Kolokwium z Analizy Funkcjonalnej, 26 listopada 2013

Problem 1 *For a sequence $y = \{y_n\}_{n\in\mathbb{N}}$ we define a function T by the formula*

$$T(y) = \{x_n\}_{n\in\mathbb{N}},$$

where

$$x_n = \frac{1}{2}\sum_{j=1}^{n}\left(\frac{1}{n}\right)^j y_j.$$

a) *Show that $T : \ell_1 \to c_0$ is a continuous linear operator.*

b) *Calculate the norm of the operator $T : \ell_1 \to c_0$.*

c) *Let $y = \{y_n\}_{n\in\mathbb{N}} \in c_0$. Prove that there exists a sequence $x = \{x_n\}_{n\in\mathbb{N}} \in c_0$ such that*

$$x_n - \frac{1}{2}\sum_{j=1}^{n}\left(\frac{1}{2}\right)^j x_j = y_n.$$

Problem 2 *Let*

$$X = \{f \text{ - Lebesgue measurable function} : \sup_{1\leqslant p\leqslant\infty} ||f||_{L_p(\mathbb{R}^n)} < +\infty\}$$

a) *Show that $\|\cdot\|_X = \sup_{1\leqslant p\leqslant\infty}\|\cdot\|_{L_p(\mathbb{R}^n)}$ is a norm on a space X.*

b) *Prove that X with $||\cdot||_X$ is complete.*

c) *We define function $\|\cdot\|_G$ in a following way*

$$\|\cdot\|_G = G(\|\cdot\|_{L_{p_1}(\mathbb{R}^n)}, \|\cdot\|_{L_{p_2}(\mathbb{R}^n)}, \ldots, \|\cdot\|_{L_{p_d}(\mathbb{R}^n)})$$

Find function $G : \mathbb{R}^d \to \mathbb{R}^+$ and sequence $\{p_i\}_{i=1}^d$, $1 \leqslant p_i \leqslant \infty$ such that $\|\cdot\|_G$ is a norm on X equivalent to $||\cdot||_X$.

Problem 3 *Find the norm of the functional $f \in C([0,1])^*$ defined for any $x \in C([0,1])$ by*

$$f(x) = x(0) - \int_0^{\frac{1}{3}} x(3t)\cdot t^{\frac{2}{3}}\,dt.$$

Problem 4 *Let $\phi : c \to c$ be given by*

$$\phi(\{a_n\}_{n\in\mathbb{N}}) = \lim_{k\to\infty}\frac{\sum_{i=1}^k 2^{i+1}a_{2^i}}{\sum_{i=1}^k 2^i}.$$

Prove the existence of Φ - an extension of ϕ to ℓ^∞ such that

$$2\liminf_{n\to\infty} x_n \leqslant \Phi(\{x_n\}_{n\in\mathbb{N}}) \leqslant 2\limsup_{n\to\infty} x_n$$