

Orbit-finite linear programming

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For today, solvability of orbit-finite system of linear inequalities only

An example

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an infinite set of atoms

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\mathbb{A}

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$$\{v_\alpha \mid \alpha \in \mathbb{A}\} \cup \{e_{\alpha,\beta} \mid \alpha \neq \beta \in \mathbb{A}\} \quad \text{set of variables}$$

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total weight of incoming edges

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weight of the vertex

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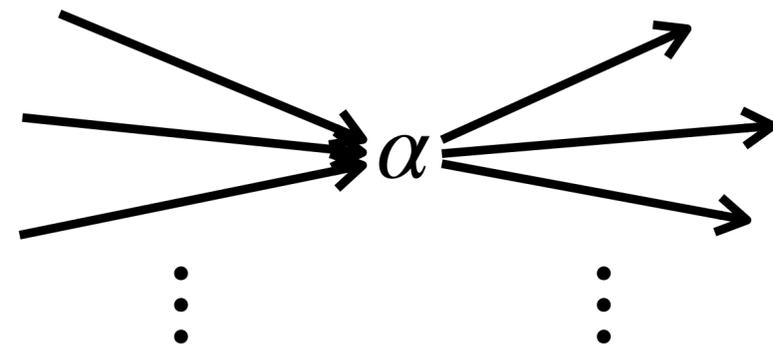
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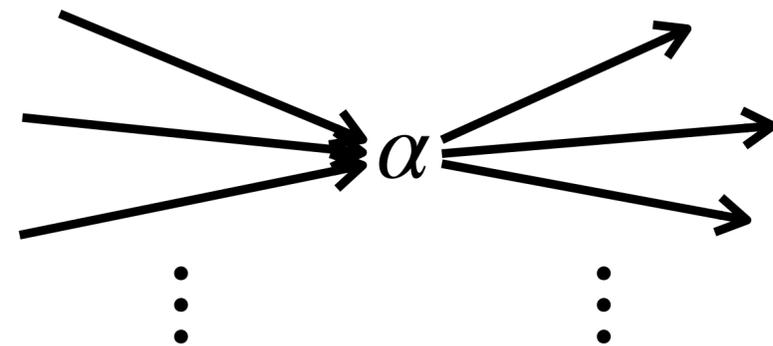
$$\sum_{\alpha \in \mathbb{A}} v_\alpha \geq 1$$

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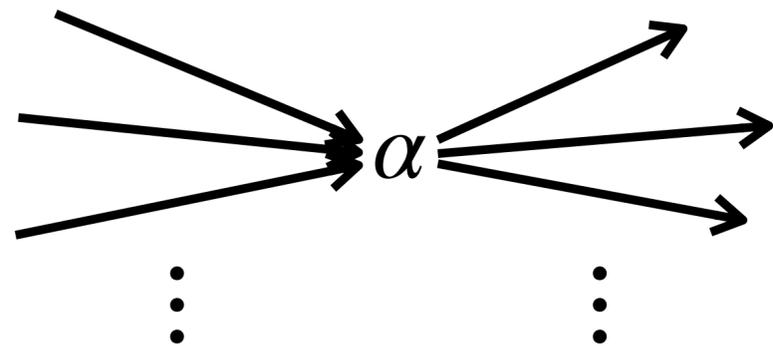
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total weight of vertices ≥ 1



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An example (orbit-finiteness)

The system is

- 1. invariant under permutations of \mathbb{A}
- 2. finite up to permutations of \mathbb{A}

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a generalisation
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orbit-finite = finitely representable

Results

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Decision problem

Input : An orbit-finite system of linear inequalities

Question : Does it have an orbit-finite solution?

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Theorem 2 : Existence of rational solutions is decidable

Theorem 3 : Existence of integer solutions is undecidable

Solving the example

$$\sum_{\beta \neq \alpha} e_{\beta, \alpha} + v_{\alpha} \leq \sum_{\beta \neq \alpha} e_{\alpha, \beta} \quad (\alpha \in \mathbb{A})$$
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Solving the example

x , a finite solution

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Solving the example

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Finite $F \subseteq \mathbb{A}$, such that x assigns non-zero values only to v_α and $e_{\alpha,\beta}$ with $\alpha, \beta \in F$

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is a solution invariant under all permutations of F

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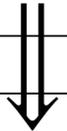
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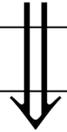
x' assigns :

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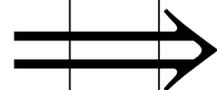
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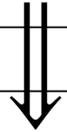
$$n = |F|$$

$$n \cdot v \geq 1$$

$$(n - 1) \cdot e + v \leq (n - 1) \cdot e$$

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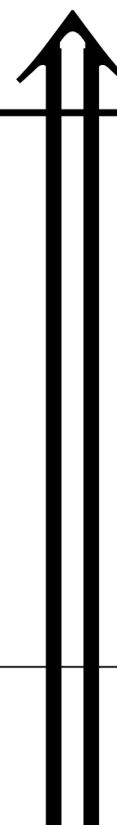
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Unsolvable!



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computation with infinite alphabets

register automata

Petri nets with data

orbit-finite sets

sets with atoms

orbit-finite dimensional vector spaces

equivariant linear algebra

symbolic computation

Open questions

- Ongoing {
1. Tight complexity
 2. Description of solutions sets
 3. Farkas lemma, duality
 4. Orbit-finite integer l.p. for atom-dimension ≤ 2
 5. Richer atoms (for eg. ordered atoms)