How to hide in a network?*

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Abstract

We propose and study a strategic model of hiding in a network, where the network designer chooses the links and his position in the network facing the seeker who inspects and disrupts the network. We characterize optimal networks for the hider, as well as equilibrium hiding and seeking strategies on these networks.

1 Introduction

This paper studies the problem of choosing the network structure in order to hide in it facing a hostile authority. Heads of criminal and terrorist organizations use networks of bilateral contacts to control and coordinate their groups. The links benefit the heads as they allow for organizing large groups of individuals that can achieve more ambitious and disruptive tasks. On the other hand, these groups are often observed by the authorities, who have access to (at least a large fragment of) the structure of such networks but have no information about the identity of their leader. The authorities may inspect selected individuals in the group and learn about their identity and the identity of their neighbours, thus discovering the leader. Therefore links in the network also create a threat. What are the network structures that allow the heads of the groups to minimize the risk of being detected? What are the optimal hiding and seeking strategies in the network? We propose a game-theoretic model to address such questions.

Related literature The problem of strategic hiding and seeking attracted attention of game theorists since the dawn of the discipline. In an early paper,von Neumann (1953), proposed a game with two players, a hider and a seeker, where the hider chooses a cell

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in a matrix while the seeker tries to catch him by choosing either a row or a column in the matrix. The hider is caught if the cell he chose is in the row or the column chosen by the seeker. The objective of the hider is to remain uncaught while the objective of the seeker is to catch the hider. Since then, the games of this sort are called "hide and seek games". In the context of graph theory, Fisher (1991) proposed a hide and seek game on a graph, where the hider and the seeker choose a vertex each in the graph. The hider is caught if the seeker chooses the same node as him or a node connected to the node chosen by him. The equilibrium strategies in this game are closely related to fractional domination and fractional packing, important concepts in fractional graph theory (c.f. Scheinerman and Ullman (1997)). There is a very large and important body of literature concerning dynamic search in graphs. Although not closely related to our paper, it is worth mentioning here. See Alpern et al. (2013) for an excellent overview.

To our knowledge, the earliest work to consider a strategic model of network formation with hiding and seeking is the paper by Baccara and Bar-Isaac (2008). The authors propose and study a model of covert network organization in face of a hostile authority. The model is a multistage game. In the first stage the authority sets a probability distribution on the set of nodes called the detection policy. This is observed by the nodes that choose nodes they reveal information about themselves to. This creates a directed information network that constitutes the structure of the covert organization. In the third stage, the nodes play a repeated prisoner's dilemma combined with a punishment game. The higher the number of nodes choosing coordination, the higher the benefits of the organization but also the higher the probability of detection. The information network allows the nodes to punish those connected to them (and thus creating incentives for cooperation) but also exposes them to detection (when a node is detected, all its maximal connected component is detected). The authors characterize the relation between detection policies and information network structure and characterize the equilibrium networks. More recently, Waniek et al. (2017) and Waniek et al. (2018), proposed and studied models of hiding individuals and communities in social networks in face of hostile authorities. The nodes are embedded in an existing, exogenous, network. The authorities observe the network and use various tools of most central nodes detection and communities of nodes detection to analyse the network. Knowing the actual tool used by the authorities, a subset of nodes (e.g. the leaders of a criminal organization) choose modifications to the links in the network in order to either reduce their centrality or prevent being detected as a community under the tool. The nodes' choices are constrained by a limited budget. The authors show that finding optimal such choices is computationally hard for most cases of tools used by the authorities. They also provide support for extending the scope of criminal network analysis to detection of nodes whose centralities or communities membership are fragile to network modifications.

Contribution We propose a model of strategic hiding in a network in face of a hostile authority. Given a set of nodes, the hider chooses a network over these nodes as well as his node in the network. The network chosen by the hider is observed by the seeker (representing the hostile authority) but the location choice is not observed. The seeker chooses one of the nodes in the network to inspect. The inspected node is removed from the network. If the hider hides in the inspected node or one of its neighbours, he is caught by the seeker and suffers a penalty. If the hider is not caught, he enjoys the benefits from the network. The benefits are a convex and increasing function of the number of nodes (including himself) that the hider can access (directly or not) in the network. This form of network benefits, first proposed in its general form by Goyal and Vigier (2014), is in line with the celebrated Metcalfe's law (Shapiro and Varian (2000)), where the function is identity. The objectives of the seeker are to minimize the payoff of the hider and the proposed model takes the form of a two-stage zero-sum game.

Although very stylised and simple, the model allows us to capture the trade-off between secrecy and network benefits. It can be viewed as a model of strategic criminal network creation, where a single individual chooses the structure of the organization in order to remain undetected when the authorities strike and interrogate one of the members of the organization. It could be also viewed as a model of strategic formation of a network of bank accounts, in order to hide illegal money. When the authorities inspect one of such accounts, they gain access to other accounts and are able to detect the illegal money in the inspected account or the accounts associated with it.

We provide optimal networks for the hider and characterize optimal strategies of the two players on these networks. In general, the optimal networks consists of a number of singleton nodes and a connected component which is either a cycle or a core-periphery network. If the component is a cycle, in equilibrium the hider mixes uniformly across its nodes. If the component is a core-periphery network, the hider mixes uniformly across the periphery nodes. This provides theoretical support to the claim that the hider chooses networks where his centrality is small and indistinguishable from the centralities of the other nodes.

The hide and seek stage in our model is similar to the hide and seek games on graphs of Fisher (1991), with the difference that in their case the penalty from being caught is 0 and benefits from not being caught are fixed and independent of the graph. Unlike in the model of Baccara and Bar-Isaac (2008), in our model the authorities choose their seeking strategy knowing the network and only one node chooses the network topology to hide himself. This is similar to the models of Waniek et al. (2017) and Waniek et al. (2018). However, unlike in their model, the authorities are strategic and they take into account the incentives and strategic behaviour of the hider when choosing the seeking strategy.

2 The model

There are two players, a *Hider* (H) and a *Seeker* (S) and a set of nodes $V = \{1, \ldots, n\}$. The game proceeds in two stages:

Network construction In the first stage the hider chooses a graph $G = \langle V, E \rangle$ over V with a set of undirected edges $E \subseteq \binom{V}{2}$ (an edge between two nodes $i, j \in V$ is a two element subset of V denoted by ij).[1]

Hiding and seeking stage In the second stage both players observe the chosen network and choose one node each. The choices are made simultaneously and independently. None of the players observe the choice of the other when making his choice.

The node chosen by the hider is his position in the network. The seeker uses his choice to capture the hider and to damage the network. The node chosen by the seeker and all its neighbours are inspected by him. In addition, the node chosen by the seeker is removed from the network. If one of the inspected nodes is the node chosen by the hider, the hider is caught. If caught, the hider gets payoff $-\beta$, where $\beta \geq 0$. Otherwise, his payoff is equal to function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of the size of his component in the residual

[1] Given set X and a natural number $k \leq |X|$, we use the standard notation $\binom{X}{k}$ to denote the set of all k-element subsets of X.

network. The payoff to the seeker is equal to minus the payoff of the hider (so the game is zero-sum). We assume f to be strictly increasing with $f(0) = 0$ and convex. An example of function f in line with these assumptions is the identity function, $f(x) = x$ for all $x \in \mathbb{R}_{\geq 0}$. This results in network benefits that are in line with Metcalfe's law, were the utility of a node from the network is the number of nodes he is connected to (directly or indirectly).

Formally, given a set of nodes $U \subseteq V$, let $\mathcal{G}(U)$ be the set of all undirected graphs over U and let $\mathcal{G} = \bigcup_{U \subseteq V} \mathcal{G}(U)$ be the set of all undirected graphs that can be formed over V or any of its subsets. A strategy of the hider is a pair $(G, h) \in \mathcal{G}(V) \times V$, where G is the graph and h is the hiding place chosen by H in G. A strategy of the seeker is a function $s : \mathcal{G}(V) \to V$ which, given a graph G over V determines the node $s(G)$ he chooses.

Before defining the payoffs we need to introduce a number of auxiliary notions. Given a set of nodes $U \subseteq V$ and a graph $G = \langle U, E \rangle$ over U, a maximal set of nodes $C \subseteq U$ such that any two nodes $i, j \in C$ are connected in G is a *component* of G.[2] The set of all components of G is denoted by $\mathcal{C}(G)$. In addition, given $i \in U$, let $C_i(G) = C \in \mathcal{C}(G)$, such that $i \in C$, be the component in G containing i. Given a set of nodes $U \subseteq V$, a graph $G = \langle U, E \rangle$ over U, and a set of nodes $U' \subseteq U$, let $G[U'] = \langle U', E[U'] \rangle$ with $E[U'] = \{ij \in E : \{i, j\} \subseteq U'\}$ be the *subgraph of G induced by U'*. Given a node $v \in V$ let $G - v = G[U \setminus \{v\}]$ be the *residual network* obtained from G by removing node v and all its links from G. Given node $v \in V$, let $N_G(v) = \{u \in V : vu \in E\}$ be the *neighbourhood of v in G*.

Given the strategy profile $((G, h), s)$, the payoff to the hider is

$$\Pi^{\mathsf{H}}_{\mathrm{W}}(G, h, s) = \begin{cases} -\beta & \text{if } h \in N_G(s(G)) \cup \{v\} \\ f(|C_i(G - s(G))|) & \text{otherwise.} \end{cases} \tag{1}$$

The payoff to the seeker is $\Pi^{\mathsf{S}}_{\mathrm{W}}((G, h), s) = -\Pi^{\mathsf{H}}_{\mathrm{W}}((G, h), s)$.

3 The analysis

Our objective is to provide optimal networks for the hider as well as to characterize the hiding and the seeking strategies on these networks. As we show in our main result (Theorem 1) these networks consist of a number of singleton nodes and a connected component which either is a cycle or has a particular core-periphery topology, that we describe below.

A *core-periphery* network over a set $V = P \cup C$ of n nodes is a network defined as follows. There are $k \geq \lceil n/2 \rceil$ *core* nodes in set $C = \{\mathrm{c}_1, \ldots, \mathrm{c}_k\}$ and $m \leq \lfloor n/2 \rfloor$ *periphery* nodes in set $P = \{\mathrm{p}_1, \ldots, \mathrm{p}_m\}$. Nodes of the core are connected forming a graph containing a cycle over these nodes, while each periphery node, p_i with $1 \leq i \leq m$, is connected to core node c_i. Nodes of the core which are not connected to a periphery node are called *orphaned*. A core-periphery network where $p = \lfloor n/2 \rfloor$, i.e. p takes its maximal value, is called *maximal*.

Whether a cycle or a core-periphery topology is better for the hider depends on the convexity of f, measured by the following quantity:

$$T(n, s) = f(n - s - 2) - (n - s - 3)(f(n - s - 1) - f(n - s - 2)) + \beta.$$

[2] Two nodes $i, j \in U$ are connected in $G = \langle U, E \rangle$ if there exists a sequence of nodes $i_1, \ldots, i_l$ such that $i_0 = i$, $i_n = j$, and for all $k \in \{1, \ldots, l\}$, $i_{k-1} i_k \in E$.

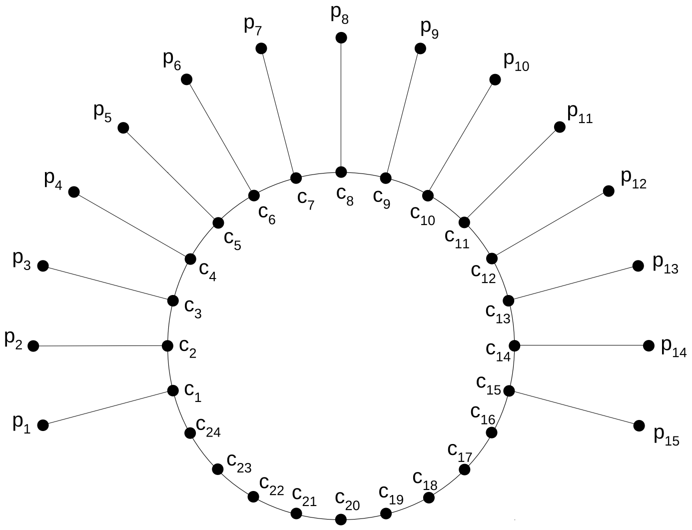

Figure 1: A core-periphery network over 39 nodes, with 15 periphery nodes and 9 orphaned core nodes.

As we show in the proof of the theorem, if $T(n,s) \leq 0$, the cycle topology is better, and otherwise, a core-periphery topology is better.

To specify the optimal networks, we introduce the following quantities, that allow us to determine an optimal number of singleton nodes. Given $n \in \mathbb{Z}_{\geq 0}$ and $s \in \{0,\ldots,n\}$, let

$$\bar{Q}(n,s) = \begin{cases} \frac{\beta B(s) - f(1)f(n-s)}{\beta + B(s) + f(1) + f(n-s)}, & \text{if } n-3 \leq s \leq n \\ \frac{\bar{A}(n,s)B(s) - f(1)f(n-s)}{\bar{A}(n,s) + B(s) + f(1) + f(n-s)}, & \text{if } 0 \leq s \leq n-4 \text{ and } \bar{A}(n,s) > -f(1), \\ \bar{A}(n,s), & \text{otherwise} \end{cases}$$

where

$$B(s) = \left(\frac{1}{s}\right)\beta - \left(1 - \frac{1}{s}\right) f(1)$$

and

$$\bar{A}(n,s) = \begin{cases} \frac{3\beta}{n-s} - \left(1 - \frac{3}{n-s}\right) f(n-s-1), & \text{if } T(n,s) \leq 0, \\ \frac{2\beta}{n-s} - \left(1 - \frac{2}{n-s}\right) f(n-s-2), & \text{if } T(n,s) > 0 \text{ and } n-s \text{ is even}, \\ \frac{2(f(n-s-2)+\beta)^2}{(f(n-s-1)+\beta)(n-s-3)+2(f(n-s-2)+\beta)} & \\ -f(n-s-2), & \text{if } T(n,s) > 0 \text{ and } n-s \text{ is odd}. \end{cases}$$

As we show in the proof of Theorem 1, $\bar{Q}(n,s)$ is an exact lower bound on the payoff to the seeker in a network with exactly s singleton nodes. Since the game is zero-sum, the hider maximises his payoff when seeker's payoff is minimised. Therefore an optimal network has $s \in S^*(n)$ singleton nodes, where

$$S^*(n) = \arg \min_{s \in \{0,\ldots,n\}} \bar{Q}(n,s).$$

We are ready to state the main result of the paper.

Theorem 1. *For any number of nodes, $n \geq 1$, and any $\beta \geq 0$ there exists an equilibrium of the game, $((G,h),s)$ such that*

- *G has exactly $s \in S^*(n)$ singleton nodes and either $s \leq n-4$ or $s = n$.*
- *If $T(n,s) \leq 0$ and $n-s \geq 4$ then G has a cycle component over the remaining $n-s$ nodes.*
- *If $T(n,s) > 0$, $n-s \geq 4$, and $n-s$ is even then G has a maximal core-periphery component over $n-s$ nodes.*
- *If $T(n,s) > 0$, $n-s \geq 4$, and $n-s$ is odd then G has a core-periphery component with three orphaned nodes over $n-s$ nodes.*
- *The hider mixes between hiding in the singleton nodes and in the connected component. When hiding in the singleton nodes, he mixes uniformly across all these nodes. When hiding in the connected component, he mixes uniformly across all the nodes (when it is a cycle), mixes uniformly across the periphery nodes (when it is a maximal core-periphery network), and mixes between hiding in periphery nodes, mixing uniformly across them, and the middle orphaned node (otherwise).*
- *The seeker mixes between seeking in the singleton nodes and in the connected component. When seeking in the singleton nodes, he mixes uniformly across all these nodes. When seeking in the connected component, he mixes uniformly across all the nodes (when it is a cycle), mixes uniformly across the core nodes (when it is a maximal core-periphery network), and mixes between seeking in the neighbours of periphery nodes, mixing uniformly across them, and the middle orphaned node (otherwise).*

Equilibrium payoff to the hider is $-\bar{Q}(n,s)$.

The proof is lengthy and we provide a brief description of the general technique before giving the details. We start by constructing a strategy of the seeker that, for each network over the set of nodes V, provides a (mixed) seeking strategy on that network. This strategy determines payoffs the seeker can secure for each possible network over V. Since the game is zero-sum, minus these payoffs are an upper bound on the payoff the hider can get, for each network. Next for each $s \in \{0,\ldots,n\}$, we construct a network that is optimal for the hider across all possible networks with exactly s singleton nodes. In the case of $T(n,s) \leq 0$, as well as in the case of s being even, these networks yield payoffs to the hider that meet the upper bound determined in the first part of the proof. In the case of $T(n,s) > 0$ and odd s, the upper bound from the first part of the proof is not exact. Therefore in this step we establish both, the optimal networks and the exact upper bound on the hider's payoff.

Proof of Theorem 1. Before proceeding with the proof we introduce auxiliary notions and notation. In particular, we introduce a partition of nodes into a number of different sets that will play a crucial role in further construction.

Given a (possibly disconnected) network G over the set of nodes V, node $i \in V$ is a *singleton node* if $|N_G(i)| = 0$. The set of singleton nodes of G is denoted by $S(G)$. Node $i \in V$ is a *leaf* if $|N_G(i)| = 1$. The set of leaves of G is denoted by $L(G)$. Given node $i \in V$, let $l_i(G) = |N_G(i) \cap L(G)|$ denote the number of leaf-neighbours of i. Let $D(G) = \{i \in L(G) : N_G(i) \subseteq L(G)\}$ be the set of leaves connected to a leaf only. Such leaves constitute two node components in G. Set $D(G)$ can be partitioned in a natural way into two equal size subsets, $D_1(G)$ and $D_2(G)$, $D_1(G) \cup D_2(G) = D(G)$, such that for each $l \in \{1,2\}$, and any two distinct nodes, i, j, in $D_l(G)$, nodes i and j are not connected in G. In other words, for any 2-node component of G, one of its nodes is in $D_1(G)$ and the other one is in $D_2(G)$. Fix any such partition.

Let

$$M(G) = \{i \in V \setminus D_2(G) : l_i(G) = 1\}$$

be the set of nodes which are not in $D_2(G)$ and are connected to exactly one leaf in G and let

$$SL(G) = \{i \in L(G) : N_G(i) \cap M(G) \neq \varnothing\}$$

be the set of leaves connected to an element of $M(G)$ (clearly $D_2(G) \subseteq SL(G)$). Such leaves are called *singleton leaves*. Let $R(G) = V \setminus (S(G) \cup SL(G) \cup M(G))$ be the set of nodes in G which are neither a singleton, nor a singleton leaf, nor a neighbour of a singleton leaf.

We start with the construction of a strategy of the seeker that secures a certain payoff on each network for him. Take any network H over V and let $s = |S(H)|$ and $m = |M(H)|$. Moreover, let $HR = H[R(H)]$ be the subnetwork of H generated by the set of nodes $R(H)$ (i.e. network H restricted to the nodes on $R(H)$). In particular, when $R(H) = \varnothing$, HR is the empty network with empty sets of nodes and links. Consider mixed strategies of player S, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, of the following form

$$\boldsymbol{\sigma} = \lambda_{\mathrm{S}}\boldsymbol{\sigma}^{\mathrm{S}} + (1 - \lambda_{\mathrm{S}})\left(\lambda_{\mathrm{R}}\boldsymbol{\sigma}^{\mathrm{R}} + (1 - \lambda_{\mathrm{R}})\,\boldsymbol{\sigma}^{\mathrm{M}}\right) \tag{2}$$

where

$$\sigma_i^{\mathrm{S}} = \begin{cases} \frac{1}{s}, & \text{if } i \in S(H), \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma_i^{\mathrm{M}} = \begin{cases} \frac{1}{m}, & \text{if } i \in M(H), \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma_i^{\mathrm{R}} = \begin{cases} \frac{l_i(HR)+1}{n-s-2m}, & \text{if } i \in R(H) \setminus L(HR), \\ 0, & \text{otherwise,} \end{cases}$$

$\lambda_{\mathrm{R}}, \lambda_{\mathrm{S}} \in [0, 1]$. Clearly $\boldsymbol{\sigma}^{\mathrm{S}}$ is a valid probability distribution as long as $S(H) \neq \varnothing$, that is $s > 0$. Similarly, $\boldsymbol{\sigma}^{\mathrm{M}}$ is a valid probability distribution as long as $M(H) \neq \varnothing$, that is $m \geq 1$. It is also easy to see that $\boldsymbol{\sigma}^{\mathrm{R}}$ is a valid probability distribution as long as $R(H) \neq \varnothing$. To see that notice that $R(H)$ contains exactly $n - s - 2m$ nodes and $\boldsymbol{\sigma}^{\mathrm{R}}$ could be obtained from a uniform distribution on $R(H)$ by moving the probability mass assigned to leaves in HR from leaves to their neighbours (this is well defined, because there are no two-node components in HR). Lastly, notice that if $S(H) \neq \varnothing$, then either all the non-singleton nodes in H have degree 1, in which case $M(H) \neq \varnothing$, or there exists a node in H of degree 2 or more, in which case either $M(H) \neq \varnothing$ or $R(H) \neq \varnothing$. Hence if $S(H) \neq \varnothing$, then either $\boldsymbol{\sigma}^{\mathrm{M}}$ or $\boldsymbol{\sigma}^{\mathrm{R}}$ is a valid probability distribution. By these observations, $\boldsymbol{\sigma}$ is a valid probability distribution as long as $\lambda_{\mathrm{S}} = 1$, if $s > n-2$, $\lambda_{\mathrm{S}} = 0$, if $s = 0$, $\lambda_{\mathrm{R}} = 0$, if $R(H) = \varnothing$, and $\lambda_{\mathrm{R}} = 1$, if $m = 0$.

The idea behind the strategies $\boldsymbol{\sigma}$ is as follows. With probability λ_{S}, player S seeks in the set of singleton nodes, $S(H)$, and with probability $(1 - \lambda_{\mathrm{S}})$ he seeks outside this set. Conditional on seeking outside $S(H)$, with probability λ_{R} player S seeks in the set of nodes $R(H)$ and with probability $(1 - \lambda_{\mathrm{R}})$ he seeks in the set $SL(H) \cup M(H)$.

When seeking in $S(H)$, S mixes uniformly across all the singleton nodes. When seeking in $SL(H) \cup M(H)$, S mixes uniformly across all the nodes neighbouring a singleton leaf, that is all the nodes in $M(H)$. Lastly, when seeking in the set of nodes $R(H)$, S mixes using strategy $\boldsymbol{\sigma}^{\mathrm{R}}$. Notice that this guarantees that player H is caught with probability at least $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}3/(n-s-2m)$, if he hides in a node from $R(H)$ that is neither a singleton node nor a singleton leaf in the graph HR, induced by the set of nodes $R(H)$. To see that, take any node $i \in R(H)$ that is not a singleton node in HR nor a singleton leaf in HR. Suppose that i is not a leaf in HR, i.e. $i \in R(H) \setminus L(HR)$. Then i has at least two neighbours in $R(H)$ and the probability that seeker seeks at i or at one of i's neighbours is at least $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}3/(n-s-2m)$. Suppose that i is a leaf in HR but not a singleton leaf there, i.e. $i \in L(HR) \setminus SL(HR)$. Then i has a neighbour $j \in R(H)$ that has at least one more leaf neighbour in HR. Thus the probability that seeker puts a seeking resource at i or at one of i's neighbours is also at least $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}3/(n-s-2m)$.

We now narrow down the possible strategies, $\boldsymbol{\sigma}$, by setting the value of λ_{R}. This is done under the assumption that $S(H) \neq V$, that is $s \leq n-2$ and there exist non-singleton nodes in H. Let

$$\begin{aligned}\varrho &= \frac{(n-s-2m)(f(n-s-2)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\\ &= 1-\frac{3m(f(n-s-1)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\end{aligned}$$

and

$$\lambda_{\mathrm{R}} = \begin{cases} 0, & \text{if } R(H)=\varnothing,\\ \varrho, & \text{otherwise.}\end{cases}$$

Clearly $\varrho \in [0,1]$ and $\lambda_{\mathrm{R}} \in [0,1]$. The value of λ_{R} fixed above guarantees that H is caught with probability at least $\lambda_{\mathrm{R}}3/(n-s-2m)$, when he hides outside singleton nodes of H, even if he hides in a singleton node of HR or a singleton leaf of HR, i.e. in $i \in S(HR) \cup SL(HR)$. In this case, i must have a neighbour, j', in $M(H)$. For otherwise i would be a singleton node in G or a singleton leaf in G and so j would belong to $S(H) \cup M(H)$ and not to $R(H)$. The probability of S putting a seeking resource in j' is

$$\begin{aligned}&(1-\lambda_{\mathrm{S}})(1-\lambda_{\mathrm{R}})\left(\frac{1}{m}\right) \geq\\ &(1-\lambda_{\mathrm{S}})\min\left(1, \frac{3m(f(n-s-1)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\right)\left(\frac{1}{m}\right) =\\ &(1-\lambda_{\mathrm{S}})\left(\frac{3(f(n-s-1)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\right) >\\ &(1-\lambda_{\mathrm{S}})\left(\frac{3(f(n-s-2)+\beta)}{3m(f(n-s-1)+\beta)+(n-s-2m)(f(n-s-2)+\beta)}\right) =\\ &(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}\left(\frac{3}{n-s-2m}\right). \quad (3)\end{aligned}$$

Thus i is caught with probability at least $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}3/(n-s-2m)$.

Suppose that $R(H) \neq \varnothing$. Conditional on H hiding in a node of $R(H)$, using any of the strategies $\boldsymbol{\sigma}$ defined above, player S obtains payoff of at least

$$\begin{aligned}&L^{\mathrm{R}}(n,m,s) =\\ &(1-\lambda_{\mathrm{S}})\left(\lambda_{\mathrm{R}}\left(\left(\frac{3}{n-s-2m}\right)\beta - \left(1-\frac{3}{n-s-2m}\right)f(n-s-1)\right)\right.\\ &\qquad\left. - (1-\lambda_{\mathrm{R}})f(n-s-2)\right) - \lambda_{\mathrm{S}}f(n-s)\end{aligned}$$

(regardless of the strategy of the hider). To see why, notice that in the case of $R(H) \setminus (S(HR) \cup L(HR)) = \varnothing$ all nodes in $R(H)$ are either singleton leafs or singleton nodes in HR and, as we argued above, have at least one neighbour in $M(H)$. Using $\boldsymbol{\sigma}$, when searching outside $S(H)$, S mixes uniformly across the nodes in $M(H)$ only and so the probability of capture for any node in $S(HR) \cup L(HR)$ is at least $1/m$ and, in the case of not capturing, at least two nodes are removed from the component of the node chosen by H. In the case of $R(H) \setminus (S(HR) \cup L(HR)) \neq \varnothing$, the probability of capture is at least $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}} 3/(n-s-2m)$ and, in the case of not capturing the hider, S guarantees that the component of the hider has at most $n-s-1$ nodes, with probability $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}$, has at most $n-s-2$ nodes, with probability $(1-\lambda_{\mathrm{S}})(1-\lambda_{\mathrm{R}})$, and has at most $n-s$ nodes, with probability λ_{S}. Lastly, conditional on H hiding in a node of $M(H) \cup SL(H)$, using any of the strategies $\boldsymbol{\sigma}$ defined above, player S obtains the payoff at least

$$L^{\mathrm{M}}(n,m,s) = (1-\lambda_{\mathrm{S}})\Bigg((1-\lambda_{\mathrm{R}})\Bigg(\left(\frac{1}{m}\right)\beta - \left(1-\frac{1}{m}\right)f(n-s-2)\Bigg) - \lambda_{\mathrm{R}} f(n-s-1)\Bigg) - \lambda_{\mathrm{S}} f(n-s),$$

(regardless of the strategy of the hider) as the probability of capture is at least $(1-\lambda_{\mathrm{S}})(1-\lambda_{\mathrm{R}})1/m$ and, in the case of not capturing the hider, S guarantees that the component of the hider has size at most $n-s-2$, with probability $(1-\lambda_{\mathrm{S}})(1-\lambda_{\mathrm{R}})$, has size at most $n-s-1$, with probability $(1-\lambda_{\mathrm{S}})\lambda_{\mathrm{R}}$, and has size at most $n-s$, with probability λ_{S}.

It is elementary to verify that the value of λ_{R} is such that $L^{\mathrm{R}}(n,m,s) = L^{\mathrm{M}}(n,m,s)$, for any $s \in \{0,\ldots,n-2\}$. Hence the lower bound on the payoff of player S in H when H hides outside singleton nodes is $L(n,m,s) = L^{\mathrm{R}}(n,m,s) = L^{\mathrm{M}}(n,m,s) = (1-\lambda_{\mathrm{S}})\,A(n,m,s) - \lambda_{\mathrm{S}} f(n-s)$ where

$$A(n,m,s) = \begin{cases} \frac{\beta}{m} - \left(\frac{m-1}{m}\right) f(n-s-2), \text{ if } R(H) = \varnothing, \\ \left(\frac{D(n,s)D(n-1,s)}{3D(n,s)-2D(n-1,s)}\right)\left(\frac{3T(n,s)}{m(3D(n,s)-2D(n-1,s))+(n-s)D(n-1,s)} - 1\right) \\ +\beta, \text{ otherwise,} \end{cases}$$

where

$$D(n,s) = f(n-s-1) + \beta$$

and

$$T(n,s) = D(n-1,s) - (n-s-3)(D(n,s) - D(n-1,s))$$

(in particular, the derivation above is valid for the extreme cases of $m = 0$ and $m = (n-s)/2$). Notice that $A(n,m,s)$ is strictly increasing in m, if $T(n,s) < 0$, is strictly decreasing in m, if $T(n,s) > 0$, and is constant if $T(n,s) = 0$.

To complete the definition of strategy $\boldsymbol{\sigma}$ we establish the value of λ_{S}. Conditional on H hiding in a node of $S(H)$, using any of the strategies $\boldsymbol{\sigma}$ defined above, player S obtains payoff of at least $L^{\mathrm{S}}(n,m,s) = \lambda_{\mathrm{S}} B(s) - (1-\lambda_{\mathrm{S}})\,f(1)$ where

$$B(s) = \left(\frac{1}{s}\right)\beta - \left(1 - \frac{1}{s}\right) f(1)$$

(regardless of the strategy of the hider) as the probability of capture is λ_{S}/s and, in the case of not capturing the hider, S gets payoff $-f(1)$. Let

$$\lambda_{\mathrm{S}} = \begin{cases} 1, & \text{if } V = S(H), \\ \frac{A(n,m,s)+f(1)}{A(n,m,s)+B(s)+f(1)+f(n-s)}, & \text{if } V \neq S(H) \text{ and } A(n,m,s) > -f(1), \\ 0, & \text{otherwise.} \end{cases}$$

To see that $\lambda_{\mathrm{S}} \in [0,1]$, notice that $B(s) > -f(1) \geq -f(n-s)$, for any $\beta \geq 0$ and $0 \leq s \leq n-1$. It is elementary to verify that the value of λ_{S} is such that in the case of $V \neq S(H)$ (i.e. $s \leq n-2$): if $A(n,m,s) > -f(1)$ then $L^{\mathrm{s}}(n,m,s) = L(n,m,s)$, for any $s \in \{0,\ldots,n-2\}$, and if $A(n,m,s) \leq -f(1)$ then $L^{\mathrm{s}}(n,m,s) \geq L(n,m,s)$, for any $s \in \{0,\ldots,n-2\}$. Hence, in the case of $V \neq S(H)$, the lower bound on the payoff of player S in H is $Q(n,m,s) = (1-\lambda_{\mathrm{S}})A(n,m,s) - \lambda_{\mathrm{S}} f(n-s)$. In the case of $V = S(H)$, there are only singleton nodes in H and $\boldsymbol{\sigma}$ mixes uniformly across them and $\lambda_{\mathrm{S}} = 1$. Thus the lower bound on the payoff of S in H is

$$Q(n,m,s) \quad = \quad \begin{cases} B(n), & V = S(H) \\ \frac{A(n,m,s)B(s) - f(1)f(n-s)}{A(n,m,s)+B(s)+f(1)+f(n-s)}, & \text{if } V \neq S(H) \text{ and } A(n,m,s) > -f(1), \\ A(n,m,s), & \text{otherwise,} \end{cases}$$

and it can be secured by strategy $\boldsymbol{\sigma}$.

Recall that $A(n,m,s)$ is increasing in m, when $T(n,s) < 0$, decreasing in m when $T(n,s) > 0$, and constant, when $T(n,s) = 0$. This, together with Lemma 1 in the appendix implies that in the case of $V \neq S(H)$, for any $0 \leq s \leq n-1$, $Q(n,m,s)$ is decreasing in m, when $T(n,s) > 0$, is increasing in m, when $T(n,s) < 0$, and is constant in m, when $T(n,s) = 0$. Hence, in the case of $V \neq S(H)$, for any $s \in \{0,\ldots,n-1\}$, $Q(n,m,s)$ is minimised at $m = (n-s)/2$, when $T(n,s) > 0$, and is minimised at $m = 0$, when $T(n,s) < 0$.

In the last step of the proof we turn to construction of networks that are optimal for the hider. Firstly, we construct optimal networks for the hider, given the number of singleton nodes in the network, s.

Notice that in the case of $s \geq n-3$, if the hider hides in $R(H)$ and the seeker seeks there, then the hider is caught with probability 1, regardless of the topology of HR. It is easy to verify that a strategy profile where the seeker chooses a node in $R(H)$ with probability $\kappa = (B(s) + f(n-s))/(B(s) + f(1) + f(n-s) + \beta)$ and, with probability $1-\kappa$, mixes uniformly across the nodes in $B(s)$, while the hider hides in a node in $R(H)$ with probability $\kappa = (B(s) + f(1))/(B(s) + f(1) + f(n-s) + \beta)$ and, with probability $1-\kappa$, mixes uniformly across the nodes in $B(s)$ is an equilibrium. Thus the exact lower bound on the payoff of the seeker in a network with exactly s singleton nodes is equal to

$$\frac{\beta B(s) - f(1)f(n-s)}{\beta + B(s) + f(1) + f(n-s)}.$$

For the remaining part of the proof, we restrict attention to $s \leq n-4$.

Consider the case of $n-s$ being even first. Let

$$\bar{A}(n,s) = \begin{cases} A(n,(n-s)/2,s), & \text{if } T(n,s) > 0, \\ A(n,0,s), & \text{if } T(n,s) \leq 0. \end{cases}$$

and let

$$\kappa = \begin{cases} \frac{B(s)+f(1)}{\bar{A}(n,s)+B(s)+f(n-s)+f(1)} & \text{if } \bar{A}(n,s) > -f(1), \\ 1, & \text{otherwise.} \end{cases} \tag{4}$$

Suppose that the hider chooses a maximal core-periphery network, in the case of $T(n,s) > 0$, or a cycle, in the case of $T(n,s) \leq 0$, over $n-s$ nodes in the first round. In the second, hide and seek, round the hider hides in the component of size $n-s$ with probability κ, mixing uniformly on the periphery nodes (in the case of the component being a core-periphery network) and mixing uniformly over all its nodes (in the case of the component being a cycle), and hides in the singleton nodes with probability $1-\kappa$, mixing uniformly on them. By similar arguments to those used for λ_{S}, above, $\kappa \in [0,1]$ and so the strategy is valid. If the seeker seeks in the singleton nodes, this yields payoff of at least $\kappa f(n-s) - (1-\kappa)B(s)$ to the hider and, since the game is zero-sum, of at most minus this value to the seeker. Similarly, if the seeker seeks in the core-periphery component, this yields payoff of at least $-\kappa\bar{A}(n,s) + (1-\kappa)f(1)$ to the hider and of at most minus this value to the seeker. With the value of κ, above, both these guarantees are equal, in the case of $\bar{A}(n,s) > -f(1)$, and the latter is greater, otherwise. Hence, the strategy guarantees payoff $-\kappa A(n,s) + (1-\kappa)f(1)$ to the hider. It is elementary to verify that $-\kappa\bar{A}(n,s) + (1-\kappa)f(1) = -\bar{Q}(n,s)$. As we showed above, $\bar{Q}(n,s)$ is the minimal payoff the seeker can get on any network with exactly s singleton nodes. Since the game is zero-sum, $-\bar{Q}(n,s)$ is the maximal payoff the hider can get on any network with exactly s singleton nodes and the network constructed above allows the hider to attain this payoff.

Next, consider the case of $n-s$ being odd. Let

$$\bar{A}(n,s) = \begin{cases} A(n,(n-s-3)/2,s), & \text{if } T(n,s) > 0, \\ A(n,0,s), & \text{if } T(n,s) \leq 0. \end{cases}$$

and let κ be defined as in (4). If $T(n,s) \leq 0$ than choosing a cycle over $n-s$ nodes and using the same hiding strategy as in the case of $n-s$ being even, the hider secures the highest possible payoff on a network with exactly s singleton nodes. Suppose that $T(n,s) > 0$. The difficulty here is that $(n-s)/2$ is not an integer and therefore, in the case of $T(n,s) \leq 0$, the hider cannot attain the upper bound on his payoff determined by the lower bound on the payoff to the seeker, $\bar{Q}(n,s)$. Recall that if $T(n,s) > 0$ then for any $0 \leq s \leq n-4$, $Q(n,m,s)$ is decreasing in m. We will show, for any $0 \leq s \leq n-4$, that the hider can attain payoff $-Q(n,(n-s-3)/2,s)$, by choosing the right network and the right hiding strategy and that this is the maximal payoff he can secure when $n-s$ is odd.

Suppose that the hider chooses a core-periphery network with three orphaned nodes over $n-s$ nodes (c.f. Figure 2). Consider a strategy of the hider

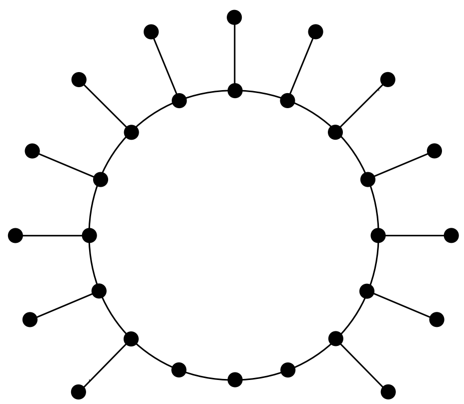

Figure 2: A core-periphery network over 23 nodes with 3 orphaned nodes.

$$\boldsymbol{\eta} = \kappa(\mu\boldsymbol{\eta}^{\mathrm{M}} + (1-\mu)\boldsymbol{\eta}^{\mathrm{R}}) + (1-\kappa)\boldsymbol{\eta}^{\mathrm{S}},$$

where

$$\eta_i^{\mathrm{M}} = \begin{cases} \frac{1}{m}, & \text{if } i \in SL(G), \\ 0, & \text{otherwise,} \end{cases}$$

(i.e. $\boldsymbol{\eta}^{\mathrm{M}}$ mixes uniformly on the periphery nodes of G),

$$\eta_i^{\mathrm{R}} = \begin{cases} 1, & \text{if } i \text{ is the middle orphaned node in } G, \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta_i^{\mathrm{S}} = \begin{cases} \frac{1}{s}, & \text{if } i \in S(G), \\ 0, & \text{otherwise.} \end{cases}$$

(i.e. $\boldsymbol{\eta}^{\mathrm{S}}$ mixes uniformly on the singleton nodes of G), and

$$\mu = \frac{(n-s-3)f(n-s-2) + (n-s-3)\beta}{(n-s-3)f(n-s-1) + 2f(n-s-2) + (n-s-1)\beta}.$$

It is immediate to see that $\mu \in [0,1]$ and so the hiding strategy is valid. If the seeker seeks in the orphaned nodes of the core-periphery component, this yields payoff of at least $\kappa(\mu f(n-s-1) - (1-\mu)\beta) + (1-\kappa)f(1)$ to the hider and, since the game is zero-sum, of at most minus this value to the seeker. Similarly, if the seeker seeks in periphery nodes or their neighbours in the core-periphery component, this yields payoff of at least $\kappa(\mu(-2\beta/(n-s-3) + (1-2/(n-s-3))f(n-s-2)) + (1-\mu)f(n-s-2)) + (1-\kappa)f(1)$ to the hider and of at most minus this value to the seeker. With the value of μ, above, both these guarantees are equal. It is elementary to verify that $\kappa(\mu f(n-s-1) - (1-\mu)\beta) + (1-\kappa)f(1) = -\kappa A(n,(n-s-3)/2,s) + (1-\kappa)f(1) = -Q(n,(n-s-3)/2,s)$. Since $Q(n,(n-s-3)/2,s)$ is a lower bound on the payoff that the seeker can secure in a network with exactly s singleton nodes and at most $(n-s-3)/2$ singleton leaves, minus this value is the highest payoff that the hider can secure in a network with exactly s singleton nodes and at most $(n-s-3)/2$ singleton leaves.

The only networks that could yield a higher payoff to the seeker are networks with exactly s singleton nodes and $(n-s-1)/2$ singleton leaves. In any such network, H, the set $R(H)$ consist of exactly one node and this node is connected to at least two nodes in $M(H)$ (it cannot be connected to one node in $M(H)$, because in this case its neighbour would have two leaf-neighbours and could not be a member of $M(H)$). Let $\tilde{\boldsymbol{\sigma}} = \lambda\boldsymbol{\sigma}^{\mathrm{S}} + (1-\lambda)\boldsymbol{\sigma}^{\mathrm{M}}$, where $\boldsymbol{\sigma}^{\mathrm{M}}$ and $\boldsymbol{\sigma}^{\mathrm{S}}$ are the mixed strategies of the seeker, defined earlier in the proof,

$$\lambda = \begin{cases} \frac{X(n,s)+f(1)}{B(s)+X(n,s)+f(1)+f(n-s)}, & \text{if X(n,s) ¿ -f(1),} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$X(n,s) = \frac{2\beta}{n-s-1} - \left(1 - \frac{2}{n-s-1}\right) f(n-s-2).$$

Using this strategy, with probability λ, S mixes uniformly on the nodes in $M(H)$ and with probability $(1-\lambda)$, S mixes uniformly on the singleton nodes of H. Payoff to S conditioned on H hiding in a singleton node is at least $\lambda B(s) - (1-\lambda)f(1)$ and payoff to S conditioned on H hiding outside singleton nodes is at least $(1-\lambda)X(n,s) - \lambda f(n-s)$. It is easy to verify that the value of λ is such that both these payoffs are equal (in the case of $X(n,s) > -f(1)$) or the latter is higher, for any value of λ. Therefore payoff to S from using $\tilde{\boldsymbol{\sigma}}$ against any strategy of H is at least

$$Y(n,s) = \begin{cases} \frac{B(s)X(n,s) - f(1)f(n-s)}{B(s)+X(n,s)+f(1)+f(n-s)}, & \text{if } X(n,s) > -f(1), \\ X(n,s), & \text{otherwise,} \end{cases}$$

and so the upper bound on the payoff to the hider on any network with s singleton nodes and $(n-s-1)/2$ singleton leaves is at most $-Y(n,s)$. To see that $-Q(n,(n-s-3)/s,s) > -Y(n,s)$ notice that

$$X(n,s) - A(n,(n-s-3)/2,s) = \frac{2(f(n-s-1)-f(n-s-2))(f(n-s-2)+\beta)(n-s-3)}{(n-s-1)(f(n-s-1)(n-s-3)+2f(n-s-2)+\beta(n-s-1))} > 0$$

and so $X(n,s) > A(n,(n-s-3)/2,s)$. This, together with Lemma 1 (in the Appendix), implies that $Y(n,s) > Q(n,(n-s-3)/2,s)$. Hence $-Q(n,(n-s-3)/2,s)$ is the exact upper bound on the payoff that a hider can secure by choosing a network with exactly s singleton nodes when $n-s$ is odd. Network that achieves this bound consist of a core-periphery network over $n-s$ nodes with three orphaned nodes and of s singleton nodes.

The analysis above shows that the exact lower bound on the payoff of the seeker in a network with exactly s singleton nodes is

$$\bar{Q}(n,s) = \begin{cases} \frac{\beta B(s) - f(1)f(n-s)}{\beta + B(s) + f(1) + f(n-s)}, & \text{if } n-3 \leq s \leq n, \\ Q(n,0,s), & \text{if } 0 \leq s \leq n-4 \text{ and } T(n,s) \leq 0, \\ Q(n,(n-s)/2,s), & \text{if } 0 \leq s \leq n-4,\ T(n,s) > 0 \text{ and } n-s \text{ is even}, \\ Q(n,(n-s-3)/2,s), & \text{if } 0 \leq s \leq n-4,\ T(n,s) < 0 \text{ and } n-s \text{ is odd}. \end{cases}$$

Thus an optimal network for the hider has exactly s^* singleton nodes, where s^* minimises $\bar{Q}(n,s)$, and a component over $n-s^*$ nodes that is either a cycle (if $T(n,s^*) \leq 0$) or a complete core-periphery network (if $T(n,s^*) > 0$ and n is even) or a core-periphery network with three orphaned nodes (if $T(n,s^*) < 0$). □

To get an intuition about the result in Theorem 1, notice that the hider weighs between the cost of being caught and the value he gets in the residual network, after the seeker's action. More links in the network allow for obtaining higher connectivity which secures larger value after the the seeker's action but, at the same time, it leads to higher exposure. Fixing the number of singleton nodes, s, the choice between a cycle and a core-periphery network is influenced by the convexity of function f, as measured by the quantity $T(n,s)$. The probability of being caught in a cycle of size $n-s$ is $3/(n-s)$, as each node has exactly two neighbours, while in the case of not being caught, only one node is lost from the cycle component. The probability of being caught in a maximal core-periphery network, on the other hand, is $2/(n-s)$ (if the hider hides mixing uniformly across the periphery nodes), while in the case of not being caught, two nodes are lost from the cycle component (if the seeker seeks mixing uniformly across the core nodes). If f is sufficiently convex, the marginal loss from an additional node being removed from a component is high and, therefore, a cycle is preferred over the core-periphery network. If f is not sufficiently convex, on the other hand, the marginal loss from an additional node being removed from a component is not sufficiently high and the hider prefers to opt for the safer, core-periphery, network. Notice that the quantity measuring the convexity of f, $T(n,s)$, increases in s, as $T(n,s+1) - T(n,s) = f(n-s-1) - f(n-s-2) - (f(n-s-2) - f(n-s-3)) \geq 0$ (because f is convex and strictly increasing). When the size of the connected component is larger, the marginal loss from loosing a single node is higher, due to convexity of f.

As we show in the proof of Theorem 1, equilibrium payoff to the seeker in an optimal network with at least one singleton node is a convex combination of $B(s)$ and $-f(1)$ and so it is at least $-f(1)$. Thus if the payoff the seeker can secure in a connected component of size n, $\bar{A}(n, 0) < -f(1)$, it is optimal for the hider to choose a connected network without singleton nodes. If on the other hand, the cost of being caught, β, is sufficiently high then $\bar{A}(n, 0) > -f(1)$ and the payoff the hider can secure in a connected network, $-\bar{A}(n, 0)$, is less than the payoff he gets if he is not caught in a singleton node. This motivates the hider to construct a network with smaller component and $s \geq 1$ singleton nodes. If the cost of being caught is sufficiently high, it is optimal for the hider to choose a disconnected network with $s = n$ singleton nodes.

To conclude the analysis, we provide a characterization of possible optimal networks in the special case of $f(x) = x$, so that the value the hider gets from a connected network is the number of nodes in his component. This is in line with the important network value function based on Metcalfe's law.

Proposition 1. *Let $f(x) = x$, for all $x \in \mathbb{R}_{\geq 0}$. For any number of nodes, $n \geq 1$, and any $\beta \geq 0$ there exists an equilibrium of the game, $((G, h), s)$ such that*

- *G has exactly $s^* \in S^*(n)$ singleton nodes and $S^*(n) \subseteq \{0, 1, n\}$.*
- *If $n - s^* \geq 4$ and is even then G has a maximal core-periphery component over $n - s^*$ nodes*
- *If $n - s^* \geq 4$ and is odd then G has a core-periphery component with three orphaned nodes over $n - s^*$ nodes.*

Proof. Notice that if $f(x) = x$, then $T(n, s) = \beta + 1 > 0$, for all $s \in \{0, \ldots, n\}$. From proof of Theorem 1 we know that for any $s \in \{0, \ldots, n - 4\}$ a network with a core-periphery component of size $n - s$ (either maximal or with three orphaned nodes) is optimal among networks with exactly s singleton nodes. By Theorem 1, if $n \leq 5$ then $s^* \in \{0, 1, n\}$. Suppose that $n \geq 6$. We show in the Appendix (Lemma 2) that if $n \geq 6$, then $\bar{Q}(n, s)$ is either minimised at $s = 0$ or it is minimised at $s = n$ and on $\{1, \ldots, n\}$ it is minimised at $s = 1$ or at $s = n$. This completes the proof. □

4 Conclusions

We proposed and studied a strategic model network design and hiding in the network facing a hostile authority that attempts to disrupt the network and capture the hider. We characterized optimal networks for the hider as well as optimal hiding and seeking strategies in these networks. Our results suggests that the hider chooses networks that allow him to be anonymous and peripheral in the network. We also developed a technique for solving such models in the setup of zero-sum games.

There are at least two avenues for future research. Firstly, different forms of benefits from the network could be considered. For example, the utility of the hider could dependent not only on the size of his component but also on his distance to the nodes in the component. Given our results, we conjecture that this would make the core periphery components with better connected core more attractive. But answering this problem precisely requires formal analysis. Secondly, the seeker could be endowed with more than one seeking unit and the units could be used either simultaneously or sequentially. Our initial investigation suggests that solving such an extension might be an ambitious task.

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Appendix

A Proofs

Lemma 1. *Function*

$$\varphi(Z) = \begin{cases} \frac{B(s)Z - f(1)f(n-s)}{Z+B(s)+f(n-s)+f(1)}, & \text{if } Z > -f(1), \\ Z, & \text{otherwise,} \end{cases}$$

is strictly increasing in Z.

Proof. Notice that $\varphi(-f(1)) = -f(1)$ when $Z = -f(1)$. Moreover, φ is increasing in Z if $Z < -f(1)$. Let $Z > -f(1)$. Taking the derivative of φ with respect to Z we get

$$\varphi'(Z) = \frac{(B(s) + f(1))(B(s) + f(n-s))}{(Z + B(s) + f(n-s) + f(1))^2}$$

and it is immediate to see that $\varphi'(Z) > 0$ and φ increases in Z when $B(s) > -f(1)$ and $B(s) \geq -f(n-s)$. Notice that $B(s) = (\beta + f(1))/s - f(1) > -f(1)$ for any $\beta \geq 0$ and $s > 0$. Also $f(n-s) \geq f(1)$ for all $s \in [0, n-1]$. Thus, by the observation on function φ, above, $\varphi(Z)$ increases when Z increases. □

Lemma 2. *Let $f(x) = x$, for all $x \in \mathbb{R}_{\geq 0}$. For any natural $n \geq 6$, $t \in \{0, 1\}$ and any $s \in \{t+1, \ldots, n\}$, $\bar{Q}(n, s) > \min(\bar{Q}(n, n), \bar{Q}(n, t))$*

Proof. When $f(x) = x$,

$$\bar{A}(n, s) = 2\binom{\beta - 2}{n - s} + 4 - (n - s), \text{ for } 0 \leq s \leq n - 2,$$

and

$$\bar{Q}(n, s) = \begin{cases} \bar{A}(n, s), & \text{if } \bar{A}(n, s) \leq -1 \text{ or } s = 0, \\ AB(n, s), & \text{if } 1 \leq s \leq n - 2 \text{ and } \bar{A}(n, s) > -1 \\ B(n), & \text{otherwise,} \end{cases} \tag{5}$$

with

$$AB(n, s) = (1 - \varrho)\bar{A}(n, s) - \varrho(n - s)$$

where ϱ solves

$$(1 - \varrho)\bar{A}(n, s) + \varrho(s - n) = \varrho B(s) - (1 - \varrho). \tag{6}$$

Notice that $\bar{A}(n, s)$ is increasing in s on $[0, n-2]$ and it is equal to β at $s = n - 2$. Thus there exists a unique $\tilde{s} \in [0, n-2]$ such that $A(n, \tilde{s}) = -1$. Solving (6) we get

$$\varrho = \frac{s(2(\beta - 2) - (n - s)(n - s - 5))}{s(2(\beta - 2) - (n - s)(n - s - 5)) + (n - s)(s(n - s - 1) + \beta + 1)}.$$

Notice that $2(\beta - 2) - (n - s)(n - s - 5) > 0$ if and only if $\bar{A}(n, s) > -1$, and $(n - s)(s(n - s - 1) + \beta + 1) > 0$ for $s \leq n - 1$. Thus if $\bar{A}(n, s) > -1$ then $\varrho \in (0, 1)$. In addition $B(s) > -1$, for all $s > 0$, so if $\varrho \in (0, 1)$ then $AB(n, s) > -1$.

By the observations above, if $\bar{A}(n, 1) \leq -1$ then $\bar{Q}(n, 0) < \bar{Q}(n, 1) < \bar{Q}(n, s)$ for all $s \in \{2, \ldots, n\}$ and the claim of the lemma holds.

For the remaining part of the proof suppose that $A(n, 1) > -1$. This implies $2(\beta - 2) > (n - 1)(n - 6)$ and, consequently, $\beta > 2$ if $n \geq 6$. We will show that $\bar{Q}(n, s)$ is either decreasing or first increasing and then decreasing on $[0, n-1]$. On $[0, \tilde{s}]$, $\bar{Q}(n, s) = \bar{A}(n, s)$ and, as we argued above, $\bar{Q}(n, s)$ is increasing. Consider the interval $[\tilde{s}, n-1]$. Notice that since $B(s) > -1 \geq n - s$, for $0 < s \leq n-1$, and $\bar{A}(n, \tilde{s}) = -1$ so $AB(n, \tilde{s}) = -1$. In addition, $AB(n, n) = B(n)$. We will show that $AB(n, s)$ is either decreasing or first increasing and then decreasing on $[0, n]$. Inserting ϱ into (5) we get

$$\bar{Q}(n, s) = \frac{(n^2(\beta+1) - 2n(s(\beta-1) + 2(\beta+1)) + s^2(\beta-3) + 6s\beta - 2(\beta+1)(\beta-2))}{s(4s-\beta+5) - n(4s+\beta+1)}.$$

Notice that $\bar{Q}(n, \tilde{s}) = A(n, \tilde{s}) = -1$. Taking the derivative of $\bar{Q}$ with respect to s we get

$$\frac{\partial \bar{Q}}{\partial s} = \frac{(\beta+1)W(s)}{(s(4s-\beta+5) - n(4s+\beta+1))^2},$$

where

$$W(s) = Xs^2 - 2Ys + \left(n + \frac{\beta-2}{2}\right)Y - \left(\frac{\beta-2}{2}\right)(n-4)(\beta+1),$$

with $X = 4n - \beta - 15$ and $Y = 4n^2 + n(\beta - 19) - 8(\beta - 2)$.

The sign of $\partial\bar{Q}/\partial s$ is the same as the sign of $W(s)$. Notice that $W(n) = -2(\beta - 2)(n + \beta - 5) < 0$, as $n \geq 6$ and $\beta > 2$. When $X > 0$, then $W(s)$ is an $\bigcup$-shaped parabola and, since $W(n) \leq 0$, either W is negative or W is first positive and the negative on $[0, n]$. Thus in this case $\bar{Q}$ is either increasing or first increasing and then decreasing on $[0, n]$. Similar observation holds when $X = 0$. Suppose that $X < 0$. In this case $W(s)$ is an $\bigcap$-shaped parabola and it has a maximum at $s^* = Y/X$. Suppose that $s^* \in (0, n-2)$. Since $X < 0$ so $Y < 0$. Moreover, for $n \geq 6$, $X < 0$ implies $\beta > 5$ and, consequently, Moreover,

$$\begin{aligned} W(s^*) &= -Ys^* + \left(n + \frac{\beta-2}{2}\right)Y - \left(\frac{\beta-2}{2}\right)(n-4)(\beta+1) \\ &= \left(n - s^* + \frac{\beta-2}{2}\right)Y - \left(\frac{\beta-2}{2}\right)(n-4)(\beta+1) < 0. \end{aligned}$$

Thus W is either negative or first positive then negative on $[0, n]$, for any natural $n \geq 5$. Hence $\bar{Q}$ is either decreasing or first increasing and then decreasing on $[0, n]$, for any natural $n \geq 6$.

By the analysis above, when $A(n, 1) > -1$ then $AB(n, s)$ is either decreasing or first increasing and then decreasing in s on $[0, n]$ and $AB(n, n) = B(n)$. Hence, by the definition of $\bar{Q}(n, s)$, the claim of the lemma follows immediately. $\square$