Lecture 6

Black-Scholes PDE

Lecture Notes
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Let the dynamics of underlining $S_t$ be given in the risk-neutral measure $\mathbb{Q}$ by

$$dS_t = rS_t dt + \sigma S_t dW_t.$$ 

If the contingent claim $X$ equals

$$X = h(S_T)$$

for some function $h$, then the price of $X$ at time $t$ is given by

$$V_t = V(S_t, t),$$

where $V(s, t)$ is given by the formula

$$V(s, t) = e^{-r(T-t)} \mathbb{E}_\mathbb{Q}(h(S_T)|S_t = s).$$
We call \((\psi, \phi)\) the replication strategy for contingent claim \(X\), if the value process is given by

\[ V_t = \psi_t B_t + \phi_t S_t, \]

where \(B_t\) is the bank account.
Goal

Find the function $V(s, t) : (0, \infty) \times [0, T] \to [0, \infty)$.

With the function $V(s, t)$ we can:

- compute the price of the contingent claim: at $t$ it equals
  \[ V(S_t, t). \]

- find the replicating strategy
  \[
  \phi_t = \frac{\partial V}{\partial s}(S_t, t), \\
  \psi_t = e^{-rt}(V(S_t, t) - \phi_t S_t).
  \]
The Black-Scholes PDE

**Theorem.** If $X = h(S_T)$ then there exists a function $V : (0, \infty) \times [0, T] \to \mathbb{R}$ such that

$$V_t = V(S_t, t).$$

This function is a solution to the Black-Scholes partial differential equation

$$\frac{\partial V(s, t)}{\partial t} + rs \frac{\partial V(s, t)}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V(s, t)}{\partial s^2} - rV(s, t) = 0$$

with the terminal condition

$$V(s, T) = h(s)$$

for any $s > 0$ and $t \in [0, T]$. 
Applications of B-S PDE

We can compute $V$ analytically for:

- vanilla call and put options,
- binary call and put options,
- call and put options on the foreign exchange market.

We cannot compute analytically but can solve B-S PDE numerically for:

- options with non-standard payoffs,
- American call and put options,
- Asian options,
- some other complicate contingent claims.
Derivation of the Black-Scholes PDE

\[ V_t = V(S_t, t). \]

Once we are at \( t \), the value \( V_t \) is no longer random as it is \( \mathcal{F}_t \) measurable.

Applying Itô’s formula gives

\[
dV_t = d(V(S_t, t)) = \left( \sigma S_t \frac{\partial V(S_t, t)}{\partial s} \right) d\tilde{W}_t \]

\[
+ \left( rS_t \frac{\partial V(S_t, t)}{\partial s} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V(S_t, t)}{\partial s^2} + \frac{\partial V(S_t, t)}{\partial t} \right) dt.
\]

The discounted value process \( \hat{V}_t = e^{-rt} V_t \) is a \textit{martingale} under the risk-neutral measure \( \mathbb{Q} \).
Compute $d\hat{V}_t$

\[
d\hat{V}_t = d(e^{-rt}V_t) = \left( e^{-rt} \sigma S_t \frac{\partial V(S_t, t)}{\partial s} \right) d\tilde{W}_t \\
+ e^{-rt} \left( rS_t \frac{\partial V(S_t, t)}{\partial s} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V(S_t, t)}{\partial s^2} + \frac{\partial V(S_t, t)}{\partial t} - rV(S_t, t) \right) dt.
\]

**Observation.** $\hat{V}_t$ is a martingale with respect to $Q$, so the term by $dt$ must be zero.

Hence the function $V(s, t)$ satisfies

\[
rs \frac{\partial V(s, t)}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V(s, t)}{\partial s^2} + \frac{\partial V(s, t)}{\partial t} - rV(s, t) = 0,
\]

which is the Black-Scholes PDE.
Boundary conditions are required to establish uniqueness of the solution to the Black-Scholes PDE. Their role is to impose some economically justified constraints on the solution of the PDE.

We have to be able to find conditions without knowing the formula for the function $V$.

- **Terminal condition:**
  \[
  V(s, T) = h(s), \quad s > 0.
  \]

- **Left-boundary:** what happens to $V(s, t)$ when $s$ approaches 0.

- **Right-boundary:** what happens to $V(s, t)$ when $s$ approaches $\infty$. 
Call option

For a call option with payoff \((S_T - K)^+\) good boundary conditions are:

\[
V(s, t) \approx 0, \quad \text{for } s \text{ very small,}
\]
\[
V(s, t) \approx s, \quad \text{for } s \text{ very large.}
\]

\[
\begin{aligned}
\frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V(s, t)}{\partial s^2} + rs \frac{\partial V(s, t)}{\partial s} - rV(s, t) + \frac{\partial V(s, t)}{\partial t} &= 0, \\
V(s, T) &= (s - K)^+, \quad s > 0,
\end{aligned}
\]

\[
\begin{aligned}
\lim_{s \to 0} V(s, t) &= 0, \quad t \in [0, T], \\
\lim_{s \to \infty} \frac{V(s, t)}{s} &= 1, \quad t \in [0, T].
\end{aligned}
\]
Put option

For a put option with payoff \((K - S_T)^+\) good boundary conditions are:

\[
V(s, t) \approx Ke^{-r(T-t)}, \quad \text{for } s \text{ very small},
\]

\[
V(s, t) \approx 0, \quad \text{for } s \text{ very large}.
\]

\[
\begin{cases}
\frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V(s, t)}{\partial s^2} + rs \frac{\partial V(s, t)}{\partial s} - rV(s, t) + \frac{\partial V(s, t)}{\partial t} = 0, \\
V(s, T) = (K - s)^+, \quad s > 0, \\
\lim_{s \to 0} V(s, t) = Ke^{-r(T-t)}, \quad t \in [0,T], \\
\lim_{s \to \infty} V(s, t) = 0, \quad t \in [0,T].
\end{cases}
\]
General form of the Black-Scholes PDE

\[ V(s, t) \] is a unique solution to the Black-Scholes PDE

\[
\begin{align*}
\frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V(s, t)}{\partial s^2} + rs \frac{\partial V(s, t)}{\partial s} - rV(s, t) + \frac{\partial V(s, t)}{\partial t} &= 0,
\end{align*}
\]

\[
\begin{cases}
V(s, T) = h(s), & s > 0, \\
\lim_{s \to 0} V(s, t) = r_1(t), & t \in [0, T],
\end{cases}
\]

\[
\begin{cases}
\lim_{s \to \infty} \frac{V(s, t)}{r_2(s, t)} = 1 & \text{or} \quad \lim_{s \to \infty} V(s, t) = r_2(t), & t \in [0, T],
\end{cases}
\]

where \( r_1 \) and \( r_2 \) are appropriately chosen to match \( h \).
Change of variables

Consider the following \textit{time dependent} change of variables:

\[
x := \log s + (r - \frac{1}{2}\sigma^2)(T - t),
\]
\[
\tau := \frac{\sigma^2}{2}(T - t),
\]
\[
y(x, \tau) := e^{-r(T-\frac{2\tau}{\sigma^2})}V\left(e^{x-(\frac{2r}{\sigma^2}-1)\tau}, T - \frac{2\tau}{\sigma^2}\right).
\]

The Black-Scholes PDE is transformed into the \textit{heat equation}

\[
\frac{\partial y}{\partial \tau}(x, \tau) - \frac{\partial^2 y}{\partial x^2}(x, \tau) = 0.
\]
time $t = 0$ corresponds to $\tau = \frac{\sigma^2 T}{2}$,

time $t = T$ corresponds to $\tau = 0$,

terminal condition $V(s, T) = h(s)$ changes to the initial condition

$$y(x, 0) = e^{-rT} h(e^x),$$

unlike $s$, the variable $x$ is between $-\infty$ and $\infty$,

new PDE is much simpler to solve numerically than the Black-Scholes PDE.
The function $y$ from the previous slide is a unique solution to the following PDE

$$\frac{\partial y}{\partial \tau}(x, \tau) - \frac{\partial^2 y}{\partial x^2}(x, \tau) = 0,$$

$$y(x, 0) = e^{-rT} h(e^x), \quad x \in \mathbb{R},$$

$$\lim_{x \to -\infty} y(x, \tau) = e^{-r(T - \frac{2\tau}{\sigma^2})} r_1 \left( T - \frac{2\tau}{\sigma^2} \right), \quad \tau \in \left[ 0, \frac{\sigma^2}{2} T \right],$$

$$\lim_{x \to \infty} \frac{y(x, \tau)}{e^{-r(T - \frac{2\tau}{\sigma^2})} r_2 \left( e^{-\left(\frac{2x}{\sigma^2} - 1\right)\tau} T - \frac{2\tau}{\sigma^2} \right)} = 1,$$

or

$$\lim_{x \to \infty} y(x, \tau) = e^{-r(T - \frac{2\tau}{\sigma^2})} r_2 \left( T - \frac{2\tau}{\sigma^2} \right), \quad \tau \in \left[ 0, \frac{\sigma^2}{2} T \right].$$
When assets pay dividend with a constant dividend yield $d$ then $V(s, t)$ is a unique solution to the following Black-Scholes PDE

$$
\begin{align*}
\frac{1}{2} \sigma^2 s^2 \frac{\partial^2 V(s, t)}{\partial s^2} + (r - d)s \frac{\partial V(s, t)}{\partial s} - rV(s, t) + \frac{\partial V(s, t)}{\partial t} &= 0, \\
V(s, T) &= h(s), \quad s > 0, \\
\lim_{s \to 0} V(s, t) &= r_1(t), \quad t \in [0, T], \\
\lim_{s \to \infty} \frac{V(s, t)}{r_2(s, t)} &= 1 \quad \text{or} \quad \lim_{s \to \infty} V(s, t) = r_2(t), \quad t \in [0, T],
\end{align*}
$$

where $r_1$ and $r_2$ are appropriately chosen to match $h$. 
Change of variables

The time dependent change of variables for dividend-paying assets:

\[ x := \log s + (r - d - \frac{1}{2}\sigma^2)(T - t), \]
\[ \tau := \frac{\sigma^2}{2}(T - t), \]
\[ y(x, \tau) := e^{-r(T - \frac{2\tau}{\sigma^2})}V(e^{-\left(\frac{2(r-d)}{\sigma^2} - 1\right)\frac{\tau}{\sigma^2} \cdot T - \frac{2\tau}{\sigma^2}}) \].

Then the Black-Scholes PDE is transformed into the heat equation

\[ \frac{\partial y}{\partial \tau}(x, \tau) - \frac{\partial^2 y}{\partial x^2}(x, \tau) = 0. \]
The function $y$ from the previous slide is a unique solution to the following PDE

\[
\frac{\partial y}{\partial \tau}(x, \tau) - \frac{\partial^2 y}{\partial x^2}(x, \tau) = 0,
\]

\[
y(x, 0) = e^{-rT}h(e^x), \quad x \in \mathbb{R},
\]

\[
\lim_{x \to -\infty} y(x, \tau) = e^{-r(T-\frac{2\tau}{\sigma^2})}r_1\left(T - \frac{2\tau}{\sigma^2}\right), \quad \tau \in \left[0, \frac{\sigma^2}{2}T\right],
\]

\[
\lim_{x \to \infty} \frac{y(x, \tau)}{e^{-r(T-\frac{2\tau}{\sigma^2})}r_2\left(e^{x-(\frac{2(r-d)}{\sigma^2}-1)\tau}T - \frac{2\tau}{\sigma^2}\right)} = 1,
\]

or

\[
\lim_{x \to \infty} y(x, \tau) = e^{-r(T-\frac{2\tau}{\sigma^2})}r_2\left(T - \frac{2\tau}{\sigma^2}\right), \quad \tau \in \left[0, \frac{\sigma^2}{2}T\right].
\]
Another change of variables

For some problems the proposed changes of variables are not very convenient because straight lines $S = c$ became time dependent in the new variables. Then the following time independent change of variables can be useful:

$$x := \log s, \quad \tau := \frac{\sigma^2}{2}(T - t),$$

$$y(x, \tau) := \exp\left(\frac{1}{2}(q_d - 1)x + \left(\frac{1}{2}(q_d - 1)^2 + q\right)\tau\right)V\left(e^x, T - \frac{2\tau}{\sigma^2}\right),$$

where $q := \frac{2r}{\sigma^2}$, $q_d := \frac{2(r - d)}{\sigma^2}$.

In this case the Black-Scholes PDE is transformed also into the heat equation

$$\frac{\partial y}{\partial \tau}(x, \tau) - \frac{\partial^2 y}{\partial x^2}(x, \tau) = 0.$$
Boundary conditions

For the change of variables from the previous slide the boundary conditions are as follows

\[
y(x, 0) = e^{\frac{1}{2}(q_d - 1)x} h(e^x), \quad x \in \mathbb{R},
\]

\[
\lim_{x \to -\infty} y(x, \tau) = \exp\left(\left(\frac{1}{4}(q_d - 1)^2 + q\right)\tau\right) r_1 \left(T - \frac{2\tau}{\sigma^2}\right), \quad \tau \in \left[0, \frac{\sigma^2}{2T}\right],
\]

\[
\lim_{x \to \infty} \frac{y(x, \tau)}{\exp\left(\frac{1}{2}(q_d - 1)x + \left(\frac{1}{4}(q_d - 1)^2 + q\right)\tau\right) r_2 \left(e^x, T - \frac{2\tau}{\sigma^2}\right)} = 1, \quad \tau \in \left[0, \frac{\sigma^2}{2T}\right].
\]