4th homework of the Functional Analysis, 16.1.2015

A. Świerczewska-Gwiazda, P. B. Mucha and L. Bartczak

The deadline: January, 23rd 2015, h14.30.
The place of submition: envelope on the door of the room no 5020.

Ex 1. Let $X \subset Y$ be closed subspaces of a Hilbert space, $X \neq Y$. Show that there exists $y \in Y$ such that $\|y\| = 1$, $\|y - x\| \geq 1$ for all $x \in X$.

Ex 2. (Apollonius' identity) Show that for arbitrary elements of (real or complex) Hilbert space x, y, z the following identity holds

$$2\|x - z\|^2 + 2\|y - z\|^2 = \|x - y\|^2 + 4\left\| z - \frac{x + y}{2} \right\|^2.$$

Ex 3. (Radon-Riesz property) Let $\{x_n\}$ be a sequence of elements of Hilbert space H, such that

$$x_n \rightharpoonup x \text{ weakly in } H, \text{ and } \|x_n\| \to \|x\|.$$

Show that then in fact $x_n \to x$ strongly in H.

Ex 4. (Hermite polynomials) Consider the real space $L^2(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{e}^{-x^2}\mathrm{dx}.$$

Show that $\{1, x, 2x^2 - 1\}$ forms an orthogonal system.

Ex 5. Let X be the set of real absolutely continuous functions on $[0, 1]$, for which $f' \in L^2(0, 1)$. Decide whether X is prehilbert, or even Hilbert space, if

$$\langle f, g\rangle = f(0)g(0) + \int_0^1 f'(x)g'(x)\mathrm{dx}.$$

Ex 6. Describe the orthogonal complement to A in $L^2(-1, 1)$, where

a) $A = \{f \in L^2(-1, 1), f \text{ is even}\}$,

b) $A = \{f \in L^2(-1, 1), \int_{-1}^{1} f(x) = 0\}$.

What do the corresponding orthogonal projections look like?