Portfolio optimization - a practical approach

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1 Introduction

The construction of the best combination of investment instruments (investment portfolio) is a principal goal of investment policy. This is an optimization problem: select the best portfolio from all admissible portfolios. To approach this problem we have to choose the selection criterion first. The seminal paper of Markowitz [8] opened a new era in portfolio optimization. The paper formulated the investment decision problem as a risk-return tradeoff. In its original formulation it was, in fact, a mean-variance optimization with the mean as a measure of return and the variance as a measure of risk. To solve this problem the distribution of random returns of risky assets must be known. In the standard Markowitz formulation returns of these risky assets are assumed to be distributed according to a multidimensional normal distribution $N(\mu, \Sigma)$, where μ is the vector of means and Σ is the covariance matrix. The solution of the optimization problem is then carried on under implicit assumption that we know both μ and Σ. In fact this is not true and the calculation of μ and Σ is an important part of the solution.

Due to the paradigm of Markowitz μ and Σ should be the moments of the distribution of future returns from risky assets. The market provides only the information about historic (past) returns. This means that we have to predict the moments of future returns using past returns, which can be justified only if the time series of market returns is a realization of an i.i.d. sequence of random variables. In fact, a number of market observations (so called stylized facts) shows that returns deviate from the i.i.d. assumptions.

In addition, normal distribution seems to be a very coarse approximation of real returns (in a number of recent papers it is rather the t-Student distribution which fits better to reality). The error due to the fact that market returns are not normal and deviate form i.i.d. assumption is called model risk (or model error).

Another source of errors in calculating μ and Σ stems from the finiteness of the sample. This kind of error (called estimation error or estimation risk) is particularly important in practical calculations where the sample is of a limited size. The effect of the estimation error to the portfolio problem has been studied since 1980's (see Merton [9], Jobson and Korkie [6], Michaud [10], Chopra and Ziemba [3]). Particularly the opinion of Michaud, who called portfolio optimizers – error maximizers, indicates practical difficulties in applying Markowitz method. All the above mentioned papers and a number of recent publications emphasize that the main source of errors in portfolio optimization is the inaccurate estimation of expected value μ of future returns. Merton claimed that to obtain a reasonable estimate of the mean we need about 100 years of monthly data. DeMiguel, Garlappi and Uppal [4] estimated that for a portfolio of 50 assets 600 months (50 years) of data is required.

In what follows we shall describe methods which use a reasonable set of market data and produce "good" optimal portfolios, i.e. portfolios well diversified and with assets shares stable with respect to estimation errors. We restrict our analysis to the elliptic distributions which are fully characterized by their first two moments (mean and covariance). The normal distribution and t-Student distribution are both examples of elliptic distributions. Hence the class is sufficiently rich for practical purposes.

2 Expected value of returns

There is the consensus among practitioners that the mean of historical returns is not a correct estimator of the future returns. There is a limited number of alternative estimators proposed in the literature. First, we should mention Bayes-Stein estimators where predicted mean is given by the formula

$$\mu_s = \delta\mu_0 + (1-\delta)\bar{\mu},$$

where $\bar{\mu}$ is the sample mean, μ_0 is the prior belief and δ is a constant ($0 < \delta < 1$). This is a shrinkage estimator; it shrinks the sample mean to the given target μ_0 (prior belief). Although improvement from using shrinkage is considerable, it is critically dependent on the right choice of the prior belief μ_0 (cf. Jorion [7]).

Bayesian statistics has been used in a more involved form by Black and Litterman [1]. Future returns are assumed to be normally distributed with a random mean (called risk-premium) μ. Sharpe theory and market equilibrium lead to the following prior beliefs about the risk premium

$$PDF(\mu) = N(\mu_{eq}, \tau\Sigma),$$

where μ_{eq} is the mean equilibrium return (benchmark), Σ is the sample covariance and τ is the subjective belief about the accuracy of benchmark predictions.

The distribution of risk premium is modified by investor's predictions about future market behavior. Investor expresses her views about the expected returns of portfolios with compositions given by the matrix P:

$$P\mu \approx q + \varepsilon,$$

where q is the vector of portfolios' returns and ε in the prediction error with the distribution $N(0, \Omega)$. Investor's views play a role of an observation in Bayesian statistics. The posterior distribution of μ, given the predictions, is normal and has the mean given by the following formula:

$$\mu_{eq} + \Sigma P^T(\Omega/\tau + P\Sigma P^T)^{-1}(q - P\mu_{eq}). \quad (1)$$

This is the famous Black-Litterman formula for estimating mean of future returns.

Practical application of formula (1) requires the following calibration steps:

1. Calculate benchmark returns μ_{eq} from the Sharpe formula

$$\mu_{eq} = \lambda\Sigma\theta_{eq},$$

   where λ is the market risk aversion, θ_{eq} is the market portfolio and Σ is the sample covariance. The first two parameters λ and θ_{eq} are subject to expert choice.

2. Choose a value of parameter τ expressing the confidence in the benchmark predictions.

3. Calculate matrices P and Ω (a number of algorithms is described in literature).

3 Covariance matrix

In the Introduction we have mentioned that the main source of errors in portfolio optimization is the estimation of the expected value of future returns. The Black-Litterman formula gives a good prediction of the mean value of expected returns and greatly improves the stability of optimal potfolios. But we still observe some minor instabilities, which are induced mainly by the covariance matrix estimation errors. They are attributed to the random noise affecting market data. To minimize the effect of these errors a number of classical statistical methods can be used: factor models, robust estimators and spectral methods to mention the most commonly used.

3.1 Factor models

The correlation structure of assets depends on $N^2/2$ parameters, where N is the number of assets. To reduce the number of parameters we can assume that the correlations among assets depend on a small number of factors. A simple model was described by Sharpe: asset returns depend only on market return (one factor). This model gives the following formula for the asset covariance matrix

$$\Sigma = \sigma_m^2 \beta \beta^T + \Sigma_\epsilon,$$

where

- σ_m is the covariance of market portfolio,
- $\beta = (\beta_1, \ldots, \beta_N)$, a β_i is the correlation of asset i with market portfolio,
- Σ_ϵ is a diagonal residual covariance matrix.

The number of unknown parameters is reduced to $3N + 1$.

The Sharpe model can be easily extended to multi-factor models. Chen, Karceski and Lakonishok [2] analyzed effectiveness of different multi-factor models in portfolio optimization: the improvement over one-factor model was not substantial.

3.2 Robust estimators

Robust estimators have been known more than 40 years. They were not frequently used in practice because of their numerical complexity. Finally about 10 years ago Rousseeuw and Van Driessen [11] introduced a new algorithm known as FastMCD which is sufficiently effective for practical calculations in finance. The idea of the algorithm is based on the assumption that real data contain, among correct measurements, a fraction of point which come from random noise. The purpose of the algorithm is to exclude noise and preserve correct data. Assuming a priori that a certain fraction of data (usually 5-10 %) is incorrect, the estimator finds the covariance matrix which fits best to remaining "correct" data.

FastMCD method opened also the possibility of practical use of many other robust estimators (M-estimators, S-estimators). Many of them have been successfully applied in portfolio optimization. For a review of recent results see the paper of Welsch and Zhou [12].

3.3 Spectral methods

Spectral methods are based on a spectral decomposition of covariance matrix. The sample covariance matrix possesses usually a number of very small eigenvalues. These small eigenvalues contribute very little to the portfolio volatility, but the corresponding eigenvectors can dramatically change the portfolio composition. At the same time small eigenvalues and corresponding eigenvectors are very vulnerable to random perturbations. The existence of small eigenvalues is a principal reason for portfolio instabilities.

The conclusion is straightforward. To stabilize optimal portfolios we remove from the spectral decomposition of the covariance matrix the part corresponding to small eigenvalues. This is known as the PCA analysis. In the implementation of this method we have to decide which eigenvalues are small. Statistical theory and in particular theory of random matrices may give some indication which eigenvalues can be treated as negligible. But the decision is always arbitrary and requires carefull tests when implemented.

4 Conclusions – how it works

The above short overview of portfolio optimization problems clearly indicates that there are several variants of the solution. Real investment decisions require a combination of mathematical modelling with deep economic understanding of the market. The group of mathematicians from University of Warsaw have had an interesting experience in portfolio optimization cooperating with The Central Bank of Poland. The goal of this project was to solve the optimal tactical asset allocation problem for the currency reserve of the Bank. It is a portfolio of about 50 billion EURO invested in multicurrency market. After 15 months of research we developed a methodology of portfolio optimization. Our conclusions are as follows:

1. Use t-Student distribution for approximation of market returns (number of degrees of freedom for the distribution is obtained by fitting to 10 years of daily data)

2. Use modified Black-Litterman formula for estimating future expected returns.

3. Cooperate with Bank experts in choosing data for Black-Litterman formula.

4. Use mix of robust estimators and spectral methods to obtain best estimates of the covariance matrix.

5. Optimize the quadratic utility function to obtain optimal portfolio.

6. Use Expected Shortfall as a risk measure.

The methodology is in use for a number of months and is evaluated by the Bank menagers very positively. On the other hand, there are still many issues which require investigation. At the first place we have to mention the stability of computed optimal portfolios. In our opinion the main effort should be concentrated on improving the estimates of covariance matrices. This includes better implementation of spectral methods. We are also working on different models of asset returns, in particular on these that combine linear methods with nonlinear autocorrelation effects, e.g. multidimensional GARCH models.

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