DISCRETE CONFLICTS WITH MULTIPLE BATTLEFIELDS

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1. RESEARCH PROJECT OBJECTIVES

Conflicts with multiple battlefields (Kovenock and Roberson (2012)) are a classic example of (adversarial) interaction between parties (or players) who must use limited resources to maximise outcomes of an encounter. Problems of this sort arise naturally in numerous applications, such as military conflicts (Blackett (1954)), political competition (Laslier and Picard (2002)), advertising (Friedman (1958)), and security (Tambe (2011)). Since the seminal work of Immorlica et al. (2011), the problem of computing optimal or approximately optimal strategies for parties in such interactions received attention in theoretical computer science as well as in artificial intelligence. This is because the theoretical models of such interaction result in large games computing equilibrium strategies is tractable.

The most famous example of such conflicts is the Colonel Blotto game, introduced by Émile Borel in 1921 (Borel (1921)). In this problem, two parties distribute their limited resources across a number of fronts (or battlefields) aiming to beat the opponent at as many fronts as possible. A player wins a battlefield if the number of her resources is strictly larger than the number of resources of the opponent. This is a zero-sum game, meaning that gain of one side of conflict is a loss of the other side. By a fundamental result in game theory (von Neumann (1928)) in every Nash equilibrium of such games both sides use an optimal strategy (a strategy that minimizes the maximal payoff the opponent can get) and payoffs to each side are the same across all equilibria. The equilibrium payoff of the side who get non-negative payoff is called the value of the game and reflects the relative strength of the stronger side in the game.

A continuous variant of Colonel Blotto game, where each party can assign any fraction of her total resources to a battlefield, was solved by Gross and Wagner (1950) (the symmetric variant) and by Roberson (2006) (asymmetric variant). In addition, a complete characterization of Nash equilibria of the continuous variant on two battlefields was obtained by Macdonell and Mastronardi (2015). The discrete variant, where each player is endowed with a natural number of resources and can only assign a natural number of resources to each battlefield, is much less understood. The most comprehensive result in this respect was obtained by Hart (2008), who found Nash equilibria for the symmetric variant of the game, as well as for the special cases of the asymmetric variant. For many cases of the asymmetric variant we do not even know the value of the game.

Another important example are the so called security games (Tambe (2011)) that, among other security applications, serve as models for counter-terrorism measures, prevention of illegal trafficking, and network security. In these models the roles of players are highly asymmetric. One of the players, the defender, has defence resources that always beat the resources of another player, the attacker (or the adversary). The resources are discrete and each battlefield can receive at most one resource of a given player. Although the study of these models advanced in some directions (like computer aided resource allocation and defence of networks), the general, basic model, of security games on multiple battlefields is not fully understood. In particular, given that values of battlefields are heterogeneous, the value of the game and players activities on different battlefields are not known. The most comprehensive result in this direction were obtained by Korzhyk et al (2011b,a), who provided full characterization of equilibrium strategies in terms of their univariate marginal probability distributions for the case of one unit of attacking resources as well as for the cases of multiple attacking resources, where the values of

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1 Colonel Blotto game is symmetric when both players have the same amount of resources and it is asymmetric otherwise.
battlefields satisfy certain additional conditions. This characterization, however, is given in terms of equilibrium payoffs of the players which is not characterized. In a recent paper, Dziubiński and Roy (2018), we obtained a complete characterization of equilibrium strategies in von Neumann’s hide and seek game (von Neumann (1953)), a game that could be viewed as a simple subclass of security games (the game is also of a separate interest and was considered recently under the name “auditing games” by Behnezhad et al. (2018)). We used this characterization to obtain an algorithm for computing equilibrium strategies and univariate marginal distributions of equilibrium strategies. The algorithm is faster than the existing algorithms developed for security games. This opens a potential for improving the characterization of equilibria and algorithms for computing equilibrium strategies in security games.

Conflicts with multiple battlefields are closely related to simultaneous auctions where two (or more) parties bid for several items (or several copies of the same item) in several concurrently run auctions. This is because, as long as the auction is standard (i.e. where the buyer who places the highest bid wins), a conflict on a single battlefield can be viewed as a single auction. In particular, each conflict on a single battlefields in Colonel Blotto game can be viewed as an all-pay auction (whoever bids the highest wins the auction, all parties pay their bids).

Simultaneous auction constitute a tractable and practical alternative to combinatorial auctions when selling multiple items is considered. Because of that they received significant attention from researchers working on topics on the interface of computer science and theoretical economics (see Roughgarden et al. (2017) for a recent survey of these attempts). In particular, Feldman et al. (2013) showed that simultaneous auctions are nearly optimal when items bundles valuations do not exhibit complements while Hassidim et al. (2011) showed that complements can lead to high inefficiencies. Besides these approximate results, we have very little understanding of the structure of equilibria in simultaneous auctions with complements. Szentes and Rosenthal (2003) proposed a simple model of such auctions, called “chopstick auctions”, where each of the buyers aims to win a majority of auctions (or else gets zero payoff). The model they propose is essentially a model of a conflict with multiple battlefields and majoritarian objectives: the objective of each player is to win a majority of battlefields. Beyond three battlefields, optimal strategies and values of the game for such conflicts are largely unknown, even in continuous case.

There is also a deeper connection between conflict with multiple battlefields and auctions. For a continuous model of Colonel Blotto game, Roberson (2006) has shown that there exists a one-to-one correspondence between the equilibrium univariate marginal distributions of the Colonel Blotto game and the equilibrium distributions of bids from a unique set of two-bidder independent and identical simultaneous all-pay auctions. Sahuguet and Persico (2006) have shown and used a correspondence between equilibria in all-pay auctions and General Lotto game. In a recent publication, Hart (2016), applies a variant of continuous Lotto games to characterize equilibria in all-pay auctions with caps, where the bids are restricted by an exogenous upper bound. Simultaneous all-pay auctions with continuous resources were fully characterized by Baye et al. (1996). In the case of discrete variant the characterization is only partially known (Cohen and Sela (2007)).

Our key objectives in the project are as follows:

(1) To develop methods for finding optimal strategies and values in unsolved cases of discrete Colonel Blotto game, “chopstick auctions” with more than three battlefields, and security games, and to solve all (or a significant fragment) of unsolved cases.

(2) To use the developed methods to obtain new, simpler and (possibly) faster, algorithms for computing equilibrium strategies and the values of these games.

(3) To generalize the model of General Lotto games and find solutions of the generalizations. The basic model is related to Colonel Blotto game played on battlefields with homogeneous values. This can be generalized by proposing a model related to Colonel Blotto games played on battlefields with heterogeneous values (such a generalization was recently proposed and studied for the continuous case by Kovenock and Roberson (2015)). Another generalization would be a model related to “chopstick auctions”, and other types of conflicts with multiple battlefields.
To use the developed methods to provide complete characterization of equilibria in all-pay auctions with discrete resources.

2. Significance of the project

Conflicts with multiple battlefields are of interests since the beginning of modern game theory. The paper by [Borel (1921)], who introduced the Colonel Blotto game is one of the first in the discipline. The significance of these models was soon recognized, which was documented by a note of [Tukey (1949)] and by Savage’s translation of Borel’s original paper [Borel (1921)]. Due to their fundamental role in such important areas as auctions, political economy and, recently, security, they remain an object of active study to this day (the research up to the year 2012 was reviewed recently by [Kovenock and Roberson (2012)]).

All the research mentioned above is restricted almost exclusively to continuous resources. There are at least two reason for that. Firstly, in some applications, where resources spend on a battlefield could be seen as time or effort, the continuous model is the most adequate. Secondly, the continuous models are often easier to solve than the discrete ones. Still, there are applications where discrete models are better suited (e.g. in military and security applications or in political campaigning, where resources are units of army, specialized equipment, or party activists, or in auctions where bids are in monetary values). Obtaining a better understanding of these models would allow us to gain more insight into the effects such as, for example, the disadvantages coming from the inability to split the resources evenly across the battlefields. We could also get insight into the quality of approximation of the discrete models with the continuous ones. Lastly, in the cases of models like “chopstick auctions”, which are still unsolved even in continuous case, the tools developed for discrete variant could provide us with new methods for solving the continuous variant.

Although [Ahmadinejad et al. (2016)] showed that equilibrium strategies for many of the games of conflicts with multiple battlefields can be found in polynomial time, the algorithm proposed there is based on the Ellipsoid method, which is often deemed to be too inefficient to be used in practice. Indeed, the works by [Behnezhad et al. (2017)] and as well as our works, [Dziubiński (2012)] and [Dziubiński and Roy (2018)], demonstrate that using an approach dedicated to a particular game and based on better understanding of equilibrium strategies can lead to algorithms which are more efficient, direct, and simpler to understand and implement. Obtaining a better understanding of equilibria in other models of conflicts with multiple battlefields and developing tools for solving them could help in obtaining better algorithms for computing their equilibria. An example of such studies is the recent paper by [Behnezhad et al. (2018)] who consider equilibria in Colonel Blotto and hide-and-seek games that guarantee the given utility with the given probability.

The project aims at closing the gap between understanding the continuous and discrete conflicts of multiple battlefields with potential for obtaining faster and simpler algorithms for computing equilibrium strategies in these games, as well as for developing tools that would allow for attacking the problems which remain hard in the continuous case. We aim at solving important open problems in the area of conflicts with multiple battlefields. One of these problems (optimal strategies and values for discrete Colonel Blotto games) is among the longest standing in the discipline of game theory. Due to important applications (military conflicts [Blackett (1954)], political competition [Laslier and Picard (2002)], security [Chia and Chuang (2011)]; [Kiekintveld et al. (2009)]; [Tambe (2011)], the significance of the objectives exceeds the mere theoretical interest.

3. Work plan

We plan to start with objective (1) and to develop and extend the method proposed by [Hart (2008)], as well as find connection between equilibria in more complex and simpler classes of conflicts with multiple battlefields, like hide and seek games and security games. The author defines an auxiliary games (called General Lotto games) the equilibria of which can be used to construct optimal strategies in the corresponding instance of Colonel Blotto game. We plan to use the full characterization of equilibria
in General Lotto games obtained in [Hart (2008)] and then completed by the author of the project in [Dzubiński (2012)]. As observed already by [Hart (2008)], converting equilibria in General Lotto Games to Colonel Blotto games is not always possible and depends on the numbers of battlefields and the number of units each player has. We plan to extend the model of General Lotto games by introducing additional constraints on the allowed strategies of the players. An initial study conducted by the author of the project shows that the right constraints result in equilibria that can be used for constructing equilibria in Colonel Blotto games. The methods described above will allow us to find examples of equilibria of the game and to establish the values of the game for different numbers of resources and battlefields. In the case of continuous variant of the game, [Roberson (2006)] obtained complete characterization of equilibria in terms of unidimensional marginal probability distributions of the numbers of resources used at each battlefield. In a recent paper [Dzubiński (2017)], we proposed a notion of spectrum of mixed strategies and used it to obtain complete characterization of equilibria for symmetric Colonel Blotto games with discrete resources such that the number of battlefields divides the number of resources each player has. We plan to extend these results to obtain complete characterization of equilibria in terms of spectrum for the remaining cases of the game.

We plan to develop an analogous methods for other types of conflicts with multiple battlefields, focusing on the conflicts with discrete resources and majoritarian objectives. We also plan to look at other models of conflict with multiple battlefields and to study their variants with discrete resources. In particular, we plan to consider security games on multiple battlefields. In a very recent paper [Dzubiński and Roy (2018)] we obtained complete characterization of equilibria in hide and seek games in terms of their univariate marginal probability distributions. This allowed us to obtain faster algorithms for computing equilibria and the value of these games. Hide and seek games could be viewed as a simple class of security games and we plan to explore the direction initiated in [Dzubiński and Roy (2018)] to obtain a complete characterization of equilibria in basic security games with multiple attacking and defending resources. This would allow us to extend the results obtained by [Korzyk et al. (2011b)].

Lastly, we plan to investigate the model of Colonel Blotto with heterogeneous resource values, using the methods developed for the homogeneous case. Research tasks related to Objective 1 are as follows:

1. Finding equilibria in Colonel Blotto game based on equilibria in General Lotto games for all the cases of the numbers of resources and battlefields for which it is feasible. Finding the value of the game for all these cases.
2. Studying restricted variants of General Lotto games in order to find equilibria and to establish the values of these games.
3. Using the equilibria in restricted General Lotto games to construct equilibria in Colonel Blotto games and extend the algorithm for computing equilibria and values for the games in question.
4. Finding the characterization of equilibria in Colonel Blotto in terms of their spectrum.
5. Attacking the Colonel Blotto game with heterogeneous resource values.
6. Developing and studying the General Lotto type of a model corresponding to conflicts with multiple battlefields and majoritarian objectives.
7. Finding equilibria and the value for conflicts with multiple battlefields with majoritarian objectives using the results obtained in task 6.
8. Studying the security games with multiple attacking and defending resources.

With sufficient progress in work on Objective 1, we plan to start work on Objective 2 to design algorithms for computing equilibria and values for the games in question.

In parallel with work on Objectives 1 and 2, we plan to work on objective 4. The work by [Myerson (1993); Sahuguet and Persico (2006); Hart (2008)] shows how continuous General Lotto Game is related to continuous all-pay auctions. We plan to use complete characterization of equilibria in Discrete General Lotto games to complete the currently known characterization of equilibria in discrete all-pay auctions. We divide the work on this Objective into two tasks:

1. Finding the equilibria and the value of the game for two player simultaneous all-pay auctions with discrete bids.
(2) Extending the model of General Lotto game to more than two players and finding the characterization of equilibria and the value of the game.

(3) Finding the equilibria and the value of the game for three and more player simultaneous all-pay auctions with discrete bids using the results obtained for task 2.

Work on Objective (3) interleaves with work on Objectives (1), (2), and (4). In particular, tasks 2, 5, and 6 of Objective (1), and task 3 of Objective (4) contribute to fulfilling Objective (3).

3.1. Risks analysis. We did some preliminary study related to tasks 1, 2, and 8 of Objective (1) and we estimate the associated risk as intermediate. We estimate the risk related to tasks 3–7 as high: the problems we plan to address there are open and potentially difficult. Objective 2 depends heavily on the results obtained in Objective 1. In some of these tasks we have promising preliminary results, while other may turn out to be very challenging. Therefore we estimate Objective (2) to be of intermediate risk. Lastly, we estimate task 1 of Objective (4) to be of intermediate risk and task 3 to be of high risk. We expect to be able to use the existing results for General Lotto games to complete task 1 while the case of more than two players, in task 3, will require extending the model of General Lotto game and obtaining new results there.

4. RESEARCH METHODOLOGY

The purpose of our project is deepening and extending knowledge and understanding of conflict with multiple battlefields where sides of conflict have discrete resources. The main objectives are finding optimal strategies for sides in such conflicts and computing their relative advantage (in terms of value of the game), and developing methods for addressing such problems. Therefore the outcomes of our research will have mainly the nature of formal results. We shall use a variety of techniques from mathematics, mainly from game theory, linear algebra, combinatorics and probability theory.

After determining the equilibrium properties of different types of the conflicts with multiple battlefields, we will use these results to drawn further, economically relevant, conclusions about these games. In particular we plan to study the comparative statics with respect to the numbers of resources of the players, the number of battlefields and their values (in the cases where the values of battlefields are heterogeneous).

We plan to conduct our research in collaboration with research units abroad: University of Cambridge, University of Warwick, and University of Bath (all three in UK) and Université Paris 1 and Paris School of Economics (in France).

REFERENCES


