3rd homework of the Functional Analysis, 5.12.2014

A. Świerczewska-Gwiazda, P. B. Mucha and L. Bartczak

The deadline: December, 12th 2014, h14.30.
The place of submition: envelope on the door of the room no 5020.

Ex 1. Let A, B be disjoint, nonempty, and convex subsets of a real Banach space X. Suppose further that A is closed, and B is compact. Prove that there exists a functional $\varphi \in X^*$ such that

$$\sup_{x\in A} \varphi(x) < \inf_{x\in B} \varphi(x).$$

Ex 2. (Uniqueness of weak* limit) Suppose that $x_n^* \rightharpoonup^* x^*$ (weakly*) in X^*, and $x_n^* \rightharpoonup^* y^*$ (weakly*) in X^*. Show that $x^* = y^*$.

Ex 3. Let $\Omega \subset \mathbb{R}^d$ be a bounded and open set. Suppose $\{f_n\} \subset L^\infty(\Omega)$, $f \in L^\infty(\Omega)$. Prove that

$$f_n \rightharpoonup^* f \text{ in } L^\infty(\Omega) \iff \begin{cases} (a) \quad \sup_{n\in\mathbb{N}} \|f_n\|_{L^\infty} < +\infty \\ (b) \quad \forall V \subset \Omega \ \lim_{n\to\infty} \int_V \big(f_n(x) - f(x)\big) = 0, \\ \qquad\qquad \text{where } V = \Pi_{i=1}^d (a_i, b_i) \text{ (hypercuboid).} \end{cases}$$

Ex 4. Let $U := \Pi_{i=1}^d (a_i, b_i) \subset \mathbb{R}^d$ ($a_i < b_i$ for each i) and let $f \in L^p(U)$ for some $1 < p < +\infty$. Let us extend f periodically on whole $\mathbb{R}^d$ (with U as a basic period), and set $f_k(x) := f(kx)$, $k = 1, 2, \dots$ Prove that

$$f_k \rightharpoonup \tilde{f} := \frac{1}{|U|} \int_U f(x)\mathrm{d}x \text{ in } L^p(U).$$

Ex 5. Let X be a Banach space, and suppose that $x_n \rightharpoonup x$ (weakly) in X, and that every subsequence of x_n has a strongly convergent subsequence. Show that then in fact $x_n \to x$ (strongly) in X.

Ex 6. Let us assume that X is a reflexive normed space, and U is a closed linear subspace of X. Show that

a) X is complete,

b) U is reflexive.

Ex 7. Let us assume that X, Y are Banach spaces, and $T : X \to Y$ is linear operator. Show that the following statements are equivalent:

a) T is bounded,

b) for any sequence $\{x_n\}_{n=1}^\infty$ weakly convergent to some x in X, it holds that $Tx_n \rightharpoonup Tx$ in Y.

Ex 8. Show that the assumption of completeness of the target space in the Open mapping theorem may not be omitted. More precisely, give an example of a linear surjective operator $T : X \to Y$, such that X is a Banach space, Y is a normed vector space and T is not open.