal{F}f)(\xi^k)| = 0$.

(a) Let $\{\xi^k\}_{k\in\mathbb{N}}$ be such that $\lim_{k\to+\infty} \xi^k = \xi$. Then, for a fixed x we have

$$\lim_{k\to+\infty} \left|\mathrm{e}^{-ix\xi^k} - \mathrm{e}^{-ix\xi}\right| = 0.$$

By Lebesgue dominated convergence theorem it holds that

$$\lim_{k\to+\infty} \left|(\mathcal{F}f)(\xi^k) - (\mathcal{F}f)(\xi)\right| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(x)| \cdot \lim_{k\to+\infty} \left|\mathrm{e}^{-ix\xi^k} - \mathrm{e}^{-ix\xi}\right|\mathrm{d}x = 0,$$

where we used the latter theorem due to the fact that the function under the integral can be estimated by $2\,|f| \in \mathbf{L^1}(\mathbb{R}^n)$.

(b) Note that the smooth functions with compact support are dense in $\mathbf{L^1}$, that is,

$$\overline{\mathbf{C_c^\infty}(\mathbb{R}^n)}^{\mathbf{L^1}(\mathbb{R}^n)} = \mathbf{L^1}(\mathbb{R}^n).$$

Thus, it is enough to show that for all $f \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^n)$ it holds that $\lim_{|\xi^k|\to+\infty}(\mathcal{F}f)(\xi^k) = 0$. Let f be such a function and j be such that $\left|\xi^k_j\right| = \max\{|\xi^k_i| \,:\, i = 1,\dots,n\}$, where ξ^k_j are coordinates of ξ^k. Then, $\left|\xi^k_j\right| \geq |\xi^k|/\sqrt{n}$. We apply the formula for integration by parts and as a result obtain

$$\begin{aligned}\left|(\mathcal{F}f)(\xi^k)\right| &= \left|\frac{-1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{\partial}{\partial x_j}f(x)\frac{1}{-i\xi_j}\mathrm{e}^{-ix\xi^k}\mathrm{d}x\right| \\ &\leq \frac{1}{(2\pi)^{n/2}}\cdot\left\|\frac{\partial}{\partial x_j}f\right\|_{\mathbf{L^1}}\cdot\frac{\sqrt{n}}{|\xi^k|} \overset{k\to+\infty}{\longrightarrow} 0.\end{aligned}$$

□

Definition 8.3. We say that $f:\mathbb{R}^n\to\mathbb{C}$ is vanishing faster than polynomial, if

$$\lim_{|x|\to+\infty} x^\alpha f(x) = 0, \quad \forall\alpha\in\mathbb{N}^n,$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$ and $x^\alpha = x_1^{\alpha_1}\cdot\dots\cdot x_n^{\alpha_n}$. A space

$$\mathcal{S}(\mathbb{R}^n) = \left\{f\in\mathbf{C}^\infty(\mathbb{R}^n) \,:\, D^\beta f \text{ vanishes faster that polynomial}, \;\; \forall\beta\in\mathbb{N}^n\right\}$$

is called Schwarz space.

Remark 22. Schwarz space consists of smooth functions, which vanish faster than polynomial together with all its derivatives. However, one has to note that

$$\mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^n) \subsetneq \mathcal{S}(\mathbb{R}^n).$$

Remark 23. If $f\in\mathcal{S}(\mathbb{R}^n)$, then

$$x^\alpha f, D^\alpha f \in \mathcal{S}(\mathbb{R}^n), \quad \forall n\in\mathbb{N}.$$

Lemma 8.4. *Let $f\in\mathcal{S}(\mathbb{R}^n)$. Then,*

(a) $\mathcal{F}f\in\mathbf{C}^\infty(\mathbb{R}^n)$ and $D^\alpha(\mathcal{F}f) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha f)$,

(b) $\mathcal{F}(D^\alpha f) = i^{|\alpha|}\cdot\xi^\alpha\mathcal{F}(f)$.

Proof of Lemma 8.4.
(a) Formal calculations leads to

$$\begin{aligned}D^\alpha(\mathcal{F}f) &= \frac{\partial^\alpha}{\partial\xi^\alpha}\cdot\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}f(x)\mathrm{e}^{-ix\xi}\mathrm{d}x = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}f(x)\cdot\frac{\partial^\alpha}{\partial\xi^\alpha}\mathrm{e}^{-ix\xi}\mathrm{d}x \\ &= (-i)^{|\alpha|}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}f(x)x^\alpha\mathrm{e}^{-ix\xi}\mathrm{d}x = (-i)^{|\alpha|}\mathcal{F}(x^\alpha f)(\xi).\end{aligned}$$

In the second equality we have passed with differentiation under the integral sign. Since $x^\alpha f\in\mathcal{S}(\mathbb{R}^n)\subset\mathbf{L^1}(\mathbb{R}^n)$ this is justified by Lebesgue dominated convergence theorem.

(b) If follows from the formula for integration by parts that

$$\begin{aligned}\mathcal{F}(D^\alpha f) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}(D^\alpha f)(x)\mathrm{e}^{-ix\xi}\mathrm{d}x\\ &= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)\cdot\frac{\partial^\alpha}{\partial x^\alpha}\mathrm{e}^{-ix\xi}\mathrm{d}x = (-1)^{|\alpha|}(-i)^{|\alpha|}\xi^\alpha\mathcal{F}(f).\end{aligned}$$

Since $x^\beta f \in \mathcal{S}(\mathbb{R}^n)$ boundary terms vanished in the calculation above. □

Lemma 8.5. *If $f \in \mathcal{S}(\mathbb{R}^n)$, then also $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$.*

Proof of Lemma 8.5. In Lemma 8.4 we proved that $\mathcal{F}f \in \mathbf{C}^\infty(\mathbb{R}^n)$. We need to show that $\mathcal{F}f$ and all its derivatives vanish faster than polynomial, that is,

$$\xi^\alpha D^\beta(\mathcal{F}f)(\xi) \to 0 \ \text{ for } \ |\xi| \to +\infty.$$

From Lemma 8.4 we know that

$$\xi^\alpha D^\beta(\mathcal{F}f) = (-i)^{|\beta|}(-i)^{|\alpha|}\mathcal{F}(D^\alpha(x^\beta f))$$

which implies that $D^\alpha(x^\beta f) \in \mathcal{S}(\mathbb{R}^n)$ and thus in $\mathbf{L^1}(\mathbb{R}^n)$. Fourier transform of a function from $\mathbf{L^1}(\mathbb{R}^n)$ is a continuous function vanishing at infinity, which ends the proof. □

Let us consider a function

$$\gamma : \mathbb{R}^n \to \mathbb{R}, \quad \gamma(x) = \mathrm{e}^{-x^2/2}.$$

It is a well known fact that

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\gamma(x)\mathrm{d}x = 1.$$

We also denote $\gamma_a(x) = \gamma(ax)$, for $a > 0$.

Lemma 8.6. *For γ and γ_a defined as above it holds that*

$$(\mathcal{F}\gamma)(\xi) = \mathrm{e}^{-\xi^2/2} \quad \text{and} \quad (\mathcal{F}\gamma_a)(\xi) = \frac{1}{a^n}(\mathcal{F}\gamma)\left(\frac{\xi}{a}\right).$$

Proof of Lemma 8.6. The second statement is straightforward. To prove the first claim let $n = 1$. Note that γ fulfills the following differential equation

$$y' + xy = 0, \quad y(0) = 1. \tag{8.7}$$

By Lemma 8.4

$$0 = \mathcal{F}(\gamma' + x\gamma) = i\xi(\mathcal{F}\gamma) + \left(\frac{1}{-i}(\mathcal{F}\gamma)\right)' \quad \Rightarrow \quad (\mathcal{F}\gamma)' + \xi(\mathcal{F}\gamma) = 0,$$

which means that $\mathcal{F}\gamma$ fulfills equation (8.7) with the same initial condition, since

$$(\mathcal{F}\gamma)(0) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}}\mathrm{e}^{-x^2/2}\mathrm{d}x = 1.$$

Since the solution to (8.7) is unique we obtain $\gamma = \mathcal{F}\gamma$. The case $n > 1$ leads in fact to the previous case, since

$$(\mathcal{F}\gamma)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \Pi_{k=1}^{n} \mathrm{e}^{-x_k^2/2} \mathrm{e}^{-ix_k\xi_k} \mathrm{d}x_1 \dots \mathrm{d}x_n = \Pi_{k=1}^{n} \mathrm{e}^{-\xi_k^2/2}.$$

□

Lemma 8.8. *For $f \in \mathcal{S}(\mathbb{R}^n)$ it holds that*

$$(\mathcal{F}\mathcal{F}f)(x) = f(-x), \quad \forall x \in \mathbb{R}^n.$$

Proof of Lemma 8.8. We proved that $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$ and thus $\mathcal{F}(\mathcal{F}f)$ is well defined. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Since a function

$$(x,\xi) \to f(\xi)g(x)\mathrm{e}^{-ix\xi}$$

is integrable, Fubini theorem yields

$$\int_{\mathbb{R}^n} (\mathcal{F}f)(x)g(x)\mathrm{d}x = \int_{\mathbb{R}^n} f(x)(\mathcal{F}g)(x)\mathrm{d}x. \tag{8.9}$$

This implies that for $g(x) = \mathrm{e}^{-ix\xi_o}\gamma(ax)$, where $\xi_o \in \mathbb{R}^n$ and $a > 0$ are fixed, that

$$(\mathcal{F}g)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\xi_o}\gamma(ax)\mathrm{e}^{-ix\xi}\mathrm{d}x = (\mathcal{F}\gamma_a)(\xi + \xi_o).$$

We use Lemma 8.6 and the following change of variables $u = \frac{x+\xi_o}{a}$, which leads to

$$\begin{aligned} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} (\mathcal{F}f)(x)\mathrm{e}^{-ix\xi_o}\gamma(ax)\mathrm{d}x &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} f(x)\frac{1}{a^n}(\mathcal{F}\gamma)\left(\frac{x+\xi_o}{a}\right)\mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} f(au - \xi_o)\gamma(u)\mathrm{d}u. \end{aligned}$$

Now we want to pass to the limit with $a \to 0$. Since $x \to (\mathcal{F}f)(x)\mathrm{e}^{-ix\xi_o}\gamma(ax)$ is bounded by $|\mathcal{F}f| \in \mathbf{L^1}$ and $x \to f(x)\frac{1}{a^n}(\mathcal{F}\gamma)\left(\frac{x+\xi_o}{a}\right)$ is bounded by $\|f\|_\infty$, application of Lebesque convergence theorem ends the proof (the limit of the left hand side is equal to $(\mathcal{F}\mathcal{F}f)\xi_o$ and the limit on the right hand side is equal to $f(-\xi_o)$). □

Theorem 8.10. *The Fourier transform is a bijection on $\mathcal{S}(\mathbb{R}^n)$. The inverse operator $\mathcal{F}^{-1}$ is defined by*

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi)e^{ix\xi}\mathrm{d}\xi, \quad \forall x \in \mathbb{R}^n. \tag{8.11}$$

Moreover, for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ it holds that

$$\langle \mathcal{F}f, \mathcal{F}g \rangle_{\mathbf{L^2}} = \langle f, g \rangle_{\mathbf{L^2}}.$$

Proof of Theorem 8.10. From the previous lemma we know that $\mathcal{F}^4 = \mathrm{Id}$. Thus, $\mathcal{F}$ is bijective and $\mathcal{F}^{-1} = \mathcal{F}^3$ is bijective as well. It also holds

$$(\mathcal{F}^{-1}f)(x) = (\mathcal{F}^2(\mathcal{F}f))(x) = (\mathcal{F}f)(-x),$$

which proves that (8.11) holds. From (8.9) we have that

$$\int_{\mathbb{R}^n} (\mathcal{F}f)(\xi)\overline{(\mathcal{F}g)(\xi)}\mathrm{d}\xi = \int_{\mathbb{R}^n} f(x)(\mathcal{F}(\overline{\mathcal{F}g}))(x)\mathrm{d}x.$$

For transparency purposes let denote $h = \mathcal{F}g$. Then,

$$(\mathcal{F}\bar{h})(x) \;=\; \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \overline{h(\xi)}\mathrm{e}^{-ix\xi}\mathrm{d}\xi = \frac{1}{(2\pi)^{n/2}}\overline{\int_{\mathbb{R}^n} h(\xi)\mathrm{e}^{ix\xi}\mathrm{d}\xi} = \overline{\mathcal{F}^{-1}h(x)} = \overline{g(x)}$$

and thus

$$\left\langle \mathcal{F}f, \mathcal{F}g\right\rangle_{\mathbf{L^2}} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\mathrm{d}x = \left\langle f, g\right\rangle_{\mathbf{L^2}}.$$

□

As a consequence, for all $f \in \mathcal{S}(\mathbb{R}^n)$ it holds that $\|\mathcal{F}f\|_{\mathbf{L^2}} = \|f\|_{\mathbf{L^2}}$. The operator $\mathcal{F}$ is thus continuous as a mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathbf{L^2}(\mathbb{R}^n)$, bijective and $\|\cdot\|_{\mathbf{L^2}}$-isometric.

Remark 24. For $f \in \mathbf{L^2}(\mathbb{R}^n)$ we define

$$g_R(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{B(0,R)} f(x)\mathrm{e}^{-ix\xi}\mathrm{d}x \quad \text{and} \quad (\mathcal{F}_2 f)(\xi) = \lim_{R\to+\infty} g_R(\xi),$$

where a limit in the definition above is the limit in $\mathbf{L^2}(\mathbb{R}^n)$.

Theorem 8.12. *(Hausdorff-Young inequality) Let $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in \mathcal{S}(\mathbb{R}^n)$ it holds that $\mathcal{F}f \in \mathbf{L^q}(\mathbb{R}^n)$ and*

$$\|\mathcal{F}f\|_{\mathbf{L^q}} \le \frac{1}{(2\pi)^{n/p-n/2}} \cdot \|f\|_{\mathbf{L^p}}.$$

There exists an extension of $\mathcal{F}$ to a continuous operator $\mathcal{F} : \mathbf{L^p}(\mathbb{R}^n) \to \mathbf{L^q}(\mathbb{R}^n)$, which is defined by

$$(\mathcal{F}f)(\xi) = \lim_{R\to+\infty} g_R(\xi),$$

where a limit in the definition above is the limit in $\mathbf{L^q}(\mathbb{R}^n)$.

Proof of Theorem 8.12. For $(p,q) = (1,+\infty)$ and $(p,q) = (2,2) wehave$

$$\begin{aligned} \|\mathcal{F} : \mathbf{L^1} \to \mathbf{L^\infty}\| &\le (2\pi)^{-n/2}, \\ \|\mathcal{F} : \mathbf{L^2} \to \mathbf{L^2}\| &\le 1. \end{aligned}$$

For other case the proof follows from the Riesz interpolation theorem. One takes $\theta = 2 - \frac{2}{p}$, so that $\frac{1}{q} = \frac{1-\theta}{+\infty} + \frac{\theta}{2}$. □

LECTURE 9
Adjoint operators on Hilbert spaces

Definition 9.1. Let T be a linear bounded operator $T \in \alpha(H,H)$. We say that $T^* : H \to H$ is an adjoint operator to T, if

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \forall x, y \in H.$$

Lemma 9.2. *For each y, element $T^* y$ is unique and $\|T^* y\| \leq \|T\| \cdot \|x\|$. Thus, T^* is well defined and its norm is bounded.*

Proof of Lemma 9.2. Fix $y \in H$. Note that $x \to \langle Tx, y \rangle$ is a linear bounded functional. From Riesz representation theorem

$$\forall y \in H \ \ \exists z_y \in H \quad \text{s. t.} \quad \langle Tx, y \rangle = \langle x, z_z \rangle, \quad \forall x \in H.$$

We set $T^* y = z_y$. Then, it also holds that $\|z_y\| \leq \|T\| \cdot \|x\|$. □

Lemma 9.3. *It holds that*

(a) $(T+S)^* = T^* + S^*$,
(b) $(\alpha T)^* = \bar{\alpha} T^*$,
(c) $(ST)^* = T^* S^*$,
(d) $T^{**} = T$.

Definition 9.4. Let $T \in \alpha(H,H)$. Then,

(a) T is normal if $TT^* = T^* T$,
(b) T is self adjoint if $T = T^*$,
(c) T is unitary if $T^* T = \mathrm{Id} = TT^*$.

Theorem 9.5. *If $T \in \alpha(H,H)$, then T is normal if and only if,*

$$\|Tx\| = \|T^* x\|, \quad \forall x \in H.$$

Proof of Theorem 9.5. " $\Rightarrow$ " It holds that

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^* Tx, x \rangle = \langle TT^* x, x \rangle = \|T^* x\|^2.$$

" $\Leftarrow$ " Before we prove this implication we shall show the following claim. Let $T \in \alpha(H,H)$ be such that

$$\langle Tx, x\rangle = 0, \quad \forall x \in H.$$

Then, $T = 0$. Indeed,

$$\langle T(x+y), T(x+y)\rangle = 0 \quad \Rightarrow \quad \langle Tx, y\rangle + \langle Ty, x\rangle = 0,$$
$$\langle T(x+iy), T(x+iy)\rangle = 0 \quad \Rightarrow \quad -\langle Tx, y\rangle + \langle Ty, x\rangle = 0,$$

and thus $\langle Ty, x\rangle = 0$, for all $x, y \in H$, which implies $T = 0$. Now we can proceed with the proof. We have that

$$0 = \|Tx\|^2 - \|T^*x\|^2 = \langle (T^*T - TT^*)x, x\rangle.$$

Thus, $T^*T - TT^* = 0$ due to the claim. □

Definition 9.6. For an operator T we define its spectrum as

$$\sigma(T) = \{\lambda \in \mathbb{C} \,:\, (T - \lambda \mathrm{Id}) \text{ is not invertible}\}.$$

The condition which states that $(T - \lambda\mathrm{Id})$ is not invertible is equivalent to:

(a) $\mathrm{Im}(T - \lambda\mathrm{Id}) \neq H$,
(b) $\ker(T - \lambda\mathrm{Id}) \neq \{0\}$.

Remark 25. If $\lambda \notin \sigma(T)$, then $(T - \lambda\mathrm{Id})^{-1}$ is a linear bounded functional (due to open mapping theorem).

Lemma 9.7. *If $T : H \to H$ is a bounded linear functional, then*

$$\sigma(T) \subset \overline{B(0, \|T\|)}.$$

Proof of Lemma 9.7. Assume that $|\lambda| > \|T\|$. Then,

$$(\mathrm{Id} - T/\lambda)^{-1} = \sum_{n=0}^{+\infty} \left(\frac{T}{\lambda}\right)^n$$

is a convergent series and thus the operator $(\mathrm{Id} - T/\lambda)$ is invertible, which is a contradiction.
□

Definition 9.8. Operator $T \in \alpha(H,H)$ is a compact operator, if a closure of an image of the unit ball is compact, that is,

$$\overline{T(B_X(0,1))} \quad \text{is compact.}$$

Lemma 9.9. *If T is self adjoint, then $\sigma(T) \subset \mathbb{R}$.*

Proof of Lemma 9.10. We know that $T = T^*$. Let $\lambda = \alpha - i\beta \in \sigma(T)$. Then,

$$\begin{aligned}
\|(T - \lambda\mathrm{Id})x\|^2 &= \langle Tx - \alpha x + i\beta x, Tx - \alpha x + i\beta x\rangle \\
&= \langle Tx - \alpha x, Tx - \alpha x\rangle + \langle Tx - \alpha x, i\beta x\rangle + \langle i\beta x, Tx - \alpha x\rangle + \langle i\beta x, i\beta x\rangle \\
&= \|Tx - \alpha x\|^2 + \beta^2 \cdot \|x\|^2.
\end{aligned}$$

Thus, $\|(T-\lambda \mathrm{Id})x\|^2 \geq \beta^2 \cdot \|x\|^2$, which implies $\beta = 0$. □

Theorem 9.10. *Let $T \in \alpha(H,H)$ be a self adjoint operator. Then,*

$$\|T\| = \sup_{x:\|x\|=1} \left|\left\langle Tx, x\right\rangle\right|.$$

Proof of Theorem 9.10. It is clear that

$$\sup_{x:\|x\|\leq 1} \left|\left\langle Tx, x\right\rangle\right| \leq \sup_{x:\|x\|\leq 1} \|Tx\| \cdot \|x\| \leq \sup_{x:\|x\|\leq 1} \|Tx\| = \|T\|.$$

To prove the opposite inequality define $M = \sup_{x:\|x\|\leq 1} \left|\left\langle Tx, x\right\rangle\right|$. Then,

$$\begin{aligned}
\left\langle T(x+y), x+y\right\rangle - \left\langle T(x-y), x-y\right\rangle &= 2\left\langle Tx, y\right\rangle + 2\left\langle Ty, x\right\rangle \\
&= 2\left\langle Tx, y\right\rangle + 2\overline{\left\langle x, Ty\right\rangle} = 2\left\langle Tx, y\right\rangle + 2\left\langle Ty, x\right\rangle \\
&= 2\left\langle Tx, y\right\rangle + 2\overline{\left\langle Tx, y\right\rangle} = 4\mathrm{Re}\left\langle Tx, y\right\rangle.
\end{aligned}$$

In the calculation above some terms vanish due to the fact that T is self adjoint. By parallelogram equality we have

$$\begin{aligned}
4\mathrm{Re}\left\langle Tx, y\right\rangle \leq M(\|x+y\|^2 + \|x-y\|^2) &= 2M(\|x\|^2 + \|y\|^2) \\
\mathrm{Re}\left\langle Tx, y\right\rangle &\leq M, \quad \forall x, y : \|x\|, \|y\| = 1
\end{aligned}$$

Multiplication by λ such that $|\lambda| = 1$ gives

$$\left|\left\langle Tx, y\right\rangle\right| \leq M \quad \forall x, y : \|x\|, \|y\| = 1,$$

which proves that $\|T\| = M$. □

Lemma 9.11. *Let $T \in \alpha(H,H)$ be a self adjoint and compact operator. Then, $\|T\|$ or $-\|T\|$ is an eigenvalue of T.*

Proof of Lemma 9.11. There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ ($\|x_n\| \leq 1$) such that

$$\left|\left\langle Tx_n, x_n\right\rangle\right| \to \|T\|.$$

This holds due to the definition of a norm of T. Since the T is compact one can chose a subsequence x_{n_k} which converges. For transparency purposes we omit the lower index k and denote this subsequence as $\{x_n\}_{n\in N}$. Denote

$$\lambda = \lim_{n\to+\infty} \left\langle Tx_n, x_n\right\rangle.$$

By compactness of T there exists y such that

$$y = \lim_{n\to+\infty} Tx_n \leq \|T\|$$

Therefore,

$$\begin{aligned}
\|Tx_n - \lambda x_n\|^2 &= \left\langle Tx_n - \lambda x_n, Tx_n - \lambda x_n\right\rangle \\
&= \|Tx_n\|^2 - 2\lambda\left\langle Tx_n, x_n\right\rangle + \lambda^2\|x_n\| \leq \lambda^2 + \|Tx_n\|^2 - 2\lambda\left\langle Tx_n, x_n\right\rangle.
\end{aligned}$$

A term $\|Tx_n\|^2$ converges to λ^2 and $\langle Tx_n, x_n\rangle$ converges to λ. Thus,

$$\|Tx_n - \lambda x_n\|^2 \to 0$$

and $y = \lim_{n\to+\infty} \lambda x_n$. Since T is continuous and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly, it holds that

$$Ty = \lim_{n\to+\infty} T\lambda x_n = \lim_{n\to+\infty} \lambda T x_n = \lambda y.$$

Thus, $\lambda \in \sigma(T)$. Note that since the spectrum of T is contained in $\mathbb{R}$, the only possibility for a value of λ is $\|T\|$ or $-\|T\|$. □

Theorem 9.12. *(Spectral theorem for compact self adjoint operator)*
Let $T \in \alpha(H, H)$ be a self adjoint and compact operator. Then, there exists an orthonormal system $S = \{e_k : k \in \mathbb{N}\}$ and a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ converging to zero as i tends to infty, such that

$$H = \ker(T) \oplus_{\mathbf{L^2}} \overline{\operatorname{lin}(S)} \tag{9.13}$$

and

$$Tx = \sum_{i=1}^{+\infty} \lambda_k \langle x, e_k\rangle e_k, \quad \forall x \in H,$$

where $\sup|\lambda_k| = \|T\|$.

Proof of Theorem 9.12. We claim that

$$\dim(\ker(\lambda_i \mathrm{Id} - T)) < +\infty.$$

Indeed, if there were infinitely many v such that $v \in \ker(\lambda_i\mathrm{Id} - T)$ (which is equivalent to $Tv = \lambda_i v$), then a dimension of the (scaled) unit ball would be infinite. However, in an infinite dimensional space the unit ball is not compact, which is a contradiction to the fact that T is compact. In each space $\ker(\lambda_i\mathrm{Id} - T)$ one can choose an orthonormal basis. Then, for $k \neq j$ we have $Te_k = \lambda_k e_k$ and $Te_j = \lambda_j e_j$. Since $e_k \perp e_j$,

$$\ker(\lambda_k\mathrm{Id} - T) \perp \ker(\lambda_j\mathrm{Id} - T).$$

Set λ_1 to be equal to $\|T\|$ or $-\|T\|$. The corresponding space would be $H_1 = \ker(\lambda_1\mathrm{Id} - T)$. We make a decomposition

$$H = H_1 \oplus_{\mathbf{L^2}} (H_1)^\perp$$

and restrict the operator to the subspace $(H_1)^\perp$. A norm of the restricted operator is not bigger that $\|T\|$. Iteration of this procedure leads to the decomposition given by (9.13). Now, the only fact which needs to be clarified is that the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ converges to 0. However, if $\{\lambda_i\}_{i\in\mathbb{N}}$ stabilized on some level, then it would be a contradiction with compactness of T (argument similar to the one used for proving that a kernel of $\lambda_i\mathrm{Id} - T$ is finite dimensional). □

LECTURE 10
Locally convex spaces

Definition 10.1. We define

(a) Set A is called a disc set if
$$\{\lambda : |\lambda| \le 1\} \cdot A \subset A,$$

(b) Set A is called an absolutely convex set if it is convex and (a) holds.

It can be shown that A is absolutely convex, if
$$\forall x_1, x_2 \in A, \ |\lambda_1| + |\lambda_2| \le 1 \ \Rightarrow \ \lambda_1 x_1 + \lambda_2 x_2 \in A.$$

Remark 26. In case $\mathbb{K} = \mathbb{R}$ a set A has the disc property, if it is balanced. Thus, it is sufficient to check if $-1 \cdot A \subset A$.

Remark 27. If a set A is absolutely convex and absorbing, then
$$p_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}$$
is a seminorm.

Remark 28. If p is a seminorm, then
$$A = \{x \in X : p(x) \le 1\}$$
is absolutely convex.

Definition 10.2. A topological linear space (X, τ) is a linear space X over a field $\mathbb{K}$, which is endowed in a topology τ, such that vector addition and scalar multiplication are continuous (in the sense that a preimage of each open set is open).

Definition 10.3. Let P be a family of seminorms on a linear space X and τ be generated by sets
$$U_{F,\varepsilon} = \{x : p(x) \le \varepsilon, \ \forall p \in F\},$$
where F is finite and $F \subset P$. Then (X, τ) is called a locally convex space. Sets $U_{F,\varepsilon}$ form a basis of neighborhoods of 0.

Theorem 10.4. *A topological linear space (X,τ) is locally convex, if there exists a basis of neighborhoods of* 0*, which consists of absolutely convex and absorbing sets.*

Lemma 10.5. *Let P be a family of seminorms which generates a locally convex topology τ on X. Then, the following conditions are equivalent*

(a) (X,τ) is a Hausdorff space,
(b) $\forall x \in X, x \neq 0$ there exists $p \in P$ such that $p(x) \neq 0$,
(c) There exists a family $\mathcal{U}$ of open sets such that $\bigcap_{U\in\mathcal{U}} U = \{0\}$.

Proof of Lemma 10.5. $(a) \Rightarrow (b)$ Assume that (X,τ) is a Hausdorff space and let $x \neq 0$. Then, there exist neighborhoods of zero U, V such that $(x+U)\cap V = \varnothing$. Without loss of generality we can assume that

$$V = U_{F,\varepsilon} = \{u \,:\, p(u) \leq \varepsilon, \ \ \forall p \in F\}.$$

Since $x \notin V$, there exists p such that $p(x) > \varepsilon \ \Rightarrow \ p(x) \neq 0$.
$(b) \Rightarrow (c)$ Let

$$x \in \bigcap_{F,\varepsilon} U_{F,\varepsilon} = \bigcap_{F,\varepsilon} \{x \,:\, p(x) \leq \varepsilon, \ \ \forall p \in F\}.$$

Then, for all $p \in P$ it holds that $p(x) = 0$, which implies (c).
$(c) \Rightarrow (a)$ Assume that $x \neq y$. Then, there exists an open set U (which is an element of the basis of neighborhoods of 0, see Definition 10.3) such that $x - y \notin U$. Addition (and subtraction) is a continuous operation. Thus, the preimage of U is open and we can find two open sets W, V such that $W - V \subset U$, which implies $(x+V)\cap(y+W) = \varnothing$. □

Example 14. Below we give some examples of families of seminorms.

(a) $X = \mathbf{C}_b(\mathbb{R}^n)$, $P = \{p_t,\ t \in \mathbb{R}^n\}$, where $p_t(f) = |f(t)|$ (topology of pointwise convergence).
(b) X - normed space, $P = X'$, $p_{x'}(x) = |x'(x)|$ (denoted as $\sigma(X, X')$).
(c) X - normed space, $P = X'$, $p_x(x') = |x'(x)|$ (denoted as $\sigma(X', X)$).

Lemma 10.6. *Let (X,τ) be a locally convex space and P be a family of seminorms, which generates a topology τ on X. Then,*

(a) For a seminorm $q : X \to [0,+\infty)$ the following conditions are equivalent
  $(a.1)$ q is continuous,
  $(a.2)$ q is continuous in 0*,*
  $(a.3)$ $\{x \,:\, q(x) < 1\}$ is an open neighborhood of 0*.*
(b) All $p \in P$ are continuous.
(c) The seminorm q is continuous, if and only if, there exists a finite family $F \subset P$ and a constant $M > 0$ such that

$$q(x) \leq M \max_{p\in F} p(x) \quad \forall x \in X.$$

Proof of Lemma 10.6. Implications $(a.1) \Rightarrow (a.2) \Rightarrow (a.3)$ are straightforward. Assume that $(a.3)$ holds. Then, for $x \in X$ and $\varepsilon > 0$ we define

$$U = \varepsilon \cdot \{y \,:\, q(y) < 1\} = \{y \,:\, q(y) < \varepsilon\},$$

which is an open set. Using the triangle inequality we obtain

$$\begin{aligned} q(x+y)-q(x) &\leq q((x+y)-x)+q(x)-q(x)=q((x+y)-x), \\ q(x+y)-q(x) &\geq q(x+y)-q(x+y)-q((x+y)-x)=-q((x+y)-x), \end{aligned}$$

which is equivalent to

$$|q(x+y)-q(x)| \leq q((x+y)-x).$$

Therefore,

$$q(x+U) \subset \{a \in \mathbb{R} : |a-q(x)| < \varepsilon\},$$

which is in fact a definition of continuity.
(b) According to the definition of τ, sets $\{x : p(x) < 1\}$ are open and thus, all p are continuous (which follows from the claim (a)).
(c) According to the claim (a), q is continuous, if and only if, there exists a finite set $F \subset P$ such that $U_{F,\varepsilon} \subset \{x : q(x) \leq 1\}$ holds, which means that

$$q(x) \leq (1/\varepsilon) \sup_{p \in F} p(x) \quad \forall x \in X.$$

□

Remark 29. Let P and Q be families of seminorms such that $P \subset Q \subset \{q : q \text{ is continuous}\}$. Then, topologies τ_P and τ_Q are equivalent.

Definition 10.7. A space consisting of all linear and continuous functionals on a locally convex space (X,τ) is called a dual space $(X,\tau)'$ (we shall also use a notation $(X_\tau)'$).

Theorem 10.8. *Let (X,τ_p) and (Y,τ_q) be locally convex spaces and $T : X \to Y$ be a linear operator. Then, the following conditions are equivalent*

(a) T is continuous (in the sense that $T^{-1}(U)$ is open for every open set U),
(b) T is continuous at 0,
(c) If q is a continuous seminorm, then $q \circ T$ is a continuous seminorm,
(d) For every $q \in Q$ there exists a finite family $F \subset P$ and $M > 0$ such that

$$q(Tx) \leq M \max_{p \in F} p(x).$$

The proof is similar to the proof of the previous theorem.

Remark 30. Let (X,τ_p) be a locally convex space and $l : X \to \mathbb{L}$ be a linear operator. Then, l is continuous if and only if, there exist a finite subset $\{p_1,\dots,p_n\} \in P$ and a constant $M > 0$ such that

$$l(x) \leq M \max_{i=1,\dots,n} p_i(x) \quad \forall x \in X.$$

If $\tau_{p_1} \subset \tau_{p_2}$, then we say that τ_{p_2} is stronger. In particular, it has more open sets, closed sets, continuous operators, but less convergent sequences (since it is harder to prove that $\lim_{n\to+\infty} p(u^n - u) = 0$ for each seminorm p).

Example 15. Let $X = \mathbf{C}_b(T)$ and consider a topology τ_P generated by a family of seminorms

$$p_t(x) = |x(t)| \quad \forall t \in T.$$

On the other side consider a topology τ_∞ generated by a norm $\|\cdot\|_\infty$. One can easily show that

$$p_t(x) \le \|x\|_\infty \quad \forall p_t \in P,$$

and thus the topology τ_∞ is stronger.

Remark 31. Consider a norm topology on X and a weak topology $\sigma(X, X')$ (or a weak* topology $\sigma(X', X)$). We recall that $\sigma(X, X')$ (respectively $\sigma(X', X)$) is generated by a family of seminorms defined by linear functionals. One can show that the unit ball in the norm topology is not an open set in $\sigma(X, X')$ (or $\sigma(X', X)$).

Example 16. In the following example we shall show that if x is in a closure of a set A, it does not imply that there exists a sequence of elements converging to x. Let (X, τ) be a locally convex space, where $X = l^2(\mathbb{N})$ and $\tau = \sigma(X, X') = \sigma(X', X)$. Let

$$A = \{e_m + me_n \,:\, 1 \le m < n\}.$$

Then, $0 \in \overline{A}^{\sigma(X,X')}$, but there is no sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that x_n converges to 0 in $\sigma(X, X')$.

Proof: Let U be any neighborhood of zero. If we prove that $U \cap A \ne \varnothing$, then 0 is in the closure of A indeed. Without loss of generality we can assume that

$$U = \{x \in l^2(\mathbb{N}) \,:\, \left|\langle x, y_i\rangle\right| \le \varepsilon\} \quad i = 1, \dots, r \text{ and } y_i \in l^2(\mathbb{N}).$$

Let m be such that

$$|y_i(m)| < \varepsilon/2, \text{ for } i = 1, \dots, r$$

and choose $n > m$ such that

$$m \cdot |y_i(n)| \le \varepsilon/2 \text{ for } i = 1, \dots, r.$$

Then,

$$\left|\langle e_m + me_n, y_i\rangle\right| \le |y_i(m)| + m \cdot |y_i(n)| \le \varepsilon$$

and therefore $e_m + me_n \in U$.

On the contrary, let $\{e_{m_k} + m_k e_{n_k}\}_{k\in\mathbb{N}}$ be a sequence in A. If the sequence converged to zero in $\sigma(X, X')$, it would imply $m_k, n_k \to +\infty$. The norm of each element $e_{m_k} + m_k e_{n_k}$ should be bounded (uniformly with respect to k), since each weakly convergent sequence is bounded. But,

$$\langle e_m + me_n, e_m + me_n\rangle = 1 + m^2,$$

which is unbounded.

Theorem 10.9. *(Hahn-Banach theorem) Let X be a locally convex space and U be its subspace. Assume that $l \in U'$. Then, there exists an extension $L \in X'$.*