1st homework of the Functional Analysis, 21.10.2014

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The deadline: October, 31st 2014, h14.30.
The place of submition: envelope on the door of the room no 5020.

Ex. 1. Show if given formula $\|\cdot\|_X$ is a norm on $C^1([0,1])$ and if yes, show whether the space $(C^1([0,1]), \|\cdot\|_X)$ is or is not a Banach space.

(a) $\|f\|_X = \left(\int_0^1 |f'(x)|^2\,dx\right)^{\frac{1}{2}}$ (b) $\|f\|_X = |f(0)| + \sup_{x\in[0,1]} |f'(x)|$

Ex. 2. Prove that $C([0,1])$ is closed subspace of $L^\infty([0,1])$ with a respect to the supremum norm.

Ex. 3. Let Y be a linear space and let a function $p : Y \to \mathbb{R}$ satisfy conditions:

i) $p(y) \geq 0$
ii) $p(ay) = |a|p(y)$

for any $y \in Y$ and $a \in \mathbb{R}$. Let $M = \{y \in Y : p(y) \leq 1\}$. Prove that

(a) set M is absorbent and is balanced,
(b) $p = \mu_M$ (μ_M Minkowski functional),
(c) if p is a seminorm in Y, then M is convex.

Ex. 4. Show that the space of bounded sequences which converge to zero is a Banach space when equipped with the supremum norm

$$c_0 = \big\{\{t_k\}_k \mid t_k \in \mathbb{R},\ \lim_{k\to\infty} t_k = 0\big\}.$$

Ex. 5. (Minkowski inequality.) Show that for $p \geq 1$, $\{x_k\}_k$, $\{y_k\}_k$ in l_p

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}.$$

Ex. 6. (Generalized Hölder's inequality.) Assume that for $i = 1, \ldots, k$, $p_i \in [1,\infty]$ are such that $\sum_{i=1}^{k} \frac{1}{p_i} = 1$, let $f_i \in L^{p_i}(\Omega)$. Then

$$\Big\| \prod_{i=1}^{k} f_i \Big\|_{L^1(\Omega)} \leq \prod_{i=1}^{k} \Big\| f_i \Big\|_{L^{p_i}(\Omega)}.$$

Ex. 7. Find the norm of the following functionals.

(a) $L : c_0 \to \mathbb{R}$, $\quad L\big(\{x_n\}\big) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$,

(b) $T_g : C([0,1]) \to \mathbb{C}$, $\quad T_g(f) = \int_0^1 f(x)g(x)\mathrm{dx}$, where $g \in C([0,1])$ is a given function.

Ex. 8. Let us define a function on $[0,1]^2$ by $k(s,t) = \begin{cases} |s-t|^{-\alpha}, & \text{for } s \neq t \\ 0, & \text{otherwise.} \end{cases}$

Prove that for $\alpha \in (0,1)$

$$T(f)(s) = \int_0^1 k(s,t)f(t)\,dt, \quad f \in C([0,1]),\ s \in [0,1]$$

is a well-defined bounded linear operator $T : C([0,1]) \to C([0,1])$ and that $\|T\| \leq \|k\|$.