Re-optimization of the Steiner Tree problem
The plan

- Some motivation
- Introducing the problem:
  - The Steiner Tree problem (STP)
  - The Concept of Re-optimization
- Some warm-up observations and tricks
- The state of the art
- The trick
- Adding a terminal and why it is simple
- Removing a terminal and why it is harder
- Future work
Some motivation
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The Steiner Tree Problem (STP)

Input:
Graph (Metric, Complete)
The Steiner Tree Problem (STP)

Input:
- Graph (Metric, Complete),
- Terminal Set of nodes
The Steiner Tree Problem (STP)

Input:
Graph, Metric, Complete,
Terminal Set of nodes

Output:
A tree spanning the terminal set
The Steiner Tree Problem (STP)

Input:
Graph, Metric, Complete,
Terminal Set of nodes

Output:
A tree spanning the terminal set
with a minimum weight
The Steiner Tree Problem (STP)

Input:
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APX-hard
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Reoptimization

Input:
- Instance of a problem,
- a solution to it (preferably good)
- a modification, preferably local

Output:
- Solution of modified instance
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Hardness of Reoptimization

To solve an instance of underlying NP-hard problem:

Start with a trivial instance:
Hardness of Reoptimization

To solve an instance of underlying NP-hard problem:

Start with a trivial instance

Continue modifying it locally:
Hardness of Reoptimization

To solve an instance of underlying NP-hard problem:

Start with a trivial instance

Continue modifying it locally:
Hardness of Reoptimization

To solve an instance of underlying NP-hard problem:

Start with a trivial instance

Continue modifying it locally:
Hardness of Reoptimization

To solve an instance of underlying NP-hard problem:

Start with a trivial instance
Continue modifying it locally
Until it is transformed into the instance we want to solve
Recipe for a PTAS

$\{1,2\}$-STP under edge weight increase
Recipe for a PTAS

\{1,2\}-STP under edge weight increase
Recipe for a PTAS

\{1,2\}-STP under edge weight increase

\[ \text{OPT}_{\text{new}} \]

\[ \text{OPT}_{\text{old}} + \Delta e \]
Recipe for a PTAS

\{1,2\}-STP under edge weight increase

\{1,2\}-STP: \( \Delta e = 1 \)
Recipe for a PTAS

{1,2}-STP under edge weight increase

\[ \text{OPT}_{\text{old}} \leq \text{OPT}_{\text{new}} \leq \text{OPT}_{\text{old}} + \Delta e \]

{1,2}-STP: \[ \Delta e = 1 \]
Recipe for a PTAS

\{1,2\}-STP under edge weight increase

\[ \text{OPT}_{\text{old}} \leq \text{OPT}_{\text{new}} \leq \text{OPT}_{\text{old}} + \Delta e \]

\[ \leq \text{OPT}_{\text{new}} + \Delta e \]

\{1,2\}-STP: \quad \Delta e = 1

feasible
Recipe for a PTAS

{1,2}-STP under edge weight increase

\[ \text{OPT}_{\text{old}} \leq \text{OPT}_{\text{new}} \leq \text{OPT}_{\text{old}} + \Delta e \leq \text{OPT}_{\text{new}} + \Delta e \]

\( \Delta e \leq \varepsilon \text{OPT}_{\text{new}} \)

{1,2}-STP: \( \Delta e = 1 \)
Recipe for a PTAS

{1,2}-STP under edge weight increase

\[ OPT_{old} \leq OPT_{new} \leq OPT_{old} + \Delta e \leq OPT_{new} + \Delta e \]

\[ \Delta e \leq \varepsilon OPT_{new} \]

\[ \frac{\Delta e}{\varepsilon} \geq OPT_{new} \geq |E(OPT_{new})| \]

then we can find optimum using exhaustive search

{1,2}-STP: \( \Delta e = 1 \)
Recipe for a PTAS

{1,2}-STP under edge weight increase

\[ \text{OPT}_{\text{old}} \leq \text{OPT}_{\text{new}} \leq \text{OPT}_{\text{old}} + \Delta e \leq \text{OPT}_{\text{new}} + \Delta e \]

\[ \Delta e \leq \varepsilon \text{OPT}_{\text{new}} \]

\[ \frac{\Delta e}{\varepsilon} \geq \text{OPT}_{\text{new}} \geq |E(\text{OPT}_{\text{new}})| \]

then we can find optimum using exhaustive search

This idea carries on to sharpened triangle inequality and many other Reoptimization problems with weights bounded by a constant

\[ \text{OPT}_{\text{old}} + \Delta e \]

\[ \text{OPT}_{\text{old}} \]

\[ \text{OPT}_{\text{new}} \]

\[ \Delta e \]

\[ \varepsilon \]
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The road map of the relevant results

STP approximation algorithms:

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Bilò, Zych, ENDM 2011
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The trick
The trick

\[ \text{SOL} \leq \text{OPT} - f_1 - f_2 \]
The trick

\[ \text{SOL} \leq \sigma (\text{OPT} - f_1 - f_2) + \ldots \]

* \( \sigma \) is the approximation ratio
The trick

\[ SOL \leq \sigma (OPT - f_1 - f_2) + f_1 + f_2 \leq \sigma OPT - (\sigma - 1)(f_1 + f_2) \]

* \(\sigma\) is the approximation ratio
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How to add a terminal

\[ \text{OPT}_{\text{new}} \]

\[ \text{OPT}_{\text{old}} \]
How to add a terminal

$P$ is a path in $\text{OPT}_{\text{new}}$ from $t$ to another terminal.

If we could guess such a path...

$$\text{Sol}_1 \leq \text{OPT}_{\text{new}} + P$$

$$\text{Sol}_2 \leq \sigma \text{OPT}_{\text{new}} - (\sigma - 1)P$$
How to add a terminal

\( P \) is a path in \( \text{OPT}_{\text{new}} \) from \( t \) to another terminal.

If we could guess such a path...

\[
Sol_1 \leq \text{OPT}_{\text{new}} + P \\
Sol_2 \leq \sigma \text{OPT}_{\text{new}} - (\sigma - 1)P
\]

Unfortunately we can’t.
How to add a terminal

$P$ is a **cheapest** path in $\text{OPT}_{\text{new}}$ from $t$ to another terminal.

We would like to guess most of it:

$$P = P_k + P'$$
How to add a terminal

$P$ is a *cheapest* path in $\text{OPT}_{\text{new}}$ from $t$ to another terminal.

We would like to guess most of it:

$$P = P_k + P'$$

$$R_1, ..., R_k \geq P'$$

$$P' \leq \frac{\text{OPT}_{\text{new}}}{k}$$
How to add a terminal

$P$ is a **cheapest** path in $\text{OPT}_{\text{new}}$ from $t$ to another terminal.

We would like to guess most of it:

\[
P = P_k + P' \quad \text{with} \quad P' \leq \frac{\text{OPT}_{\text{new}}}{k}
\]

\[
R_1, \ldots, R_k \geq P'
\]

\[
\text{Sol}_1 \leq \text{OPT}_{\text{new}} + P_k + P'
\]

\[
\text{Sol}_2 \leq \sigma \text{OPT}_{\text{new}} - (\sigma - 1)P_k
\]

\[
\frac{2\sigma - 1 + \frac{\sigma - 1}{k}}{\sigma} \xrightarrow{k \to \infty} \frac{2\sigma - 1}{\sigma}
\]
How to add a terminal

**P** is a **cheapest** path in **OPT**$_{\text{new}}$ from *t* to another terminal.

We would like to guess most of it:

\[
\begin{align*}
P &= P_k + P' \\
R_1, \ldots, R_k &\geq P' \\
P' &\leq \frac{\text{OPT}_{\text{new}}}{k}
\end{align*}
\]

**Sol**$_1 \leq \text{OPT}_{\text{new}} + P_k + P'$

**Sol**$_2 \leq \sigma\text{OPT}_{\text{new}} - (\sigma - 1)P_k$

Not yet exactly what we want, but ! ...
How to add a terminal

We can assume $t$ has degree at least 2 in $\text{OPT}_{\text{new}}$

...and contract two paths!

Remark:
Two paths are the reason why we have better ratios for terminal modifications than for edge cost modifications
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Why removing a terminal is harder

$$\text{OPT}_{\text{old}} \geq \text{OPT}_{\text{new}}$$

feasible
Why removing a terminal is harder

$$\text{OPT}_{\text{old}} \geq \text{OPT}_{\text{new}}$$

feasible

$$\text{OPT}_{\text{old}} - P \leq \text{OPT}_{\text{new}}$$

not feasible
Why removing a terminal is harder

\[ \text{OPT}_{\text{old}} \leq \text{OPT}_{\text{new}} \]

feasible

\[ \text{OPT}_{\text{old}} - P \leq \text{OPT}_{\text{new}} \]

not feasible

\[ \text{OPT}_{\text{old}} - P + T \leq \text{OPT}_{\text{new}} + T \]

feasible

\[ \text{Sol}_2 \leq \sigma \text{OPT}_{\text{new}} - (\sigma - 1)T \]
Why removing a terminal is harder

\[ \text{OPT}_{\text{old}} \leq \text{OPT}_{\text{new}} \]
feasible

\[ \text{OPT}_{\text{old}} - P - \text{OPT}_{\text{new}} \]
not feasible

\[ \text{OPT}_{\text{old}} - P + T \leq \text{OPT}_{\text{new}} + T \]
feasible

\[ \text{Sol}_2 \leq \sigma \text{OPT}_{\text{new}} - (\sigma - 1)T \]

How to guess T in polynomial time?
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